ANGULAR CORRELATIONS NEAR MULTIPLE-PRODUCTION THRESHOLDS

A. G. GRASHIN

Moscow Engineering Physics Institute

Submitted to JETP editor November 18, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) **36**, 1480-1484 (May, 1959)

Limiting angular correlations near threshold are found for reactions that have as their final products an infinitely heavy nucleus and two, three, or four identical fermions with spin $\frac{1}{2}$.

In the treatment of reactions near threshold it is usually assumed that the final products of a reaction are in s states, since the contributions from non-zero orbital angular momenta can be neglected. If there are among the final products $N > 2\sigma + 1$ identical fermions (σ is the spin of these fermions), then they cannot all be in s states; even at threshold wave functions satisfying the Pauli principle must in this case contain nonvanishing orbital angular momenta. Inclusion of effects of the Pauli principle leads to the appearance of angular correlations and to a change of the energy-dependences of cross-sections near threshold.[1]

For uncharged products the limiting angular correlations (output channel energy $E \rightarrow 0$) caused by exchange symmetry can be calculated in a number of cases independently of the concrete mechanism of the reaction. This can be done in cases in which because of different energy dependences only one channel with a definite symmetry can contribute at the threshold. For charged products all channels (with different orbital angular momenta, with different symmetries) have the same energy dependence at threshold, and to get the limiting angular distributions one must know the weights of the various channels. This requires further study of the concrete mechanism of the reaction.

We here calculate the angular correlations for reactions that have as their final stages the emission of N uncharged fermions with spin $\frac{1}{2}$ by an infinitely heavy nucleus ($N = 2, 3, 4$). In this case a channel with definite symmetry corresponds to a definite total spin S. The dependence of the reaction amplitude (in the momentum representation) on the momenta of the emerging particles is given near threshold by functions of the form

$$f(\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_N) = A_\alpha (k_{\alpha_1})_i (k_{\alpha_2})_k \ldots (k_{\alpha_m})_l T_{ik\ldots l}, \tag{1}$$

which are the first terms of the expansion of the exact amplitude in powers of the momenta. Here $(k_\alpha)_i \equiv k_{\alpha i}$ is the i-th component of the momentum of the α-th particle, A_α is symmetrization by the Young diagram corresponding to the given channel, and $T_{ik\ldots l}$ is a certain tensor of the m-th rank that is independent of the momenta and contains the dependence of the amplitude on directions in space that are privileged for the given reaction (the rank m is the same as the lowest degree in the expansion of the amplitude in powers of the momenta[1]). In the general case one can construct for each channel several such functions differing only in the numbering of the particles. These functions (and their linear combinations) form the basis of a certain irreducible representation of the permutation group, of dimensionality $r \geq 1$ (cf., e.g., reference 2).

Analogously, a certain representation of the same dimensionality (that associated with the first representation) is given by spin functions of the form

$$\chi(\sigma_1, \sigma_2 \ldots \sigma_N) = A_\alpha^* \prod_\alpha \chi(\sigma_\alpha), \tag{2}$$

where $\chi(\sigma_\alpha)$ are spin functions of the α-th particle (hereafter we shall write $\chi(\sigma_\alpha = \frac{1}{2}) = \xi_\alpha$, $\chi(\sigma_\alpha = -\frac{1}{2}) = \eta_\alpha$), and A_α^* is symmetrization by the associated (transposed) Young diagram, which is obtained from the diagram for the coordinate functions by interchange of rows and columns.

The cross section of the reaction is determined by the square of the "complete" amplitude, which is a certain bilinear combination of the coordinate and spin functions:

$$F(\mathbf{k}_1 \ldots \mathbf{k}_N, \sigma_1 \ldots \sigma_N) = \sum_{i,k=1}^{r} c_{ik} f_i \chi_k,$$

$$\sigma_N = \int |F|^2 \prod_\alpha k_\alpha^2 dk_\alpha d\Omega_\alpha \delta(k^2 - 2mE)$$

$$= E^{(3N-2)/2+m} \int \rho_N(\theta_1, \varphi_1 \ldots \theta_N, \varphi_N) \prod_\alpha d\Omega_\alpha. \tag{3}$$

The correlation function $\rho_N(\vartheta_i) = \overline{\rho_N(\theta_1\varphi_1 \ldots \theta_N\varphi_N)}$ is obtained by averaging the absolute square of the amplitude (3) over all directions in space and integrating over the possible distributions of the energy among the particles; it depends only on the angles ϑ_i between the various pairs of particles ($i = 1, 2, \ldots\ldots N(N-1)/2$). We emphasize that the meaning of the amplitude for a given channel (definite N, S, S_z) belongs only to the "complete" amplitude F, which involves all possible coordinate functions f_i. If, for example, we are interested only in the energy dependence of the cross-section, there is no need to construct the amplitude (3), since actually all we need to calculate is the degree m, which is the same for all the f_i. For the calculation of the angular distribution it is already essential to know the weights with which the various coordinate functions occur in the amplitude (3).

From the mathematical point of view the construction of the amplitude (3) means separation of the antisymmetric representation of the permutation group from the direct product of the two representations given by the coordinate and spin functions.[2,3] We note that although the construction of the coordinate and spin bases is not an unambiguous operation, all bases lead to the same amplitude. The reaction amplitude has been found in the following way: an orthonormal spin basis and the corresponding representation were constructed, then a coordinate basis was chosen to give the associated representation, and finally the antisymmetric function was separated out from the direct product of the two bases.

1. N = 2. The only contribution at threshold is that from the channel with S = 0, since the channel with S = 1 is suppressed by an extra power of the energy E in the cross section. The spin and coordinate functions

$$\chi = (\xi_1\eta_2 - \eta_1\xi_2)/\sqrt{2}, \qquad f = \mathrm{const}$$

give respectively an antisymmetric and a symmetric representation. The amplitude $F = f\chi$ gives $\rho_2(S = 0) = \mathrm{const}$. If in virtue of some selection rule the channel with S = 0 is closed (for example, if the total angular momentum in the final state must be different from zero), then the reaction goes in the channel with S = 1, for which

$$\chi = \xi_1\xi_2, \qquad f = (k_1 - k_2)_i T_i,$$
$$F = f\chi, \quad \rho_2(S = 1) \sim \pi - {}^8/_3 \cos\vartheta.$$

We have written out only the spin function with the maximum value of S_z, since the channels with different S_z do not interfere and give the same angular dependences.

2. N = 3. The reaction goes in the channel with $S = \frac{1}{2}$, for which the two bases (spin and coordinate)

$$\chi_1 = (\xi_1\eta_2 - \eta_1\xi_2)\xi_3/\sqrt{2},$$
$$\chi_2 = [2\xi_1\xi_2\eta_3 - (\xi_1\eta_2 + \eta_1\xi_2)\xi_3]/\sqrt{6},$$
$$f_1 = (k_1 - k_2)_i T_i, \qquad f_2 = (k_1 + k_2 - 2k_3)_i T_i/\sqrt{3}$$

give the same two-dimensional representation, with the following matrices for the interchanges of pairs of particles:

$$M_{12} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_{13} = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix},$$
$$M_{23} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

The amplitude and the correlation function have the forms

$$F = f_1\chi_2 - f_2\chi_1, \qquad \rho_3(S = {}^1/_2) \sim \pi - \frac{8}{9}\sum_i \cos\vartheta_i.$$

For the total spin $S = {}^3/_2$ the representations are one-dimensional:

$$\chi = \xi_1\xi_2\xi_3,$$
$$f = [(k_1 - k_2)_i(k_2 - k_3)_k - (k_2 - k_3)_i(k_1 - k_2)_k]T_{ik}, \quad F = f\chi,$$
$$\rho_3(S = \tfrac{3}{2}) \sim (\pi/2)\Big(1 - \frac{1}{3}\sum_i \cos^2\vartheta_i\Big)$$
$$+ \frac{8}{9}\sum_{i \neq k}\cos\vartheta_i\cos\vartheta_k - \frac{8}{9}\sum_i \cos\vartheta_i.$$

3. N = 4. The channels with S = 0 and S = 1 have the same energy dependence $\sigma_4 \sim E$.[7]

a) S = 0. The spin and coordinate bases

$$\chi_1 = (\xi_1\eta_2 - \eta_1\xi_2)(\xi_3\eta_4 - \eta_3\xi_4)/2,$$
$$\chi_2 = \{2\xi_1\xi_2\eta_3\eta_4 + 2\eta_1\eta_2\xi_3\xi_4 - (\xi_1\eta_2 + \eta_1\xi_2)(\xi_3\eta_4 + \eta_3\xi_4)\}/2\sqrt{3},$$
$$f_1 = \{(k_1 - k_2)_i(k_3 - k_4)_k + (k_3 - k_4)_i(k_1 - k_2)_k\}T_{ik},$$
$$f_2 = \{2k_{1i}k_{2k} + 2k_{2i}k_{1k} + 2k_{3i}k_{4k} + 2k_{4i}k_{3k}$$
$$- (k_1 + k_2)_i(k_3 + k_4)_k - (k_3 + k_4)_i(k_1 + k_2)_k\}T_{ik}/\sqrt{3}$$

give the same two-dimensional representation, with the matrices

$$M_{12} = M_{34} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_{13} = M_{24} = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix},$$
$$M_{23} = M_{14} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}.$$

The amplitude and the correlation function have the forms

$$F = f_1\chi_2 - f_2\chi_1,$$
$$\rho_4(S = 0) \sim \frac{\pi}{8}\Big(3 + \sum_i \cos^2\vartheta_i\Big) - \frac{1}{3}\sum_{i \neq k}\cos\vartheta_i\cos\vartheta_k$$
$$+ \frac{8 + \pi}{3\pi}\sum{}'\cos\vartheta_i\cos\vartheta_k - \frac{1}{3}\sum_i \cos\vartheta_i,$$

where

$$\sum{}' \cos\vartheta_i \cos\vartheta_k = \cos\vartheta_{12}\cos\vartheta_{34} + \cos\vartheta_{13}\cos\vartheta_{24} + \cos\vartheta_{14}\cos\vartheta_{23}.$$

b) S = 1. The spin basis

$$\chi_1 = \xi_1\xi_2(\xi_3\eta_4 - \eta_3\xi_4)/\sqrt{2}, \qquad \chi_2 = (\xi_1\eta_2 - \eta_1\xi_2)\,\xi_3\xi_4/\sqrt{2},$$

$$\chi_3 = {}^1/_2\,\{(\xi_1\eta_2 + \eta_1\xi_2)\,\xi_3\xi_4 - \xi_1\xi_2(\xi_3\eta_4 + \eta_3\xi_4)\}$$

gives a three-dimensional representation, with the matrices

$$M_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M_{34} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M_{13} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -\sqrt{2} \\ 1 & 1 & \sqrt{2} \\ -\sqrt{2} & \sqrt{2} & 0 \end{pmatrix}, \qquad M_{14} = \frac{1}{2}\begin{pmatrix} 1 & -1 & \sqrt{2} \\ -1 & 1 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \end{pmatrix},$$

$$M_{23} = \frac{1}{2}\begin{pmatrix} 1 & -1 & -\sqrt{2} \\ -1 & 1 & -\sqrt{2} \\ -\sqrt{2} & -\sqrt{2} & 0 \end{pmatrix},$$

$$M_{24} = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}. \qquad (4)$$

The coordinate basis

$$f_1 = \{(k_1 - k_2)_i(k_1 + k_2 - k_3 - k_4)_k - (k_1 + k_2 - k_3 - k_4)_i(k_1 - k_2)_k\}\,T_{ik},$$

$$f_2 = \{(k_1 + k_2)_i(k_3 - k_4)_k - (k_3 - k_4)_i(k_1 + k_2)_k + 2k_{3i}k_{4k} - 2k_{4i}k_{3k}\}\,T_{ik},$$

$$f_3 = \sqrt{2}\,\{(k_1 - k_2)_i\,(k_3 - k_4)_k - (k_3 - k_4)_i\,(k_1 - k_2)_k\}\,T_{ik}$$

gives the other three-dimensional representation, with matrices that are the same as the matrices (4) except for a factor -1. The amplitude and correlation function have the forms

$$F = f_1\chi_1 + f_2\chi_2 + f_3\chi_3,$$

$$\rho_4(S=1) \sim \frac{1}{16}\pi\left(6 - \sum_i \cos^2\vartheta_i\right) + \frac{1}{6}\sum_{i\neq k}\cos\vartheta_i\cos\vartheta_k - \frac{1}{6}\sum{}'\cos\vartheta_i\cos\vartheta_k - \frac{1}{3}\sum_i \cos\vartheta_i.$$

The amplitudes that have been found can be used for the construction of angular dependences in reactions with charged products, since although the ratio of the contributions of successive channels, $\sim (Ze^2mR_0)^2$, does not vanish for $E \to 0$, it can still be small. Unlike the case of uncharged particles, the angular part does not depend on the distribution of the energy among the particles (instead of the momenta we must insert the unit vectors $\mathbf{k}_\alpha/k_\alpha$), and therefore the correlation function is the same as the mean square of the amplitude.

In conclusion we emphasize that the basic condition for the applicability of the method used here and in reference 1 is the existence of a finite range of the reaction (the radius R_0). In this case the formulas we have obtained are the first terms of the expansions of the exact values (or their asymptotic expressions) in powers of $\xi = (2mE)^{1/2}R_0$ (cf. also reference 4). Besides this one assumes a sufficiently smooth dependence of the wave function on the energy near threshold, i.e., the absence of resonance effects in the region $\xi \le 1$. This condition can be violated, for example, if the entire system or some subsystem of the final products has an energy level (actual or virtual) near the threshold. Thus for two nucleons there exists near zero energy the virtual level $\epsilon = 0.07$ Mev (for $S = 0$), and therefore for reactions with nucleons in the final state the range of validity of the threshold relations is further restricted to small values of the ratio E/ϵ. In the range $E/\epsilon \lesssim 1$ it is already necessary to take into account effects caused by the interaction of pairs of nucleons.[5,6] Another example of a distortion of the threshold relations is the case in which the region $\xi \lesssim 1$ includes the threshold of another reaction (for two final products this effect has been studied in references 7 and 8).

I express my gratitude to Professor I. Ya. Pomeranchuk for suggesting the problem and for a discussion, and to I. Yu. Kobzarev for helpful comments.

[1] A. F. Granish, J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 719 (1958), Soviet Phys. JETP **8**, 499 (1959).

[2] L. D. Landau and E. M. Lifshitz, Квантовая механика (Quantum Mechanics), GITTL, Moscow-Leningrad 1948 [Eng. Transl. Pergamon, 1958].

[3] H. Weyl, Gruppentheorie und Quantenmechanik, 2. Aufl., 1931. [Engl. Transl., Dover, 1950].

[4] Hart. Gray, and Guier, Phys. Rev. **108**, 1512 (1957).

[5] K. Watson, Phys. Rev. **88**, 1163 (1952).

[6] A. B. Migdal, J. Exptl. Theoret. Phys. (U.S.S.R.) **28**, 3 (1955), Soviet Phys. JETP **1**, 2 (1955).

[7] G. Breti, Phys. Rev. **107**, 1612 (1957).

[8] A. I. Baz', J. Exptl. Theoret. Phys. (U.S.S.R.) **33**, 923 (1957), Soviet Phys. JETP **6**, 709 (1958).

Translated by W. H. Furry
290