

A NONLINEAR VACUUM EFFECT IN GRAVITATION THEORY

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Second-quantization theory leads to a new interaction between gravitons through the virtual quanta of other fields. For the case of a scalar field in a slowly changing metric, a vacuum cosmological term arises, and can be obtained using Schwinger's method. This can be used to evaluate an additional scattering of gravitons by a Schwarzschild field in a scalar particle vacuum. For low-energy gravitons, the effect is comparable with the nonlinear effect in the classical theory.

1. INTRODUCTION

THE ideas of nonlinear field theory are gaining an ever increasing acceptance in physics. This theory is necessary for a unified description of physical fields, of the interactions between them, and of the structure of particles.¹⁻³ At the present time, the only widely accepted version of the theory is Einstein's theory of the gravitational field, which may turn out to be a consequence of a more fundamental unified theory of matter. However, nonlinear effects can also be analogous to a quantum-mechanical vacuum.^{4,5} In lowest order of perturbation theory, the corresponding non-linear vacuum Lagrangians give effects which are known in high orders for the linear theory. In this connection, it should be emphasized that there is a fundamental difference between "bare" and vacuum nonlinearities. Both types of nonlinearity are serious obstacles to the realization of second-quantization program.^{6,7} Hence it becomes necessary either to develop new methods for second quantization,³ or to use known approximate methods which are useful in the theory of interacting fields.⁸

Leaving aside for the time being the unified field program, we will consider here interacting gravitational and scalar fields, and compare the bare and vacuum nonlinearities. The authors of an earlier work⁹ tried adding a linear vacuum term to \mathcal{L}_g ; they also used an unusual Lagrangian for the scalar field \mathcal{L}_{sc} . In calculating concrete effects, we use the interaction representation (as does Gupta⁸), but do not use tetrapods¹⁰ but follow instead reference 11 in determining \mathcal{L}_g and various physical quantities. Following Piir,¹² we do not consider $\gamma = \delta_{\mu\nu} \gamma^{\mu\nu}$ to be independent we bear in mind the expansion

$$\sqrt{-g} g^{\mu\nu} = \delta^{\mu\nu} - k\gamma^{\mu\nu}, \tag{1}$$

where in cgs units the constant k is

$$k = \sqrt{2\kappa} = \sqrt{16 \pi G / c^2} = 6.1 \cdot 10^{-14} \text{ cm}^{1/2} \cdot \text{g}^{-1/2}, \tag{2}$$

G being Newton's gravitation constant). In the following we use, in addition to cgs units, natural ones ($c = 1, \hbar = 1$).

In the usual formalism for second quantization, in the interaction representation,¹³ the field Lagrangians and other physical quantities are expanded in power series with respect to the constant k; only orders higher than the first are of interest to us. The Lagrangian \mathcal{L}_g for the gravitational field was expanded in reference 14; the first few terms of \mathcal{L}_{sc} are

$$\begin{aligned} \mathcal{L}_{sc} &= 1/2 \sqrt{-g} (g^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu} - m^2 \varphi^2) \\ &= 1/2 (\delta^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu} - m^2 \varphi^2) + k (1/4 m^2 \gamma \varphi^2 - 1/2 \gamma^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu}) \\ &\quad + k^2 m^2 \varphi^2 (1/8 \gamma^{\mu\nu} \gamma_{\mu\nu} - 1/16 \gamma \gamma) + O(k^3). \end{aligned} \tag{3}$$

Since, in the interaction representation, the commutation relations are taken in the (linear) free-field approximation, they agree for the scalar (meson) field with those derived in reference 13. For the gravitational field they are

$$[\gamma^{\lambda\rho}(-)(k), \gamma^{\sigma\tau}(l)] = (\delta^{\lambda\rho} \delta^{\sigma\tau} - \delta^{\lambda\sigma} \delta^{\rho\tau} - \delta^{\lambda\tau} \delta^{\rho\sigma}) \delta(k-l) \tag{4}$$

which is in accord with the form of the canonical energy-momentum quasi-tensor.¹¹ It is not difficult to verify that the spin of the gravitational field is two (we could equally well take the non-antisymmetrized spin of reference 11). Starting from a non-covariant Lagrangian for gravity,¹² Piir also obtained the relations (4).

2. THE VACUUM NON-LINEAR TERM IN \mathcal{L}_g (SCHWINGER'S METHOD)

In this section we obtain a vacuum correction (analogous to a cosmological term) to the Lagrangian of the gravitational field, i.e., we consider stationary fields only. This will give an interpretation of further conclusions to be drawn about cross sections.

Variation of the $g_{\mu\nu}$ in the action integral, constructed with the exact Lagrangian (3), gives

$$\delta W_{sc} = \frac{i}{2} \int (dx) \left[\left(\delta g^{\mu\nu} \frac{\partial^2}{\partial x^\mu \partial x^\nu} + m^2 \delta \sqrt{-g} \right) D^c(x-y) \right]_{x \rightarrow y} \quad (5)$$

Introducing the notation

$$\mathfrak{D} = \sqrt{-g} (g^{\alpha\beta} \partial^2 / \partial x^\alpha \partial x^\beta + m^2) \quad (6)$$

this expression can be written in the form

$$\delta W_{sc} = \frac{1}{2i} \delta \int_0^\infty ds \cdot s^{-1} \text{Sp} \exp(-is \mathfrak{D}). \quad (7)$$

We have used here the following property of Green's functions:

$$\mathfrak{D} D^c(x-y) = \delta(x-y). \quad (8)$$

It should be noted that the δ function defined in this way should be a scalar density, which is in accord with invariance of the integral. In this way we are led to the assumption that the matrix elements

$$(x' | \exp(-i \mathfrak{D} s) | x'') = (x(s') | x(0)''), \quad (9)$$

are scalar densities. According to the total differential (7), they enter into the vacuum Lagrangian of the gravitational field in the form

$$\Omega_{vac}(x) = -\frac{i}{2} \int_0^\infty ds \cdot s^{-1} (x | \exp(-i \mathfrak{D} s) | x). \quad (10)$$

The calculation of the matrix elements (9) for a constant field is really no different from Schwinger's calculation.⁴ The final result is:

$$(x(s') | x(0)'') = \frac{-is^{-2}}{(4\pi)^2 \sqrt{-g}} \exp \left[-\frac{i(x' - x'')^2}{4s \sqrt{-g}} - im^2 s \sqrt{-g} \right]. \quad (11)$$

Hence, the vacuum Lagrangian of the gravitational field is

$$\Omega_{vac}(x) = -\frac{\sqrt{-g}}{32 \pi^2} \int_{\tau_0}^\infty d\tau \cdot \tau^{-3} e^{-m^2 \tau}, \quad (12)$$

where a cut-off at a minimal proper time τ_0 has been introduced to avoid a divergence. In contrast

with electrodynamics⁴ and mesodynamics,⁵ we are faced here with a strong divergence in Ω_{vac} ; it is impossible to cut off the integral correctly, and this makes our conclusions only qualitative. Since the sign of the quantity obtained agrees with the sign of Einstein's cosmological term,² (12) leads to the following value for the cosmological constant due to the vacuum

$$\Lambda_{vac} = \frac{\kappa}{64\pi^2} [e^{-m^2 \tau_0} (\tau_0^{-2} - m^2 \tau_0^{-1} - m^4 \ln \tau_0) - m^4 (0.58 + \ln m^2)]. \quad (13)$$

Expression (12) leads formally to the gravitational equations, at least up to corrections linear in the $g_{\mu\nu}$, but since the multiplier $\sqrt{-g}$ has an infinite power series expansion in κ , it must be interpreted in a nonlinear sense. Indeed, this term gives rise to a nonlinear effect in the scattering of gravitons by a Schwarzschild field, there being some extra scattering over and above that calculated previously.¹⁴ It is clear that here we have to deal with the analogue of a mass term, although further difficulties arise, connected with the problem of quantizing the gravitational field, and due to the fact that Einstein's equations with a cosmological term have no solutions in empty space which differ but little from the Galilean ones.

3. THE SCATTERING OF GRAVITONS ON A SCHWARZSCHILD FIELD IN A SCALAR-PARTICLE VACUUM

The vacuum nonlinearity can be approached from another point of view, using the S-matrix formalism. The cosmological term does not enter into the usual theory, but on the basis of the results in Sec. 2, can be expected to affect the scattering in third order with respect to κ . We examined the simplest processes giving scattering of gravitons on a Schwarzschild field (the cross section for the latter is given in reference 14, along with cross sections for scattering of other particles on this field). It is not possible to present all the complicated expressions here. We considered the interaction of gravitons with a vacuum of scalar particles having zero rest mass (for which it is relatively easy to calculate the total matrix element); a typical matrix element was also calculated for the interaction with vacuum meson of mass $m \neq 0$. In the first case, some of the terms, which describe processes where spin is not conserved in the intermediate states, diverge in an unusual way (the denominator vanishes identically because of the relativistic relation between energy and momentum for real particles). This

might be connected with the quantization difficulties mentioned above. In the interests of regularization, we introduced a graviton "mass" μ ; the integrals were cut off at a maximum momentum L (cf. the earlier note on the minimum value of proper time). Considering only terms which do not violate spin conservation, the cross section in the first case ($m = 0$) can be written

$$d\sigma = \frac{4k^8 M^2 L^8}{3 (8\pi)^6} \left(\frac{\mu^2 + 4k^2/3}{\mu^2 + 4k^2 \sin^2(\theta/2)} \right)^2 \frac{d\Omega}{k^4 \sin^4(\theta/2)} \quad (14)$$

while for the case $m \neq 0$

$$d\sigma = \frac{4k^8 M^2 m^4 L^4}{(8\pi)^6} \frac{d\Omega}{k^4 \sin^4(\theta/2)}. \quad (15)$$

These can be compared with the expressions given in reference 14 because they have the same angular dependence at small θ . We see that for long graviton wave lengths the vacuum effect dominates over the classical one. The critical wavelength can easily be obtained. For example, from (15),

$$\lambda_c = (8\pi^2 \hbar / kmL) \sqrt{\hbar/c}. \quad (16)$$

The divergence of the cross section for scattering from a Schwarzschild field is characteristic of all particles with nonvanishing rest mass.¹⁴ The mass of a graviton can be obtained by comparing our cross sections with those in the paper just quoted. According to (15), the mass in grams is

$$\mu = kmL / 4\pi \sqrt{\hbar c} \approx 10^{-6} mL. \quad (17)$$

The meson momentum is to be obtained from the radius of a nucleon, and the mass obtained does not contradict the astronomical data (taking into account the qualitative nature of the calculation). The result (17) can be understood in light of the deep analogy between the cosmological term and the mass term in Klein's equation. Then (13) leads to

$$\mu = k\sqrt{c} / 8\pi\tau_0 \sqrt{\hbar} \quad (m = 0). \quad (18)$$

The mass μ could decrease or even vanish if account were taken of the interaction between gravitation and fields other than the scalar one con-

sidered. If the theory we started with had had a cosmological term, we could speak of its renormalization also.

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