STABILITY OF A PLANE POISEUILLE FLOW OF AN IDEALLY CONDUCTING FLUID IN A LONGITUDINAL MAGNETIC FIELD

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The necessary and sufficient conditions for stability of a flow in a magnetic field have been found. It is shown that the critical value of the magnetic field that stabilizes the flow is $0.1\ V_0\sqrt{4\pi\rho}$ where V_0 is the velocity in the center of the channel and ρ is the fluid density.

1. INTRODUCTION

In this paper we investigate the stability of flow of an ideally-conducting liquid in a longitudinal magnetic field with respect to infinitesimally small disturbances. We use here the asymptotic method of Heisenberg and Lin, the applicability of which was proven by Wasow and Tatsumi as well as by the agreement with the results of Thomas, obtained with a calculating machine. The physical interpretation of this method was confirmed in the experiments of Schubauer and Skramstad (see reference 1 and the literature therein). The inverse limiting case of poorly-conducting liquid was considered by Stuart.[2]

2. STATEMENT OF THE PROBLEM

The motion of an incompressible conducting liquid and of the field in it is described by the equations of magnetohydrodynamics

$$\rho\frac{d\mathbf{V}}{dt} = -\nabla P - \frac{1}{4\pi}\mathbf{B}\times\operatorname{curl}\mathbf{B} + \nu\nabla^2\mathbf{V};\ \operatorname{div}\mathbf{V} = 0;$$

$$d\mathbf{B}/dt = (\mathbf{B}\nabla)\mathbf{B} + \lambda\nabla^2\mathbf{B};\ \operatorname{div}\mathbf{B} = 0;\ \lambda = c^2/4\pi\sigma, \quad (2.1)$$

where ν is the kinetic viscosity and σ the conductivity of the liquid. The character of the flow is determined by the values of three dimensionless parameters: the hydrodynamic Reynolds number $R_g = V_c L/\nu$, the magnetic Reynolds number $R_m = V_c L/\lambda$, and the Alfven number $A = B_c/V_c\sqrt{4\pi\rho}$ (V_c is the characteristic velocity, B_c is the characteristic induction of the magnetic field, and L is the characteristic dimension of the flow).

At the critical values of these parameters the flow becomes absolutely unstable, i.e., infinitesimally small disturbances in the flow start increasing with time. To find the critical values of the parameters, we restrict ourselves to a consideration of the initial stage of the increase in disturbances, while they are still considerably smaller than the corresponding stationary values. We therefore seek all quantities in the form of large stationary terms and small disturbing increments

$$\mathbf{V} = \mathbf{V}_0 + \mathbf{v},\ \mathbf{B} = \mathbf{B}_0 + \mathbf{b},\ P = P_0 + p. \quad (2.2)$$

Inserting (2.2) into (2.1) we obtain a system of equation for the disturbances. The coefficients of this system are independent of the time. Its solutions are therefore of the form

$$f(\mathbf{r}, t) = f(\mathbf{r})\,e^{-i\omega t},$$

where ω is determined from the solution of the boundary-value problem for the system (2.1) with corresponding boundary conditions. The sign of the imaginary part of ω determines whether the disturbances increase or decrease with time.

We shall assume henceforth that the speed of flow and the field are directed along the x axis of a rectangular system of coordinates, and that all quantities depend only on z. The solutions have therefore the form

$$f(\mathbf{r}, t) = f(z)\exp(-i\omega t + ik_x x + ik_y y). \quad (2.3)$$

It was shown by Michael[3] that the most dangerous are two dimensional disturbances with $k_y = 0$. Using this, we obtain from (2.1) a sixth-order differential equation for the z component of the disturbance of the magnetic field.

In dimensionless symbols, it has the form

$$\left\{\left(w - \frac{\omega}{k}\right)^2 - A^2 - \frac{iw''}{kR_m}\right\}b_z'' + 2w'\left(w - \frac{\omega}{k}\right)b_z'$$

$$- k^2\left\{\left(w - \frac{\omega}{k}\right)^2 - A^2 - \frac{iw''}{kR_m}\right\}b_z$$

$$= -\frac{i}{k}\left\{\frac{(d^2/dz^2 - k^2)^2}{R_g}\left(w - \frac{\omega}{k}\right)b_z + \frac{(w - \omega/k)}{R_m}\right.$$

$$\left.\times\left(\frac{d^2}{dz^2} - k^2\right)^2 b_z\right\} + \frac{1}{k^2 R_m R_g}\left(\frac{d^2}{dz^2} - k^2\right)^3 b_z, \quad (2.4)$$

where $k = k_x$ and $w = V_0/V_c$. The z component of the velocity disturbance is connected with the z component of the disturbance of the magnetic field by the following relation

$$v_z = (w - \omega/k)\, b_z + (i/kR_m)(d^2/dz^2 - k^2)\, b_z. \qquad (2.5)$$

The boundary conditions at infinity consist of the vanishing of the disturbances of the velocity and of the field

$$\mathbf{v} = 0;\ \mathbf{b} = 0 \ \text{for}\ z = \pm\infty.$$

On solid surfaces we need specify only the vanishing of the velocity disturbance

$$\mathbf{v} = 0 \ \text{for}\ z = z_b. \qquad (2.6)$$

The magnitude of the field disturbance on a solid surface depends on the material of the solid. On a wall made of material with $\mu \sim 1$ and a conductivity σ_b, the boundary condition has the following form

$$b_z'/b_z = \pm(k^2 - i\omega\, 4\pi\sigma_b\, LV_0/c^2)^{1/2} \ \text{for}\ z = \pm 1. \qquad (2.7)$$

It is obtained by joining the solution of the magnetohydrodynamic equations inside the flow with the solution of Maxwell's equation inside a thick wall.

Thus, the problem has been reduced to the determination of the eigenvalues of ω from the solution of the non-self-adjoint differential equation (2.4) with boundary conditions (2.6) and (2.7).

3. SUFFICIENT CONDITIONS OF STABILITY

For a well conducting liquid, $R_m \gg 1$. In most problems of practical interest the instability can be expected only at $R_g \gg 1$. Therefore all the terms in the right half of Eq. (2.4) are preceded by small parameters. However, it is dangerous merely to discard these terms, for this reduces the order of the differential equation and raises the paradox of vanishing dissipation.[1]

As is known, the dissipation is important not only in the thin boundary layer near the walls, where it is necessary to satisfy a larger number of boundary conditions than there are solutions for the "ideal" equation, but also near the singular points of this equation, obtained from Eq. (2.4) by going to the limits $R_g \to \infty$ and $R_m \to \infty$. At the point $z_0\,(w = \omega/k)$ phase resonance occurs between the disturbances and the flow, and at points z_1 and z_2 resonance occurs between the disturbances and the local Alfven wave.

However, as will be shown below, a strong magnetic field prevents such a resonance. Therefore in a strong magnetic field there should be no vanishing-dissipation paradox and the liquid can be considered as ideal. Eq. (2.4) now becomes

$$\frac{d}{dz}\left\{\left[(w - \omega/k)^2 - A^2\right]\frac{d}{dz}b_z\right\} - k^2\left[(w - \omega/k)^2 - A^2\right] b_z = 0 \qquad (3.1)$$

with boundary conditions $b_z = 0$ on the walls. If the characteristic velocity of flow is taken to be at a maximum, $V_c = \max V_0$, then $0 < w < 1$.

Let us multiply (3.1) by b_z^* and integrate between the walls. Integrating by parts, we obtain

$$\omega_i \int_{z_{b_2}}^{z_{b_1}} (w - \omega_r/k)\left(|b_z'|^2 + k^2|b_z|^2\right) dz = 0; \qquad (3.2)$$

$$\int_{z_{b_2}}^{z_{b_1}} \left[(w - \omega_r/k)^2 - A^2 - \omega_i^2/k^2\right]\left(|b_z'|^2 + k^2|b_z|^2\right) dz = 0, \qquad (3.2a)$$

where $\omega = \omega_r + i\omega_i$.

It follows from (3.2) that for disturbances that increase with time to exist it is necessary that the phase velocity of the disturbances coincide with the local velocity of the flow somewhere inside this flow. Therefore

$$0 < \omega_r/k < 1. \qquad (3.3)$$

Analogously, we obtain from (3.2a)

$$\omega_i^2/k^2 < \max(w - \omega_r/k)^2 - A^2 < 1 - A^2.$$

This leads to the sufficient condition for the stability of flow:

$$A^2 > 1. \qquad (3.4)$$

In this case resonant points z_1 and z_2 are outside the flow. For disturbances that increase with time to exist, it is apparently necessary that all three resonant points be present.

As expected, a strong field $(A > 1)$ stabilizes the flow no matter what the velocity of distribution. However, the critical value of the Alfven number A_{cr}, depends on the profile, particularly on whether this profile has a point of inflection, which determines the stability of flow of a non viscous liquid.

4. ASYMPTOTIC METHOD OF SOLUTION

We confine ourselves to an examination of monotonic distributions, symmetrical with respect to the point $z = 0$, of the speed of flow between solid surfaces at a distance $2L$ apart (at $z = \pm 1$, where z is a dimensionless vertical coordinate).

Equation (2.4) is now symmetrical with respect to the point $z = 0$. Its solutions therefore break up into a group of symmetrical and a group of antisymmetrical solutions, b_z^+ and b_z^-. We confine ourselves to an examination of the symmetrical disturbances, which have a smaller number of zeros and lead to instability at lower values of R_g.

By solving the boundary-value problem we can obtain ω_i as a function of the parameters k, R_m, R_g, and A. The equation $\omega_i = 0$ determines the critical surface that separates the stability region from the instability region. To obtain this equation we put $\omega_i = 0$ in the original equations, i.e., we consider neutral disturbances.

It is obvious that the region of instability is bounded on the side of large k, since short-wave disturbances bend very strongly the lines of force and are, furthermore, attenuated by dissipation. This region is also bounded on the side of small R_g, since the field apparently cannot lead to a reduction in the critical value of this number.

Such limitations cannot be established for R_m and A, since they determine the influence of the field only in conjunction with each other. When $R_m < 1$ the role of electromagnetic retarding force, as shown by Stuart,[2] is determined by the product $R_m A^2$, and when $R_m \gg 1$, as we have seen, the flow is stable for $A > 1$.

Using these limitations, we seek approximate solutions of Eq. (2.4) in the form of asymptotic series in powers of the small parameters $1/kR_g$ and $1/kR_m$:

$$b_z(z) = \sum b_{z;\,p,\,n} (kR_m)^{-n} (kR_g)^{-p}.$$

For the zero approximation we obtain the second-order-equation

$$\frac{d}{dz}\left\{[(w-\omega/k)^2 - A^2]\frac{d}{dz} b_{z;\,0,\,0}\right\} - k^2\{(w-\omega/k)^2 - A^2\} b_{z;\,0,\,0} = 0. \quad (4.1)$$

We seek a solution of this equation in the form of a convergent series in powers of k^2 (we shall henceforth drop the index z):

$$b_{00}^{+} = \sum_{n=0}^{\infty} q_n^{+} k^{2n}, \qquad b_{00}^{-} = \sum_{n=0}^{\infty} q_n^{-} k^{2n}, \quad (4.2)$$

where

$$q_{n+1}^{\pm} = \int_0^z \frac{dz}{(w-\omega/k)^2 - A^2} \int_0^z \{(w-\omega/k)^2 - A^2\} q_n^{\pm}\, dz, \quad (4.3)$$

$$q_0^{+} = 1, \qquad q_0^{-} = \int_0^z \frac{dz}{(w-\omega/k)^2 - A^2}. \quad (4.4)$$

Thus, $b_{00}^{+} \approx 1 + k^2 q_1^{+}$. We shall call the solution b_{00}^{+} "ideal" and denote it by b_{id}^{+}. The four other linearly-independent solutions will be sought in the form

$$b_z = \exp\left\{\int g(z)\, dz\right\}, \quad (4.5)$$

$$g(z) = \sqrt{kR_m}\, g_0 + g_1 + g_2/\sqrt{kR_m} + O(1/kR_m). \quad (4.6)$$

For $g_0(z)$ we obtain the expression

$$g_0^2 = \frac{i}{2}(w-\omega/k) \pm \left[-\frac{(w-\omega/k)^2}{4} + \frac{R_g}{R_m}\{(w-\omega/k)^2 - A^2\}\right]^{1/2}. \quad (4.7)$$

Away from the points $z = z_0$ and $z = z_1, z_2$, we have when $R_m/R_g \gg 1$

$$g_{0;\,3,\,4} = \pm \frac{1}{\sqrt{kR_m}}\left[ikR_g \frac{(w-\omega/k)^2 - A^2}{w-\omega/k}\right]^{1/2}; \quad (4.8)$$

$$g_{0;\,5,\,6} = \pm\sqrt{i(w-\omega/k)}. \quad (4.9)$$

The asymptotic expressions (4.8) and (4.9) are poor near the points $z = z_0$, z_1, and z_2. Approximate solutions near these points can be obtained, as in the WKB method, by expanding the coefficients of Eq. (2.4) in a series about the resonance points and by stretching the scale, so as to retain only the essential terms. For this purpose we make the change of variable

$$z - z_p = \varepsilon\eta, \quad (4.10)$$

where

$$\varepsilon^{-3} = -e^{-3i\pi/2}\, 2w_p' kR_g, \qquad w_p' \equiv w(z_p), \quad (4.11)$$

$$\eta = -i\sqrt{-2w_p' kR_g (z - z_p)}. \quad (4.12)$$

Inserting this new variable into Eq. (2.4) and putting $R_m = \infty$ in the same equation, we obtain, with accuracy to ϵ,

$$\frac{d}{d\eta}(\eta b_0' + b_0''') = 0. \quad (4.13)$$

This equation has the following solutions:

$$b_{01} = 1 + O(\varepsilon); \quad b_{02} = \int S_{1,\,1/3}(^2/_3\eta^{3/2})\sqrt{\eta}\, d\eta, \quad (4.14)$$

$$b_{03} = \int h_1(\eta)\, d\eta, \quad (4.15)$$

$$b_{04} = \int h_2(\eta)\, d\eta, \quad (4.16)$$

where h is the Airy function (see reference 4) and S is the Lommel function (see reference 5).

The region of applicability of these asymptotic expressions differs from the region of applicability of expressions (4.8) and (4.9), for these are valid for fixed η and $kR_g \to \infty$, while (4.8) and (4.9) are valid for fixed z and $kR_g \to \infty$.

Away from the resonant points, the asymptotic representation of (4.15) and (4.16) coincides with (4.8), for in the sector

$$-\pi/6 < \arg\zeta < 7\pi/6 \quad (4.17)$$

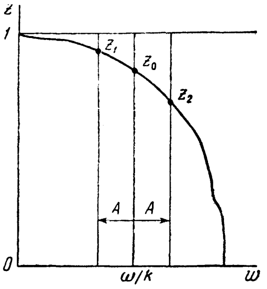

FIG. 1. Critical points on the velocity profile.

we have (see reference 4)

$$b_{03}(\zeta) \approx \zeta^{-3/4} \exp\{{}^2/_3 e^{-i\pi/4} \zeta^{3/2}\}[1 + O(\zeta^{-3/2})],$$

$$\zeta = (-2w'_p k R_g)^{1/3}(z - z_p). \qquad (4.18)$$

We thus have obtained asymptotic expressions for the linearly-independent solutions of Eq. (2.4). However, the resonant points are branch points of these functions.

Therefore, for neutral disturbances, it is necessary to find the correct path around the branch point. For disturbances that increase with time, the asymptotic expression for solutions (4.4), (4.8), and (4.9) has no singularities. If one considers a neutral disturbance as the limiting case of an increasing disturbances as $\omega_i \to 0$, this corresponds to going around the singular points in the complex z plane from above, since upon passing through the point $w = \omega_r/k$, the argument $w = \omega_r/k - i\omega_i/k$ (where $\omega_i > 0$) changes by $-\pi$ (see Fig. 1). If a neutral disturbances are considered as the limiting case of an attenuating disturbance, the phase changes by π. This paradox was resolved by Landau and Wasow. Landau[6] solved the initial-condition problem in an analogous case with the aid of a unilateral Laplace transformation, while Wasow[7] investigated the problem with allowance for dissipation. It was found in both cases that one must go around the resonant points from the $\omega_i > 0$ side. The same can be concluded from the condition (4.17), for when it is satisfied the four linearly-independent solutions of Eq. (4.13) go into two "ideal" solutions and two rapidly fluctuating ones.

As seen in Sec. 2, for neutral or increasing disturbances to exist it is necessary, apparently, that all three resonant points be located inside the flow. Therefore the point z_1 should be located near the wall. A numerical calculation for $z_1 > 1$ also discloses no neutral disturbances.

Since $\arg(w - \omega/k)$, $\arg(w - \omega/k - A)$, and $\arg(w - \omega/k + A)$ change by $-\pi$ in going about the resonant points, the real part of (4.7) does not reverse its sign. Therefore the solutions b_{04} and b_{06} increase exponentially from the wall inside the flow, and their contribution to the symmetrical combinations $a_1 b_{03} + a_2 b_{04}$ and $c_1 b_{05} + c_2 b_{06}$ is negligible near the wall. The point z_0 is separated from the boundary by a finite distance, and one can use for b_{05} the expression

$$b_{05} = \exp\left\{\int g_{05}\, dz\right\}. \qquad (4.19)$$

On an ideally conducting wall, the boundary conditions for b are of the form

$$b = 0; \quad b' = 0; \quad i\omega R_m b' + b''' = 0. \qquad (4.20)$$

Inserting (4.4), (4.15), and (4.19) into (4.20), and neglecting R_g/R_m compared with unity, we obtain the dispersion equation

$$b^+_{id}/b^{+'}_{id} = (b_{03}/b'_{03})\{1 + (b^{+'''}_{id}/b^{+'}_{id})\, i\omega R_m\}. \qquad (4.21)$$

When $R_m = \infty$ (4.21) becomes

$$b^+_{id}/b^{+'}_{id} = b_{03}/b'_{03}. \qquad (4.22)$$

5. STABILITY OF PLANE FLOW FOR $R_m = \infty$

The right half of Eq. (4.22) is expressed in terms of the Airy function $h_1(0 - i\zeta)$:

$$\frac{b_{03}}{b'_{03}} = \frac{\int_{-\infty}^{\zeta} h_1(0 - i\zeta)\, d\zeta}{\zeta h_1(0 - i\zeta)}(z - z_1) = \Phi(\zeta). \qquad (5.1)$$

The function $\Phi(\zeta)$ is calculated by numerical integration and is shown in Table I and Fig. 2. It is important that the imaginary part of this function has a maximum equal to 0.387 at $\zeta \approx 2.4$, and tends to zero when $\zeta \to 1.5$ and $\zeta \to \infty$. This behavior of $\mathrm{Im}\,\Phi(\zeta)$ results in two branches of

TABLE I. The function $\Phi(\zeta)$

ζ	Re Φ (ζ)	Im Φ (ζ)	ζ	Re Φ (ζ)	Im Φ (ζ)
1.4	1.054	−0.03784	2.8	0.2316	0.3373
1.6	0.9677	0.09275	3.0	0.1644	0.2926
1.8	0.8476	0.2125	3.2	0.1190	0.2470
2.0	0.7195	0.3082	3.4	0.09002	0.2049
2.2	0.5789	0.3683	3.6	0.07436	0.1678
2.3	0.5087	0.3822	3.8	0.06408	0.1393
2.4	0.4421	0.3871	4.0	0.06026	0.1161
2.5	0.3803	0.3835	4.2	0.05917	0.09858
2.6	0.3241	0.3727			

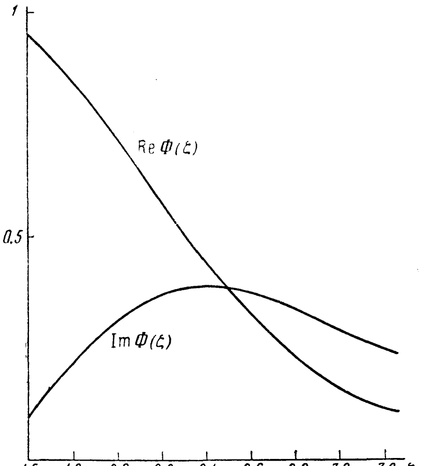

FIG. 2. The function $\Theta(\zeta)$.

the neutral curve. Asymptotically, when $\zeta \to \infty$,

$$\Phi(\zeta) \approx \zeta^{-3/2} e^{-i\pi/4}[1 + {}^3/_4 \zeta^{-3/2} e^{i\pi/4} + \zeta^{-3}(-{}^5/_{64} + 2i)]. \quad (5.2)$$

The imaginary part of the "ideal" solution b_{id}^{+} equals the sum of the half residues at the resonant points z_1 and z_2. When going around these points, art $(w - \omega/k \pm A)$ changes by $-\pi$. Therefore

$$\operatorname{Im} b_{id}^{+} = \frac{k^2\pi}{2A}\left\{\frac{1}{w_1'}\int_0^{z_1}[(w - \omega/k)^2 - A^2]\,dz - \frac{1}{w_2'}\int_0^{z_2}[(w - \omega/k)^2 - A^2]\,dz\right\}. \quad (5.3)$$

It follows from Sec. 2 that the neutral disturbances are possible only when $A < 1$. Therefore Eq. (5.3) can be calculated approximately by expanding it in a series* in powers of A:

$$\operatorname{Im} b_{id}^{+} = \frac{k^2\pi}{w_1'}\left\{\frac{w_1''}{w_1'^2}H_1 - \left(\frac{3w_1''^2}{w_1'^4} - \frac{w_1'''}{w_1'^3}\right)A + F_1 A^2\right\},$$

$$H_1 = \int_0^{z_1}(w - \omega/k)^2\,dz, \quad (5.4)$$

where

$$F_1 = -\frac{12\,w_1'' w_1'''}{w_1'^5}H_1 + \frac{20\,w_1''^3}{w_1'^6}H_1 - \frac{2\,w_1^{(VI)}}{3\,w_1'^4} - \frac{2\,w_1'' z_1}{w_1'^2} + \frac{26}{3\,w_1'}.$$

The derivative of the symmetrical solution is

$$\text{F} \quad b_{id}^{+\prime} = \frac{k^2}{(w - \omega/k)^2 - A^2}\int_0^{z}\{(w - \omega/k)^2 - A^2\}\,dz. \quad (5.5)$$

Let us consider disturbances arising in a thin layer near the wall, i.e., let us put $1 - z_1 \ll 1$, which is not a limitation, since it follows from (5.6) that actually the distance between this point and the wall is always small. This permits us to express the critical values of the parameters in terms of the flow characteristics near the wall.

Inserting (5.4), (5.5), and (5.3) into (4.22) and considering that

$$(1 - z_1) \sim -(\omega/k - A)/w_b'; \quad (w_b' \equiv w'(1)), \quad (5.5a)$$

we get

$$-\frac{\pi w_b'}{w_1'^3}\,\frac{w_1'' H_1 + (w_1'''/w_1' - 3\,w_1''^2/w_1'^2)\,A}{H_b}\left(\frac{\omega}{k} + A\right) = \operatorname{Im}\Phi(\zeta). \quad (5.6)$$

The left half of (5.6) increases with increasing ω/k, i.e., with decreasing z_1, provided

$$(w_1''' w_1' - 3\,w_1''^2)/w_1'^4 < 0. \quad (5.7)$$

This condition is satisfied for convex profiles of the Poiseuille-flow type. Therefore, the left half of (5.6) has a minimum at $z_1 = 1$ and equals

$$-(2\pi A/w_b'^2)\,[w_b'' + A(w_b'''/w_b' - 3\,w_b''^2/w_b')]. \quad (5.8)$$

If (5.8) is greater than max $\operatorname{Im}\Phi(\zeta) = 0.387$, the flow is stable. From this condition it is possible to determine the critical Alfven number

$$A_{cr}^{-1} = -8.12\,w_b''/w_b'^2 - [(66 + 48.7/H_b)\,w_b''^2/w_b'^4 - 16.2/H_b - w_b'''/w_b'^3]^{1/2}. \quad (5.9)$$

If $|\omega_b'''| \ll |\omega_b''^2/\omega_b'^2|$, we obtain from (5.9)

$$A_{cr}^{-1} = -(8.12 + \sqrt{66 + 48.7/H_b})\,w_b''/w_b'^2 \approx -21\,w_b''/w_b'^2, \quad (5.10)$$

since $H_b = \int_0^1 (w - \omega/k)^2\,dz \sim 1/2$. Finally

$$A_{cr} \approx -0.05\,w_b'^2/w_b''. \quad (5.10a)$$

For a Poiseuille flow, it follows from (5.10) that

$$A_{cr}^{-1} \approx -20.6\,w_b''/w_b'^2 \approx 10.$$

A more exact numerical calculation given in Sec. 6 yields a value 9.8.

The smaller the critical magnetic Alfven number, the stronger the stabilizing influence of the field. It therefore follows from (5.10) that the greater the curvature of the profile, the more stable the flow and the less the field required for its stabilization. For example, when Poiseuille flow is established, the profile at the inlet of the tube is almost exponential near the wall, $|w''/w'^2| \sim 1$, and the critical Alfven number is half that of a steady-state parabolic profile.

If $A < A_{cr}$, then the flow is unstable when $R_g > R_g^{cr}(A)$. The curve of neutral stability $k(R_g)$ for flow in a field has a different asymp-

*Actually, it follows from the results [see (5.10)] that $A_{cr} \sim 1/10$.

totic value than without the field, since the "ideal" solution has no solutions with $k = 0$ and $\omega/k = A$.

Since $\operatorname{Im}\Phi(\zeta)$ is a function with a maximum, Eq. (5.6) has two solutions for $A < A_{cr}$, $\zeta = \zeta_1$ or ζ_2. The left half of Eq. (5.6) is bounded from below. Therefore $\infty > \zeta_1 > \zeta_2 > 0$ and $(1 - z_1) \sim (kR_g)^{-1/3}$, i.e., the resonant point z_1 is located at all times within a viscous layer near the wall. To find analytically the asymptote of the neutral-stability curve, we note that in the term $\sim \ln(1 - z_1)$ in the expression for $\operatorname{Re} b_{id}^{+}$ near the wall is particularly large. Therefore

$$\operatorname{Re} b_{id}^{+} \sim 1 - k^2 \frac{H_1}{2w_1' A} \ln(1 - z_1),$$

$$(1 - z_1) \sim \exp\{2 w_b' A / k^2 H_b\}. \tag{5.11}$$

Since ζ is finite, we have

$$R_g = -\zeta^3 / 2 k w_1' (1 - z_1)^3 \sim \exp\{-6 w_b' A / k^2 H_b\}. \tag{5.12}$$

For Poiseuille flow $H \sim 8/15$ and $R_g \sim \exp(22.3\, Ak^{-2})$.

The critical Reynolds number is determined by the point where the two branches merge. The phase velocity ω/k of such "critical" disturbances is determined from (5.6), $\operatorname{Im}\Phi(\zeta) = 0.387$ where on the right side. The corresponding value of the wave number k is determined from Eqs. (4.22), (4.4), (5.1) and (5.5a) for $\zeta \sim 2.4$

$$\left(1 + k^2 \operatorname{Re} \int_0^1 \frac{dz}{(w - \omega/k)^2 - A^2} \int_1^z \left\{\left(w - \frac{\omega}{k}\right)^2 - A^2\right\} dz\right) \frac{\omega/k + A}{k^2 H_b} = -\frac{0{,}442}{w_b'}. \tag{5.13}$$

The integral in (5.13) can be found approximately by using the transformation

$$J = \int_0^1 \frac{dz}{(w - \omega/k)^2 - A^2} \int_0^z \left\{\left(w - \frac{\omega}{k}\right)^2 - A^2\right\} dz$$
$$= \int_0^1 \frac{dz}{(w - \omega/k)^2 - A^2} \int_0^1 \left\{\left(w - \frac{\omega}{k}\right)^2 - A^2\right\} dz$$
$$- \int_0^1 \frac{dz}{(w - \omega/k)^2 - A^2} \int_{z_1}^1 \left\{\left(w - \frac{\omega}{k}\right)^2 - A^2\right\} dz. \tag{5.14}$$

Since $\omega/k < 1$, the internal integral in the second term is small in the vicinity of the poles of the integrand. Therefore

$$J \approx \int_0^1 \frac{dz}{(w - \omega/k)^2 - A^2} H_b$$
$$\approx -\frac{H_b A^{-1}}{2w_1'} \ln \frac{\omega/k - A}{\omega/k - A - 4A/w_1'}; \tag{5.15}$$

$$k = +\sqrt{-H_b / w_1'}$$
$$\times \left\{-\frac{1}{2A} \ln \frac{\omega/k - A}{\omega/k - A(1 + 4/w_1')} + \frac{0{,}44}{\omega/k + A}\right\}^{-1/2}. \tag{5.16}$$

In spite of the fact that k depends on the difference of the small quantities ω/k and A, the accuracy of the expression is not bad, since this dependence is logarithmic.

Using these expressions, it is possible to obtain from (4.18a), with accuracy to within the order of magnitude, the critical hydrodynamic Reynolds number

$$R_g^{cr} = \frac{\zeta_{cr}^3}{-2 w_1' k (1 - z_1)^3}$$
$$\approx \frac{13{,}8\, w_b'^3}{2 w_1' k (\omega/k - A)^3} \sim \frac{\text{const}}{(A_{cr} - A)^3}. \tag{5.17}$$

We note that one cannot put $A = 0$ in any of the resultant expressions, for in this case the singular points will merge into one and the expansion in the vicinity of z_1 loses its meaning [see (4.13)].

6. STABILITY OF THE POISEUILLE FLOW. NUMERICAL METHOD

In conclusion let us give the results of the calculation of the stability of a parabolic velocity profile. In this case

$$b_{id}^{+} = 1 + k^2 \left\{\frac{z^2}{10} - \frac{1}{A}\left(\frac{2}{15} z_0^4 - \frac{A^2}{5}\right) \ln\left(\frac{z^2 - z_0^2 + A}{z^2 - z_0^2 - A} \frac{z_0^2 + A}{z_0^2 - A}\right) - \frac{z_0^2}{15} \ln \frac{(z^2 - z_0^2)^2 - A^2}{z_0^4 - A^2} + \frac{2}{15} i\pi z_0^2\right\}. \tag{6.1}$$

Inserting this expression into (4.21), we obtain

$$\frac{b_{id}^{+}}{b_{id}^{+\prime}} = \{\operatorname{Re}\Phi(\zeta) - \operatorname{Im}\Phi(\zeta)\,\delta\}$$
$$+ i\{\operatorname{Im}\Phi(\zeta) + \operatorname{Re}\Phi(\zeta)\,\delta\}, \tag{6.2}$$

where

$$\delta = -\frac{4}{kR_m}\left\{\frac{z_0^6 + 3z_0^4 - 9z_0^2 - A^2 z_0^2 + 5 + 3A^2}{(1 - z_0^2)(1 - z_1)^2(1 + z_1)^2(1 - z_0^2 + A)^2} - \frac{1}{z_0^4 - 2z_0^2/3 - A^2 + 1/5}\right\},$$

and z_0^2 is the coordinate of the mean resonant layer $w = \omega/k$. This equation was solved numerically for $kR_m = \infty$, 10^8, 10^7, and 2×10^6.

The neutral curve for $A = 0.08$ and $kR_m = \infty$ is given in Table II and in Fig. 3. Table III and Fig. 3 contain the neutral curve for $kR_m = 10^8$ and the initial portions of the neutral curves for $kR_m = 10^7$ and 2×10^6. It is seen that as kR_m

TABLE II
Neutral curve at A = 0.08, $kR_m = \infty$

z_0^2	First branch		Second branch	
	k^2	$R_g^{1/3}$	k^2	$R_g^{1/3}$
0,919	0,130	3460	0,145	5230
0,918	0,148	1700	0,168	2550
0,915	0,183	663	0,210	971
0,910	0,222	325	0,260	445
0,905	0,255	214	0,299	293
0,900	0,286	159	0,334	211
0,895	0,316	128	0,365	164
0,890	0,346	107	0,394	131
0,885	0,380	92,1	0,420	108
0,880	0,400	84,4	0,439	90,2

TABLE III
Neutral curve at A = 0.08

z_0^2	k^2	$R_g^{1/3}$	k^2	$R_g^{1/3}$
$kR_m = 10^8$				
0,918	0.126	2210	0,129	2520
0,915	0.186	880	0.208	1185
0,910	0.222	330	0.258	462
0,905	0.255	214	0,299	293
0.900	0,286	159	0,334	211
0,895	0,316	128	0,365	164
0,890	0,364	107	0,394	131
0,885	0,380	92,1	0,420	108
0.880	0,400	84,4	0,439	90.2
$kR_m = 10^7$				
0,880	0,393	86,6	0,402	89,3
$kR_m = 2\cdot10^6$				
0,890	0,334	113	0,365	129

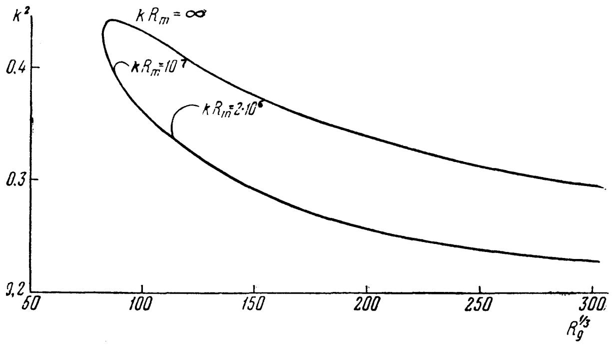

FIG. 3. A neutral curve at A = 0.08, $kR_m = \infty$.

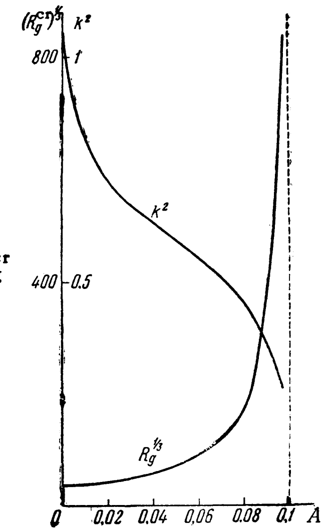

FIG. 4. Dependence of R_g^{cr} and of the wave number of critical disturbances on A.

decreases, the critical hydrodynamic Reynolds number increases. It is impossible to trace the reduction in the stabilizing influence of the field, owing to the limitations of the method.

The dependence of R_g^{cr} and of the wave number k of the critical disturbances on the Alfven number A is given in Table IV and in Fig. 4. The critical value of the Alfren number is 0.102, i.e., to stabilize the flow it is sufficient to have

$$B_c \geqslant V_c\sqrt{4\pi\rho}\,0.102 \approx 0.36\, V_c\sqrt{\rho}.$$

7. CONCLUSIONS

An analysis of infinitesimally small disturbances of plane flow in a magnetic field shows therefore that in the case of ideal conductivity of the liquid it is sufficient, to stabilize any flow, to have $A > 1$, i.e., $B_c^2/8\pi \geq \rho V_c^2/2$.

The field necessary for stabilization depends on the specific type of the velocity profile. The results of this work and of earlier research on stability are given below.

Arbitrary form of velocity profile A = 1
Tangential discontinuity of magnitude V_0 (reference 8) $A = 1/2$
Convex profile with maximum velocity V_0 $A = (\omega_b')^2/20\,|\omega_b|$.
Poiseuille flow A = 0.1

TABLE IV. Dependence of R_g^{cr} and of the wave number of critical disturbances on A

A	z_0^2	k^2	$(R_g^{cr})^{1/3}$	A	z_0^2	k^2	$(R_g^{cr})^{1/3}$
0,00		1.0845	17.45[1]	0.06	0,8608	0,5502	41,92
0;02	0,8266	0,7162	20,58	0.08	0.8784	0,4429	83.04
0.04	0,8438	0,8352	27.78	0.10	0.8956	0,2389	873.7

The difference in the magnitude of the stabilizing field is due to the fact that in flow with a tangential discontinuity the disturbances arise inside the entire layer, which is approximately replaced by the discontinuity, where the total velocity gradient is effective. In the case of a profile with an inflection or a kink, the disturbances arise between the point of inflection and the wall, and in the case of a convex profile they arise in the thin layer near the latter. Therefore the field need merely suppress the instability in this layer. If we insert in the Syrovatskiĭ[8] criterion, instead of the magnitude of the discontinuity, the speed of the extreme resonant layer for critical disturbances $z_2^2 = 0.8$, $\omega/k + A \sim 0.2$, we obtain for A_m^{cr} exactly the critical value

$$\frac{B_c^{cr}}{\sqrt{4\pi\rho}}\frac{1}{0.2V_c} = \frac{1}{2} \text{ and } \frac{B_c^{cr}}{V_c\sqrt{4\pi\rho}} = 0.1.$$

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227