THE DISINTEGRATION OF UNSTABLE SHOCK WAVES IN MAGNETOHYDRODYNAMICS

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Submitted to JETP editor October 30, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) **36**, 1272-1278 (April, 1959)

The fate of an unstable magnetohydrodynamic shock wave is considered; it is shown that such a wave must necessarily disintegrate into several waves among which there are fast and slow magnetoacoustic shock and similarity waves, Alfven discontinuities and a contact discontinuity. It is significant that disintegration of the unstable shock wave is accompanied by an increase in the entropy. The disintegration of a stable shock wave is impossible.

1. INTRODUCTION

ALTHOUGH the conditions for the stability of shock waves in ordinary and magnetohydrodynamics[1-3] are well known, it has not been explained what happens with an unstable magnetohydrodynamic shock wave if it is created in some fashion. This question is considered in the present paper in an example of a plane stationary shock wave in an ideal gas, on both sides of which the magnetic field makes a small angle with the normal to the plane of the discontinuity.

In Sec. 1, the qualitative picture of the disintegration is investigated. In Sec. 2, the problem of the disintegration is solved in zeroth approximation, with neglect of the small tangential magnetic field. In this case, the initial unstable shock wave disintegrates into two discontinuities. However, the approximate distance here between the discontinuities that are formed does not increase with time. Therefore, in making clear the possibility of such a disintegration, it is necessary to consider the following approximation. Consideration of the tangential magnetic field in first approximation is given in Sec. 3. In this approximation the initial shock wave disintegrates into four discontinuities. In Sec. 4, it is shown that in the consideration of a tangential magnetic field, the distances between the discontinuities which are formed continues to grow.

We note that, in order that the disintegration can actually take place, it is necessary not only that the distances between the discontinuities increase with time, but also that the discontinuities be stable. As follows from reference 2, satisfaction of the second condition automatically follows from the satisfaction of the first condition. Moreover, the process of disintegration must be accompanied by an increase in entropy. Satisfaction of this condition is also shown in Sec. 4.

We shall show that the value of the normal magnetic field H_x is such that on both sides of the discontinuity the Alfven velocity V_x is greater than the sound velocity c and the instability conditions are satisfied:[2,3]

$$V_{1x} < v_{1x}; \quad v_{2x} < V_{2x}, \tag{1}$$

where v_x is the velocity of the liquid relative to the front of the discontinuity; the index 1 refers to the region in front of the wave, the index 2 to that behind it. (The x-axis is directed perpendicularly to the plane of the discontinuity from the region 1 into region 2, the magnetic field lies in the xy plane, the plane of the discontinuity is at rest relative to the chosen system of coordinates, and the liquid moves parallel to the x axis in the positive direction.)

Such an unstable magnetohydrodynamic shock wave can be obtained if an ordinary stable hydrodynamic shock wave for which

$$v_{1x} > c_1; \; v_{2x} < c_2 \tag{1'}$$

is placed in a magnetic field (naturally, the liquid is assumed to be ideally conducting).

If such a wave disintegrates, then, in addition to the contact discontinuity, only plane shock and self-simulating waves* are formed. This results from the fact that a characteristic length is absent from the problem. The conditions for instability of the discontinuities that are formed superimposed restrictions on the possible picture of the

*Self-simulating in the narrow sense of this word applies to waves for which all the magnetohydrodynamic quantities depend upon the ratio x/t.

disintegration. As is well-known,[2] there exist three types of stable discontinuities: fast and slow magnetoacoustic shock waves, rotational (Alfven) discontinuities and three types of simple waves:[5] fast and slow magnetoacoustic, and rotational (magnetohydrodynamic).

The self-simulating waves are a special case of the simple waves. However, the rotational simple waves cannot be self-simulating, since the velocity in front of the front of a simple rotational wave is equal to the velocity of the back of the front; therefore, there exist only two types of self-simulating waves: fast and slow magnetoacoustic waves.

The velocity of all the enumerated waves are such that on each side there is propagated not more than three waves: in front, the fast magnetoacoustic (shock or self-simulating); behind, rotational discontinuity and, finally, the slow magnetoacoustic wave (shock or self-simulating). Waves traveling to the left are separated by the contact discontinuity from waves traveling to the right.

The amplitudes of these 7 waves should be determined such that the sum of the jumps of each of the seven magnetohydrodynamical quantities (ρ, p, v_x, v_y, v_z, H_y, H_z) on these waves is equal to the initial jump:

$$\sum_{i=1}^{7} \Delta_i \rho = \rho_2 - \rho_1; \quad \sum_{i=1}^{7} \Delta_i v_x = v_{2x} - v_{1x} \tag{2}$$

and so forth.

For simplification of the calculation, we shall assume that the initial magnetic field makes a small angle with the normal to the surface of discontinuity, i.e., that the tangential magnetic fields H_{1y} and H_{2y} are very small. Without limitation of generality, it can be assumed that $v_{1z} = v_{2z} = 0$, and the quantities v_{1y} and v_{2y} are small.

We note that even after the disintegration the component of the magnetic field H_z is identically equal to zero.

2. ZEROTH APPROXIMATION

We shall solve the problem of the disintegration by the method of successive approximations, neglecting in zeroth approximation the quantities H_{1y}, H_{2y}, v_{1y}, v_{2y}, and in the first approximation, the squares of these quantities.

In the determination of the type of waves being formed, it must be kept in mind that if a tangential magnetic field is absent in front of the fast magnetoacoustic wave, then the wave cannot be self-simulating. Actually, for each plane magnetoacoustic wave, the following relation is satisfied:[4,5]

$$dH_y / d\rho = u_\pm^2 H_y / \rho (u_\pm^2 - V_x^2), \tag{3}$$

where

$$u_\pm^2 = {}^1/_2 [V^2 + c^2 \pm \sqrt{(V^2 + c^2)^2 - 4c^2 V_x^2}\,], \quad V = H / \sqrt{4\pi\rho},$$

u_+ corresponds to the fast, and u_- to the slow acoustic wave.

Multiplying the relation (3) by H_y and noting that $u_-^2 < V_x^2 < u_+^2$, we find that $dH_y^2/d\rho > 0$ for the fast magnetoacoustic wave, and $dH_y^2/d\rho < 0$ for the slow magnetoacoustic wave. On the other hand, it is known[5-7] that the density decreases in self-simulating magnetoacoustic waves. From this it follows that in the fast magnetoacoustic wave, the magnetic field decreases, while in the slow, it increases. It is easy to show that the opposite conditions are satisfied in shock waves. Therefore, on the forward front of the fast magnetoacoustic self-simulating wave, the tangential magnetic field cannot be equal to zero.

From this it follows that the fast magnetoacoustic waves formed as a result of the disintegration are shock waves, if only they are not equal to zero in zeroth approximation.

It is easy to prove that the conditions (2) in zeroth approximation correspond to a disintegration into two shock waves traveling to the left,* while the magnetohydrodynamical quantities in the region included between these waves are equal to

$$\tilde{\rho} = \rho_1 v_{1x}^2 / V_{1x}^2; \quad \tilde{v}_x = V_{1x}^2 / v_{1x},$$

$$\tilde{p} = p_1 + \frac{\rho_1 (v_{1x}^2 - V_{1x}^2)}{3V_{1x}^2} (3c_1^2 + v_{1x}^2 - V_{1x}^2),$$

$$\tilde{v}_y = \eta \left[\frac{2}{3} \frac{v_{1x}^2 - V_{1x}^2}{v_{1x}^2} (4V_{1x}^2 - v_{1x}^2 - 3c_1^2) \right]^{1/2},$$

$$\tilde{H}_y = \eta \left[\frac{8\pi\rho_1}{3} \frac{v_{1x}^2 - V_{1x}^2}{V_{1x}^2} (4V_{1x}^2 - v_{1x}^2 - 3c_1^2) \right]^{1/2},$$

$$\eta = \pm 1. \tag{4}$$

If the velocity v_{1x} differs slightly from the velocity V_{1x} and $v_{2x} < u_{2-} = c_2$, then such a disintegration is possible only for $v_{1x} > V_{1x}$, i.e., only when the stability conditions[2,3] of the initial shock wave are not satisfied. It follows from this that for $v_{1x} < V_{1x}$ the expressions for $\tilde{H}_y$ and $\tilde{v}_y$ become imaginary.

The conditions (2) will also be satisfied when there is an Alfven discontinuity between the shock waves that are formed, making an angle π with the tangential magnetic field.

*This circumstance was noted by S. I. Syrovatskii in a lecture at the Conference on Applied Theoretical Problems of Magnetohydrodynamics.[8]

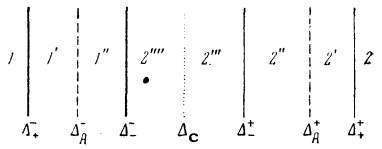
It can also be shown that there can be no other disintegrations of the initial wave which satisfy the equations (2) in zeroth approximation, if the quantity $\alpha = \sqrt{(v_{1x} - V_{1x})/V_{1x}}$ is sufficiently small and the amplitudes of all the waves are small except the slow magnetoacoustic wave traveling to the left.

We note that these waves are compressional waves, i.e., they satisfy the well-known condition of thermodynamic stability.[9-11]

The two waves found in the zeroth approximation lie at the boundary of the region of stability and have a velocity equal to zero. Therefore one cannot yet see at this stage of the calculation that the waves formed as a result of the disintegration of the initial wave are stable. In this connection, we proceed to the first approximation, i.e., we shall consider quantities of the order H_{1y}, v_{1y}, H_{2y}, v_{2y}.

3. FIRST APPROXIMATION

The schematic picture of the disintegration of the initial wave is shown in the drawing. The dotted line indicates the contact discontinuity. The fast and slow magnetoacoustic waves are located on either side of it. The Alfven discontinuities, indicated by the dashed lines, are located between the magnetoacoustic waves. The regions between the waves are denoted by the indices 1, 1′, 1″, 2‴″, 2‴, 2″, 2′, 2. We shall denote the corresponding magnetohydrodynamical quantities by H_{1y}, H'_{1y}, . . . , etc.



The discontinuities in the magnetohydrodynamical quantities on waves of small intensity are related among themselves in first approximation in the following way:

fast magnetoacoustic wave:

$$\Delta_+^+ v_y = -\Delta_+^+ H_y / \sqrt{4\pi\rho_2}; \tag{5}$$

slow magnetoacoustic wave:

$$\Delta_-^+ p = c_2^2 \Delta_-^+ \rho, \quad \Delta_-^+ v_x = \frac{c_2}{\rho_2} \Delta_-^+ \rho; \tag{6}$$

contact discontinuity:

$$\Delta_c \rho \neq 0.$$

Δ denotes the discontinuity undergone by the magnetohydrodynamical quantity in the passage of the wave. The upper index on Δ gives the direction of motion of the wave: the sign (+) to the right, the sign (−) to the left; the lower indices (+), (−), (A), and (c) correspond to the fast and slow magnetoacoustic waves, the Alfven wave and the contact discontinuity, respectively. All discontinuities not written down are equal to zero in the first approximation.

On the Alfven waves, the relations

$$H''_{1y} = \eta_{i1} H'_{1y}, \quad v''_{1y} = v'_{1y} + V'_{1y}(\eta_{i1} - 1); \tag{7}$$

$$H''_{2y} = \eta_{i2} H'_{2y}, \quad v''_{2y} = v'_{2y} + V'_{2y}(1 - \eta_{i2}). \tag{8}$$

are satisfied.

The remaining magnetohydrodynamical quantities do not change on Alfven discontinuities. The value of η_1 is equal to unity if the Alfven wave traveling to the left is absent, and is equal to -1 if this wave intersects the magnetic field at the angle π. The coefficient η_2 has analogous meaning for the Alfven wave traveling to the right.

The following general boundary conditions are satisfied on the two magnetoacoustic shock waves traveling to the left, whose intensity is not small:

$$\Delta[\rho(v_x - U)] = 0, \tag{9}$$

$$\Delta[H_x v_y - (v_x - U) H_y] = 0, \tag{10}$$

$$\Delta[\rho(v_x - U) v_y - H_x H_y / 4\pi] = 0, \tag{11}$$

$$\Delta[p + \rho(v_x - U)^2 + H_y^2 / 8\pi] = 0, \tag{12}$$

$$\Delta\left[\frac{5}{2}\frac{p}{\rho} + \frac{1}{2}(v_x - U)^2 + \frac{1}{2}v_y^2 + \frac{H_y}{4\pi\rho} - \frac{H_x H_y v_y}{4\pi\rho(v_x - U)}\right] = 0. \tag{13}$$

Let us consider in more detail Eqs. (9) — (13) on the fast magnetoacoustic wave traveling to the left. For simplicity of calculation, we shall consider the quantity α to be small in what follows. Introducing the notation $\delta H_y = H'_{1y} - \tilde{H}_y$, $\delta\rho = \rho'_1 - \tilde{\rho}$, . . . , etc., we obtain from (9) — (13)

$$\delta\rho = \rho_1 \tilde{v}_y \beta^{-2} \delta H_y / \sqrt{4\pi\rho_1} + O(\alpha H_{1y}),$$

$$\delta v_x = -V_{1x} \tilde{v}_y \beta^{-2} \delta H_y / \sqrt{4\pi\rho_1} + O(\alpha H_{1y}),$$

$$U_+ = -\tfrac{1}{2} V_{1x} \tilde{v}_y \beta^{-2} \delta H_y / \sqrt{4\pi\rho_1} + O(\alpha H_{1y}),$$

$$\delta v_y = \delta H_y / \sqrt{4\pi\tilde{\rho}} - (\tilde{v}_y^2 / 2\beta^2)\, \delta H_{1y} / \sqrt{4\pi\rho_1} + v_{1y} - H_{1y} / \sqrt{4\pi\rho_1} + O(\alpha^2 H_{1y}), \tag{14}$$

where $\beta^2 = V_{1x}^2 - c_1^2$.

In order to find the quantities δH_y, U_-, $\Delta_{kc}\rho$, $\Delta_-^+\rho$, $\Delta_+^+ H_y$, we return to the five boundary conditions (9) — (13) on the slow magnetoacoustic shock wave traveling to the left. These equations lead to the following results:

$$U_- \sim \Delta_-^+\rho \sim \Delta_c\rho \sim \alpha H_{1y}, \qquad \eta_{\beta 2}\frac{\Delta_+^+ H_y}{\sqrt{4\pi\rho_2}} = \frac{-2v_{1y}+2V_{1y}+2v_{2y}+2V_{2y}(1-\eta_2)-\eta_2 V_{2y}\sqrt{\rho_2/\rho_1}+O(\alpha^2 H_{1y})}{2+\sqrt{\rho_2/\rho_1}}, \tag{15}$$

$$\eta_{\beta 1}\frac{\tilde{v}_y^2}{2\beta^2}\frac{\delta H_y}{\sqrt{4\pi\rho_1}} = \frac{V_{1y}-v_{1y}+v_{2y}+V_{2y}}{1+2\sqrt{\rho_1/\rho_2}}+O(\alpha^2 H_{1y}). \tag{16}$$

Equations (15), (16), together with the formulas (4), (5), (6), (14) permit us to determine the values of the magnetohydrodynamical quantities in all regions for small α.

4. STABILITY OF ALL THE WAVES GENERATED

As we shall now show, the requirement of stability of the waves permits a unique determination of the values of the quantities η, η_1, and η_2.

We begin with the slow magnetoacoustic wave. For its stability, it is necessary that

$$v_{2x}'' - U_- < u_{2-}'';$$
$$v_{1x}'' - U_- > u_{1-}''; \quad v_{1x}'' - U_- < V_{1x}''. \tag{17}$$

The first and second of these inequalities reduce in zeroth approximation to the relations $v_{2X} < c_2$, $\tilde{V}_X > \tilde{u}_-$. The first of these is satisfied by virtue of (1′), and the second is an identity. In view of the smallness of H_{1y}, v_{1y}, H_{2y}, v_{2y}, these inqualities remain valid in higher approximations. The last inequality of (17) reduces to the relation

$$\eta\eta_1(V_{1y}-v_{1y}+v_{2y}+V_{2y})>0. \tag{18}$$

for small α.

The conditions for stability of the fast magnetoacoustic wave traveling to the left have the following form

$$v_{1x}-U_+ > u_{1+}, \quad v_{1x}'-U_+ < u_{1+}', \quad v_{1x}'-U_+ > V_{1x}'. \tag{19}$$

The first two inequalities of (19) reduce in zeroth approximation to the relations $v_{1X} > V_{1X}$, $\tilde{V}_X < \tilde{u}_+$. The first of these is satisfied by virtue of (1), while the second is an identity for $\tilde{H}_y \neq 0$. The last of the inequalities (19) reduces, for small α, to the expression

$$\eta H_{1y} > 0 \quad \text{if} \quad \eta = \operatorname{sign} H_{1y}. \tag{20}$$

We shall now consider the boundary conditions on the initial wave, connecting the quantities H_{1y}, v_{1y}, H_{2y}, v_{2y}. For small α, they reduce to the relations

$$v_{2y}-v_{1y}=V_{1y};\ V_{2y}=-2\alpha^2 V_{iy}/(1-\rho_1/\rho_2). \tag{21}$$

It follows from the relations (18), (21) that

$$\eta_{\beta 1}=1, \tag{22}$$

i.e., the Alfven discontinuity traveling to the left is absent.

We note that upon satisfaction of the inequalities (17), (19), waves traveling to the left diverge.

Equations (15) and (21) show that $|H_{2y}'| \gg |H_{2y}|$. This increase in the magnetic field or in the fast magnetoacoustic wave is evidence that the latter is a shock and not a self-simulating wave. For such a wave and for small α, the relation

$$\Delta_+^+\rho = (\Delta_+^+ H_y)^2/8\pi(V_{2x}^2-c_2^2)<0, \tag{22a}$$

is valid, indicating that this shock wave is a compressional wave. It then follows that it is thermodynamically stable. Proof of the mechanical stability of this wave is not carried out, since it requires the solution of the problem with account of quantities of the order H_{1y}^2. However, assuming that such stability does exist, we can determine the value of η_2. In fact, the tangential magnetic field has the same direction[7] on the two sides of the stable magnetoacoustic shock wave. This means that the quantities H_{2y} and $\Delta_+^+ H_y$ have the same sign. According to Eqs. (15), (21), this takes place for $\eta_2 = -1$.

Thus, the unstable shock wave under consideration disintegrates into four waves (if we neglect waves whose amplitudes are of the order of αH_{1y}): a fast magnetoacoustic shock wave traveling to the left with amplitude of the order α and velocity of the order V_{1y}; a slow magnetoacoustic shock wave traveling to the left with amplitude which differs slightly from the amplitude of the initial wave, and a velocity of the order αH_{1y}; an Alfven discontinuity traveling to the right and making an angle π with the magnetic field; and, finally, a fast magnetoacoustic shock wave traveling to the right with amplitude of the order H_{1y}.

The inequality $|U_+| \gg |U_-|$ leads to the result that the process of disintegration of the initial wave is accompanied by an increase in entropy. This means that the shock wave under consideration is unstable relative to disintegration not only from the viewpoint of mechanics but also in a thermodynamic sense.

In conclusion, we note that the case in which the magnetic field is strictly perpendicular to the initial unstable wave is excluded, since in this case the unstable wave cannot disintegrate spontaneously. However, upon collision of it with magnetoacoustic shock waves of small intensity, which are incident on it from two sides, it disintegrates into stable waves in which two of them have finite amplitude. This follows from the calculation carried out above; the boundary conditions (21) are replaced by the relations

$$v_{1y} = -V_{1y};\ v_{2y} = V_{2y}, \qquad (22b)$$

which hold for fast magnetoacoustic shock waves of low intensity and for $H_y \ll H_x$. The number of Alfven waves arising in the disintegration of an unstable wave is equal to zero, one or two, depending on the relation between V_{1y} and V_{2y}.

The authors take this opportunity to express their sincere gratitude to A. I. Akhiezer, A. S. Kompaneets, L. D. Landau and I. M. Lifshitz for discussions and useful advice.

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Translated by R. T. Beyer
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