ON ANALYTIC PROPERTIES OF CASUAL COMMUTATORS

V. Ya. FAĬNBERG

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

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A simple derivation is given of the integral representation for the causal commutator discovered by Jost and Lehmann[1] and generalized by Dyson,[2] which does not require the use of six dimensions. In the simpler cases (vertex part, two-particle matrix element) more detailed spectral formulas are found. On the basis of these formulas it is shown that the two-particle scattering amplitude — for real values of the energy in the center-of-mass system — is an analytic function of the square of the momentum transfer regular in the entire complex plane except for poles and cuts on the real axis.

1. Jost and Lehmann[1] discovered an integral representation for the matrix element of the causal commutator of two Heisenberg operators A and B

$$f(x) = \langle p, r \,|\, [A(x/2),\ B(-x/2)] \,|\, p', r' \rangle, \tag{1}$$

where $|p, r\rangle$ is the state characterized by a total momentum p_μ, and r denotes all other quantum numbers. However the representation for f(x) which they found was not manifestly covariant and was rigorously valid only in the symmetric case (A = B). The generalization to the nonsymmetric case was carried out in an invariant form by Dyson[2] by introducing six dimensions in momentum space.

The main results of the present work consist showing that the general four-parameter Jost-Lehmann-Dyson representation for f(x) can be made substantially more specific in the simpler cases and that it reduces to a two- and three-parameter representation respectively for the vertex part (when one of the states in (1) is the vacuum state and the other is a single-particle state) and the two-particle matrix element (when both states in (1) are single-particle states). Furthermore, a simple derivation of the general representation for f(x) is given, without recourse to six dimensions which, it seems to us, needlessly complicate the proofs.

The three-parameter representation for the two-particle matrix element permits a significant increase in the region of regularity of the scattering amplitude as a function of the momentum transfer Δ^2 as compared to the region found by Lehmann;[3] namely it is possible to show that the scattering amplitude (for real values of the energy in the center of mass system) is an analytic function of Δ^2 regular in the entire complex plane except for poles and cuts on the real axis.

It should be emphasized that we have not as yet succeeded in demonstrating the necessity of the limitations on the spectral function in the three-parameter representation of the matrix element for the scattering of two particles.

For the sake of simplicity we ignore the spin dependence of the matrix elements in the following.

2. Let us find a general representation for f(x) which makes use of the causality condition

$$f(x) = 0 \quad \text{for} \quad x^2 \equiv x_0^2 - \mathbf{X}^2 < 0. \tag{2}$$

This condition allows f(x) to be written in the form

$$f(x) = \int_0^\infty d\lambda^2 f(x)\, \delta(x^2 - \lambda^2). \tag{3}$$

Let us introduce an auxiliary function

$$\varphi_{\lambda^2}(x) = f(x)\, \delta(x^2 - \lambda^2) \tag{4}$$

such that

$$f(x) = \int_0^\infty \varphi_{\lambda^2}(x)\, d\lambda^2. \tag{4'}$$

It is easy to see that the Fourier transform $\tilde{\varphi}_{\lambda^2}(q)$ of $\varphi_{\lambda^2}(x)$ satisfies the wave equation with a mass λ in q-space:

$$(\Box_q - \lambda^2)\, \tilde{\varphi}_{\lambda^2}(q) = 0,$$
$$\Box_q \equiv -\partial^2/\partial q_0^2 + \partial^2/\partial q_1^2 + \partial^2/\partial q_2^2 + \partial^2/\partial q_3^2, \tag{5}$$

The solution of Eq. (5) may be expressed in terms of the value and the normal derivative of $\varphi_{\lambda^2}(q)$ on an arbitrary space-life surface

$$\tilde{\varphi}_{\lambda^2}(q) = \int d\sigma_\alpha \left[\tilde{\varphi}_{\lambda^2}(u), \frac{\partial}{\partial u_\alpha} \right] \Delta(q - u, \lambda^2), \tag{6}$$

where $\Delta(q, \lambda^2)$ is the odd invariant function of Eq. (5);

$$\left[\varphi, \frac{\partial}{\partial x}\right]\psi \equiv \frac{\partial\varphi}{\partial x}\psi - \varphi\frac{\partial\psi}{\partial x};$$

and σ_α is an arbitrary three-dimensional space-like surface in q-space.

According to Eq. (4) the Fourier transform $\tilde{f}(q)$ of $f(x)$ is

$$\tilde{f}(q) = \int e^{-iqx} f(x)\, d^4x = \int_0^\infty \tilde{\varphi}_{\lambda^2}(q)\, d\lambda^2. \tag{7}$$

Inserting (6) into (7) we obtain the invariant Jost-Lehmann-Dyson representation in its most general form

$$\tilde{f}(q) = \int_0^\infty d\lambda^2 \int d\sigma_\alpha \left[\tilde{\varphi}_{\lambda^2}(u), \frac{\partial}{\partial u_\alpha}\right]\Delta(q-u, \lambda^2). \tag{8}$$

If we now write

$$\Delta(q-u, \lambda^2) = \int_0^\infty \varepsilon(q-u)\,\delta((q-u)^2 - \varkappa^2)\,\overline{\Delta}(\varkappa^2, \lambda^2)\, d\varkappa^2,$$

where

$$\overline{\Delta}(\varkappa^2, \lambda^2) = -\frac{2}{(2\pi)^4}\, \mathrm{P}\int \frac{e^{-ipx}}{p^2 - \lambda^2}\, d^4p,$$

and set

$$\psi(\varkappa^2, u) = \int_0^\infty d\lambda^2\, \varphi_{\lambda^2}(u)\,\overline{\Delta}(\varkappa^2, \lambda^2),$$

then we obtain in place of Eq. (8) the following

$$\tilde{f}(q) = \int_0^\infty d\varkappa^2 \int d\sigma_\alpha \left[\psi(\varkappa^2, u)\frac{\partial}{\partial u_\alpha}\right] \times \varepsilon(q-u)\,\delta((q-u)^2 - \varkappa^2), \tag{8'}$$

i.e., the desired representation investigated in detail by Dyson.[2]* Choosing the surface $u_0 = 0$ in Eq. (8′) we get the representation found by Jost and Lehmann.[1]

From Eq. (1) it follows that $\tilde{f}(q)$ vanishes in the region

$$P_0 - (m_1^2 + (\mathbf{q} - \mathbf{P})^2)^{1/2} \leqslant q_0 \leqslant (m_2^2 + (\mathbf{q} + \mathbf{P})^2)^{1/2} - P_0, \tag{9}$$

where $P = (p + p')/2$ and m_1 and m_2 are the masses of the lowest mass intermediate states $|n_1\rangle$ and $|n_2\rangle$ for which the matrix elements

$\langle p, r | A | n_1\rangle \langle n_1 | B | p', r'\rangle$

and $\langle p, r | B | n_2\rangle \langle n_2 | A | p', r'\rangle$

fail to vanish.

Dyson has shown that the representation (8′) satisfies conditions (9) if and only if the function $\psi(\kappa^2, u)$ vanishes everywhere except for the region**

$$\varkappa^2 \geqslant \max\{0, m_1 - |P+u|, m_2 - |P-u|\}$$
$$(P \pm u)^2 \geqslant 0. \tag{10}$$

We do not repeat that proof here. Let us only stress that in the derivation of Eq. (8′) as well as in the deduction of the limitation (10) on $\psi(\kappa^2, u)$ there is no need whatsoever for introducing a space of six dimensions as was done in reference 2.

3. The general representation (8) or (8′) depends on four parameters (λ^2 or κ^2 and the three components of the vector u). This is related to the fact that $\tilde{f}(q)$ depends on four quantities (q_0, q_1, q_2, q_3). In the derivation of (8) or (8′) we have nowhere explicitly made use of the invariance properties of $f(x)$. We shall show below that it is possible to obtain more specific representations than (8) or (8′) for the simpler matrix elements of the form (1) provided use is made of consequences of relativistic invariance. In general $f(x)$ may be expressed in terms of the invariants x^2, $p \cdot x$, $p' \cdot x$, etc.:† $f(x) \equiv f(x^2, p\cdot x, p'\cdot x, \ldots)$ where the dots denote all other possible invariants beside x^2. Consequently Eq. (3) may be rewritten in the form‡

$$f(x) = \int_0^\infty d\lambda^2 f(\lambda^2, px, p'x\ldots)\,\delta(x^2 - \lambda^2). \tag{3'}$$

We now use the relation***

*The function $\psi(\varkappa^2, u)$ is related to the function $F(u, s)$ of Eq. (30), reference 2, by $\psi(\varkappa^2, u) = \partial F(u, \varkappa^2)/\partial\varkappa^2$.

**We note that the proof of necessity of the condition (10) in Dyson's work cannot be considered complete. In the proof, Dyson introduced the concept of admissible hyperboloids $(q-u)^2 - \varkappa^2 = 0$, corresponding to values of u and $\varkappa^2$ satisfying condition (10), and showed, using theorems of Jost and Lehmann[1] that for any twice inadmissible hyperboloid tha corresponding value of $\psi(\varkappa^2, u)$ is zero. A twice inadmissible hyperboloid is a hyperboloid whose both sheets (lower and upper) are in q-space inside region (9). However, as is easy to see by drawing an appropriate figure, a majority of the points u and $\varkappa^2$ that lie outside the region (10) correspond to hyperboloids $(q-u)^2 - \varkappa^2 = 0$ that have only one sheet inadmissible, i.e. the majority of the inadmissible hyperboloids is not twice inadmissible; and for just such hyperboloids Dyson did not show that the corresponding values of $\psi(\varkappa^2, u)$ are zero (this reservation applies equally to the cases of symmetric and non-symmetric regions).

†Strictly speaking the function $f(x)$ is an invariant only when multiplied by factors of the type $(2p_0)^{1/2}$; we ignore these factors.

‡Equation (3) is, of course, not unique. In particular, one may replace x_0 by $\pm(\mathbf{x}^2 + \lambda^2)^{1/2}$ everywhere in $f(\lambda^2, p\cdot x\ldots)$, such a substitution leads to the Jost-Lehmann[1] formula [see (14′)].

***This identity is easily verified starting from the parametric representation.

$$\int_{-\infty}^{+\infty} d\varkappa^2 \overline{\Delta}(x^2, \varkappa^2) \overline{\Delta}(\varkappa^2, \lambda^2) = \int_0^{\infty} \overline{\Delta}(x^2, \varkappa^2) \overline{\Delta}(\varkappa^2, \lambda^2) d\varkappa^2 = (2\pi)^{-2} \delta(x^2 - \lambda^2). \quad (11)$$

Here $\overline{\Delta}(x^2, \kappa^2) = \epsilon(x_0)\Delta(x, \kappa^2)$, where $\Delta(x, \kappa^2)$ is the commutator function with mass κ, and $\epsilon(x_0)$ is the sign function. Introducing (11) into (3′) and denoting

$$\rho(x^2, px, \ldots) = (2\pi)^2 \int_0^{\infty} d\lambda^2 f(\lambda^2, px, \ldots) \; \varepsilon(x) \; \overline{\Delta}(x^2, \lambda^2), \quad (12)$$

we find

$$f(x) = \int_0^{\infty} d\varkappa^2 \rho(\varkappa^2, px, \ldots) \Delta(x, \varkappa^2). \quad (13)$$

From here it follows that the Fourier transform $\tilde{f}(q)$ is

$$\tilde{f}(q) = \int_0^{\infty} d\varkappa^2 \int d^4u \, \Phi(\varkappa^2, u) \; \varepsilon(q-u) \, \delta((q-u)^2 - \varkappa^2), \quad (14)$$

where

$$\Phi(\varkappa^2, u) = \frac{-i}{(2\pi)^3} \int \rho(\varkappa^2, px \ldots) e^{-iux} d^4x. \quad (14')$$

The expression (14) for $\tilde{f}(q)$ coincides in form with the five parameter representation obtained by Dyson (formula (49) of reference 2, theorem "c"), which also follows directly from Eq. (8).

For purposes of application (see reference 3) the derivation of restrictions of the type (10) on the spectral function $\Phi(\kappa^2, u)$ appearing in (14) is of greatest importance. That these restrictions are sufficient is obvious. That they are necessary has not yet been shown,* however one can hardly doubt the validity of the conditions (10) for $\Phi(\kappa^2, u)$. In particular it is clear from physical considerations that the only contributions to $\tilde{f}(q)$ come from the admissible hyperboloids $(q-u)^2 - \kappa^2 = 0$, since only they correspond to possible physical processes for which the matrix element of $\tilde{f}(q)$ does not vanish.

Let us apply formula (14) to some simple matrix elements.

$$\overline{\Delta}(x^2, \varkappa^2) = (2\pi)^{-2} \int_{-\infty}^{+\infty} d\alpha \exp[i\alpha x^2 + i\varkappa^2/4\alpha].$$

All relations and integrals encountered in this paper are to be interpreted in the distribution-theory sense.

*In this connection see footnote **, page 1067.

A. Vertex Part

The antihermitian part of the vertex function is expressed in terms of a matrix element of the form

$$f(x) = \langle 0 | [A(x/2), B(-x/2)] | p \rangle, \quad (15)$$

where $|0\rangle$ and $|p\rangle$ are the vacuum and one-particle states respectively. In this case $f(x)$ depends on three invariants: x^2, $p \cdot x$, and p^2,* of which only the first two are x-dependent. Consequently a two-parameter representation for $f(q)$ should exist instead of the four- or five-parameter representations given by Eqs. (8′) and (14). Indeed, the function $\rho(\kappa^2, p \cdot x, p^2)$ in Eq. (13) may be written in the form

$$\rho(\varkappa^2, px, p^2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\alpha(px)} \tilde{\rho}(\varkappa^2, \alpha, p^2) d\alpha, \quad (16)$$

where α is some scalar parameter. From this, taking into account Eq. (14′), we obtain

$$\Phi(\varkappa^2, u) = (-i) \int_{-\infty}^{+\infty} \tilde{\rho}(\varkappa^2, \alpha, p^2) \delta(u - \alpha p) d\alpha. \quad (17)$$

We now insert (17) into (14), integrate over d^4u and obtain

$$\tilde{f}(q) = (-i) \int_0^{\infty} d\varkappa^2 \int_{-\infty}^{+\infty} d\alpha \tilde{\rho}(\varkappa^2, \alpha, p^2) \times \varepsilon(q - \alpha p) \delta((q - \alpha p)^2 - \varkappa^2). \quad (18)$$

For the Fourier transform of the retarded commutator

$$f_R(q) = \int \theta(x) f(x) \exp(-iqx) d^4x,$$

we have the following spectral representation

$$f_R(q) = \frac{1}{2\pi} \int_0^{\infty} d\varkappa^2 \int_{-\infty}^{+\infty} \frac{d\alpha \, \tilde{\rho}(\varkappa^2, \alpha, p^2)}{((q - \alpha p)^2 - \varkappa^2)}, \quad q_0 \to q_0 - i\varepsilon. \quad (19)$$

The restrictions on the spectral function are the following: $\tilde{\rho}$ is different from zero only in the region ($\mathbf{p} = 0$; $p_0 = m$; $m_1 \geq m_2$)

$$-1/2 + (m_1 - \varkappa)/m \leqslant \alpha \leqslant 1/2 - (m_2 - \varkappa)/m \quad \text{if } \varkappa \leqslant m_2;$$
$$-1/2 + (m_1 - \varkappa)/m \leqslant \alpha \leqslant 1/2 \quad \text{if } m_2 \leqslant \varkappa \leqslant m_1;$$
$$-1/2 \leqslant \alpha \leqslant 1/2 \quad \text{if } \varkappa > m_1.$$

Thus in the case of the vertex part, which, except for unimportant factors is equal to $f_R(q)$, it is possible to derive a two-parameter (α and κ^2)

*The function $\varepsilon(x_0)$ may always be written as $\varepsilon(p \cdot x)$ because p is a timelike vector with $p_0 \geqslant 0$.

representation (18) and (19) instead of (14), by using relativistic invariance.

B. Two Particle Matrix Element

In this case

$$f(x) = \langle p \,|\, [A(x/2),\ B(-x/2)] \,|\, p' \rangle \qquad (20)$$

and it is related to the matrix element for the scattering of two particles.

Repeating the considerations of section A gives*

$$\tilde{f}(q) = i \int_{-\infty}^{+\infty} d\alpha\, d\beta \int_0^{\infty} d\varkappa^2\, \tilde{\rho}(\varkappa^2, \alpha, \beta, Q^2)\, \varepsilon(q - \alpha P - \beta Q)$$

$$\times\, \delta\left((q - \alpha P - \beta Q)^2 - \varkappa^2\right), \qquad (21)$$

$$f_R(q) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\alpha\, d\beta \int_0^{\infty} d\varkappa^2\, \frac{\tilde{\rho}(\varkappa^2, \alpha, \beta, Q^2)}{(q - \alpha P - \beta Q)^2 - \varkappa^2},$$

$$q_0 \to q_0 - i\varepsilon, \qquad (22)$$

where

$$P = (p + p')/2; \quad Q = (p - p')/2; \quad PQ = 0,$$

$$\text{if } p^2 = p'^2 = m^2; \ Q^2 = -\Delta^2;$$

Δ^2 — the momentum transfer; the region in which the $\tilde{\rho}$ of Eqs. (21) and (22) is different from zero (in the frame $\mathbf{P} = 0$) is given as follows:

$$-1 + \left(\left(\frac{m_1 - \varkappa}{P_0}\right)^2 + \beta^2 \frac{\Delta^2}{P_0^2}\right)^{1/2} \leqslant \alpha$$

$$\leqslant 1 - \left(\left(\frac{m_2 - \varkappa}{P_0^2}\right)^2 + \beta^2 \frac{\Delta^2}{P_0^2}\right)^{1/2}, \qquad \varkappa \leqslant m_2,$$

$$-1 + \left(\left(\frac{m_1 - \varkappa}{P_0}\right)^2 + \beta^2 \frac{\Delta^2}{P_0^2}\right)^{1/2} \leqslant \alpha \leqslant 1 - \left|\frac{\beta\Delta}{P_0}\right|,$$

$$m_2 \leqslant \varkappa \leqslant m_1,$$

$$-1 + |\beta\Delta/P_0| \leqslant \alpha \leqslant 1 - |\beta\Delta/P_0|, \qquad \varkappa > m_1. \qquad (23)$$

Consequently the two particle matrix element has a three-parameter representation (α, β, and κ^2).

Let us note an important property of Eqs. (18), (19), (21), and (22). In the limit as p and p′ tend to zero they automatically go over into the well known formulas of Kallen-Lehmann[4] for the one-particle Green's function.

For the study of analytic properties of the matrix element for nucleon-meson scattering as a function of momentum transfer Δ^2, it is more convenient to consider instead of (20) a retarded commutator of the form

$$f_R(x) = i\theta(x) \langle 0 \,|\, [\eta(x/2),\ j(-x/2)] \,|\, p;\ k \rangle,$$

where j is the nucleon current and η is the right hand side of the equation for the nucleon field operators; p and k are the initial momenta of the nucleon and meson. The retarded scattering amplitude $f_R(q = (p' - k')/2)$ is equal, in this case (up to unimportant factors), to the Fourier transform of $f_R(x)$ and can be expressed in the form (22) with a different spectral function $\tilde{\rho}_1$:

$$f_R(q) = \frac{1}{2\pi} \int \frac{d\alpha\, d\beta\, d\varkappa^2 \tilde{\rho}_1(\varkappa^2, \alpha, \beta, p, k)}{[(p' - k') - \alpha(p + k) - \beta(p - k)]^2/4 - \varkappa^2}. \qquad (24)$$

The limitations on $\tilde{\rho}_1$ of the type (23) may be easily obtained from conditions (10) by setting $u = [\alpha(p+k) + \beta(p-k)]/2$ in the latter.

In the center of mass system ($\mathbf{p} + \mathbf{k} = 0$; $(p+k)^2 = w^2$; $m_1 = m + \mu$; $m_2 = 3\mu$) the entire dependence of $f_R(q)$ in Eq. (24) on Δ^2 is in the denominator. Carrying out considerations fully analogous to those of Lehmann[3] one can show that the scattering amplitude $f_R(q) \equiv f_R(w^2, \Delta^2)$ is regular in Δ^2 everywhere except for the following region on the real axis

$$1 - 2\Delta^2/\mathbf{k}^2$$

$$\geqslant [1 + 8\mu^3(2m + \mu)/\mathbf{k}^2(w^2 - (m - 2\mu)^2)]^{1/2}. \qquad (25)$$

To find the region in which the imaginary part of $f_R(w^2, \Delta^2)$ is regular in Δ^2 it is necessary to use the general formula (14) (see reference 3) because expressions of the form (24) turn out to be insufficient.

*This formula was derived by a different method by V. D. Skarzhinskiĭ (Thesis, Moscow State University, 1957).

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