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SCATTERING OF DIRAC PARTICLES IN THE SECOND BORN APPROXIMATION

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The problem treated is that of the elastic scattering of Dirac particles by a fixed spherically symmetrical center of force. The values of the scattering amplitudes are found in the second Born approximation.

In a paper by Sokolov and the writers[1] values of the scattering phase shifts $\delta_l^{(1)}$ and $\delta_l^{(2)}$ for the scattering of Dirac particles by an arbitrary force center were found in second approximation in the interaction potential [see Eq. (23) of reference 1].

Using the following integral representations of the spherical Bessel functions:

$$j_l^2(kr) = \frac{1}{2kr}\int_0^\pi \sin\left(2kr\sin\frac{\varphi}{2}\right)\cos\frac{\varphi}{2}P_l(\cos\varphi)\,d\varphi,$$

$$j_l(kr)\,n_l(kr) = -\frac{1}{2kr}\int_0^\pi \cos\left(2kr\sin\frac{\varphi}{2}\right)\cos\frac{\varphi}{2}P_l(\cos\varphi)\,d\varphi,$$

we can put the expressions in question in the form

$$\tan\delta_l^{(1)} = -\frac{K}{c\hbar}\int_0^\pi [\alpha P_l(\cos\varphi) + \beta P_{l+1}(\cos\varphi)]\,L(\varphi)\,d\varphi$$
$$+\left(\frac{K}{c\hbar}\right)^2\int_0^\pi\int_0^\pi [\alpha^2 P_l(\cos\varphi)P_l(\cos\psi)$$
$$+\alpha\beta P_l(\cos\varphi)P_{l+1}(\cos\psi) + \alpha\beta P_{l+1}(\cos\varphi)P_l(\cos\psi)$$
$$+\beta^2 P_{l+1}(\cos\varphi)P_{l+1}(\cos\psi)]\,M(\varphi,\psi)\,d\varphi\,d\psi,$$

$$\tan\delta_l^{(2)} = -\frac{K}{c\hbar}\int_0^\pi [\alpha P_l(\cos\varphi) + \beta P_{l-1}(\cos\varphi)]\,L(\varphi)\,d\varphi$$
$$+\left(\frac{K}{c\hbar}\right)^2\int_0^\pi\int_0^\pi [\alpha^2 P_l(\cos\varphi)P_l(\cos\psi)$$
$$+\alpha\beta P_l(\cos\varphi)P_{l-1}(\cos\psi) + \alpha\beta P_{l-1}(\cos\varphi)P_l(\cos\psi)$$
$$+\beta^2 P_{l-1}(\cos\varphi)P_{l-1}(\cos\psi)]\,M(\varphi,\psi)\,d\varphi\,d\psi, \quad (1)$$

where

$$L(\varphi) = \frac{1}{2}\cos\frac{\varphi}{2}\int_0^\infty \sin\left(2kr\sin\frac{\varphi}{2}\right)V(r)\,r\,dr,$$

$$M(\varphi,\psi) = \frac{1}{2}\cos\frac{\varphi}{2}\cos\frac{\psi}{2}\int_0^\infty \cos\left(2kr\sin\frac{\varphi}{2}\right)V(r)\,r\,dr$$
$$\times\int_0^r \sin\left(2kr'\sin\frac{\psi}{2}\right)V(r')\,r'dr'. \quad (2)$$

The rest of the notation is that of reference 1. In particular, $\hbar k$ is the momentum, $E = c\hbar K$ is the energy, and $m = \hbar k_0/c$ is the mass of the particle.

Assuming the phase shifts small, we set $\tan\delta_l \approx \delta_l$, and instead of the exact expressions for the scattering amplitudes $f(\vartheta)$ and $g(\vartheta)$ we content ourselves with the linear approximations

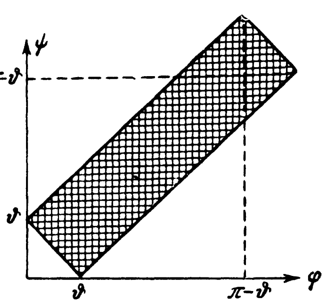
$$f(\vartheta) = k^{-1}\sum_{l=0}^\infty [(l+1)\,\delta_l^{(1)} + l\delta_l^{(2)}]\,P_l(\cos\vartheta),$$

$$g(\vartheta) = k^{-1}\sum_{l=1}^\infty [\delta_l^{(1)} - \delta_l^{(2)}]\,P_l^1(\cos\vartheta),$$

where $P_l^1(\cos\vartheta) = -\sin\vartheta\, dP_l(\cos\vartheta)/d\cos\vartheta$, and $P_l(\cos\vartheta)$ is the Legendre polynomial. Substituting here the values of $\delta_l^{(1)}$ and $\delta_l^{(2)}$ from Eq. (1) and carrying out the summation over l by the use of Eqs. (8a) — (8e) of the Appendix, we get

$$f(\vartheta) = -\frac{K}{c\hbar k}\frac{\alpha + \beta\cos\vartheta}{2\sin\left(\frac{\vartheta}{2}\right)}\int_0^\infty \sin\left(2kr\sin\frac{\vartheta}{2}\right)V(r)\,r\,dr$$
$$+\frac{2K^2}{\pi c^2\hbar^2 k}\iint_\Omega [\alpha^2 + \alpha\beta(\cos\varphi + \cos\psi)$$
$$+\beta^2\cos\vartheta]\,R^{-1}M(\varphi,\psi)\,d\varphi\,d\psi; \quad (3)$$

$$g(\vartheta) = -\frac{K}{c\hbar k}\frac{\beta\sin\vartheta}{2\sin\left(\frac{\vartheta}{2}\right)}\int_0^\infty \sin\left(2kr\sin\frac{\vartheta}{2}\right)V(r)\,r\,dr$$
$$+\frac{2K^2}{\pi c^2\hbar^2 k}\iint_\Omega \left[\alpha\beta\tan\frac{\vartheta}{2}(\cos\varphi + \cos\psi)\right.$$
$$\left.+\beta^2\sin\vartheta\right]R^{-1}M(\varphi,\psi)\,d\varphi\,d\psi, \quad (4)$$



where the region Ω is defined by the conditions (9) and shown graphically in the diagram, and

$$R = [1 - \cos^2\varphi - \cos^2\psi - \cos^2\vartheta + 2\cos\varphi\cos\psi\cos\vartheta]^{1/2}. \quad (4')$$

In the particular case of Coulomb scattering $V(r) = -Ze^2/r$ and

$$M^c(\varphi, \psi) = \frac{\pi Z^2 e^4}{8k}\cos\frac{\varphi}{2}\cot\frac{\psi}{2}\Big\{2\delta\left(2k\sin\frac{\varphi}{2}\right) - \delta\left[2k\left(\sin\frac{\varphi}{2} - \sin\frac{\psi}{2}\right)\right] - \delta\left[2k\left(\sin\frac{\varphi}{2} + \sin\frac{\psi}{2}\right)\right]\Big\}.$$

Using this, we can get the well known formula of McKinley and Feshbach (cf., e.g., reference 1) from Eqs. (3) and (4). If various special spherically symmetrical charge distributions inside the nucleus are specified, one can calculate by numerical integration the deviations from pure Coulomb scattering caused by the finite dimensions of the nucleus.

A particularly interesting case is that of high energies, for which one can neglect the rest mass of the particle in the Dirac equation in comparison with its total energy. This does not make any important change in the picture of the scattering, and in the final results it affects only terms of the order $(mc^2/E)^2$. It can be shown that when the rest mass is neglected the phase shifts corresponding to a prescribed total angular momentum are exactly equal,[2] i.e., $\delta_l^{(1)} = \delta_{l+1}^{(2)}$. In our case this can be seen directly from the expressions (1) and (2) if we set $\alpha = \beta = 1$. Using this fact, one can easily show that when the rest mass is neglected

$$g_{ur}(\vartheta) = \tan\left(\frac{\vartheta}{2}\right) f_{ur}(\vartheta),$$

and the differential cross-section takes the form

$$d\sigma_{ur}/d\Omega = \sec^2(\vartheta/2)\,|\,f_{ur}(\vartheta)|^2, \quad (5)$$

where

$$f_{ur}(\vartheta) = -\frac{K}{c\hbar k}\frac{\cos^2(\vartheta/2)}{\sin(\vartheta/2)}\int_0^\infty \sin\left(2kr\sin\frac{\vartheta}{2}\right)V(r)\,r\,dr + \frac{2K^2}{\pi c^2\hbar^2 k}\iint_\Omega (1 + \cos\varphi + \cos\psi + \cos\vartheta)\,R^{-1}M(\varphi, \psi)\,d\varphi\,d\psi.$$

When we go to the nonrelativistic case we must set $\alpha = 1$, $\beta = 0$, and we have from Eqs. (3) and (4)

$$f_{nr}(\vartheta) = -\frac{K}{c\hbar k}\frac{1}{2\sin(\vartheta/2)}\int_0^\infty \sin\left(2kr\sin\frac{\vartheta}{2}\right)V(r)\,r\,dr + \frac{2K^2}{\pi c^2\hbar^2 k}\iint_\Omega R^{-1}M(\varphi, \psi)\,d\varphi\,d\psi,$$

$$g_{nr}(\vartheta) = 0. \quad (6)$$

It can be shown that in the case of pure Coulomb scattering in nonrelativistic approximation the second term in the expression (6) for f_{nr} makes the contribution zero. This is in agreement with the fact that the classical Rutherford formula is exact within the framework of nonrelativistic quantum mechanics.

APPENDIX

By means of the formalism of the Dirac δ function one can calculate a number of sums containing products of three Legendre polynomials. We construct a δ function from the orthonormal Legendre polynomials:

$$\sum_{l=0}^{\infty}\left(l + \frac{1}{2}\right)P_l(\cos\omega)\,P_l(\cos\vartheta) = \delta(\cos\omega - \cos\vartheta) \quad (7)$$

and set

$$\cos\omega = \cos\varphi\cos\psi + \sin\varphi\sin\psi\cos\gamma.$$

Using the fact that

$$\frac{1}{\pi}\int_0^\pi P_l(\cos\omega)\,d\gamma = P_l(\cos\varphi)\,P_l(\cos\psi),$$

we integrate Eq. (7) from 0 to π. Examination of the limits of integration and use of the fundamental property of the δ function give the following value for the sum of products of three Legendre polynomials:

$$\sum_{l=0}^{\infty}\left(l + \tfrac{1}{2}\right)P_l(\cos\varphi)\,P_l(\cos\psi)\,P_l(\cos\vartheta) = \begin{cases} 1/\pi R & \text{inside } \Omega, \\ 0 & \text{outside } \Omega, \end{cases} \quad (8a)$$

where the region Ω is defined by the conditions

$$\varphi + \psi + \vartheta \leqslant 2\pi, \quad \varphi + \psi - \vartheta \geqslant 0, \quad \varphi - \psi + \vartheta \geqslant 0, \quad -\varphi + \psi + \vartheta \geqslant 0 \quad (9)$$

and has the shape shown in the diagram, and R is given by Eq. (4′).

Using the well known recurrence relations between the Legendre polynomials, we can get from Eq. (8a) the values of the following sums:

$$\sum_{l=0}^{\infty} P_l(\cos\varphi)\,[(l+1)\,P_{l+1}(\cos\psi) + lP_{l-1}(\cos\psi)]\,P_l(\cos\vartheta) = \begin{cases} 2\cos\psi/\pi R & \text{inside } \Omega, \\ 0 & \text{outside } \Omega; \end{cases} \quad (8b)$$

$$\sum_{l=0}^{\infty}[(l+1)\,P_{l+1}(\cos\varphi)\,P_{l+1}(\cos\psi) + lP_{l-1}(\cos\varphi)\,P_{l-1}(\cos\psi)]\,P_l(\cos\vartheta) = \begin{cases} 2\cos\vartheta/\pi R & \text{inside } \Omega, \\ 0 & \text{outside } \Omega. \end{cases} \quad (8c)$$

Starting from the value of the sum

$$\sum_{l=1}^{\infty}[P_{l+1}(\cos\omega)-P_{l-1}(\cos\omega)]\,P_l^1(\cos\vartheta)$$
$$=2\sin\vartheta\,\delta(\cos\omega-\cos\vartheta),$$

we can calculate by an analogous method the following sums:

$$\sum_{l=1}^{\infty} P_l(\cos\varphi)\,[P_{l+1}(\cos\psi)-P_{l-1}(\cos\psi)\,P_l^1(\cos\vartheta)$$
$$=\begin{cases} 2(\cos\varphi-\cos\psi\cos\vartheta)/\pi R\sin\vartheta & \textbf{inside }\Omega,\\ 0 & \textbf{outside }\Omega;\end{cases} \qquad (8d)$$

$$\sum_{l=1}^{\infty}[P_{l+1}(\cos\varphi)\,P_{l+1}(\cos\psi)-P_{l-1}(\cos\varphi)\,P_{l-1}(\cos\psi)]$$
$$\times P_l^1(\cos\vartheta)=\begin{cases} 2\sin\vartheta/\pi R & \textbf{inside }\Omega,\\ 0 & \textbf{outside }\Omega.\end{cases} \qquad (8e)$$

[1] Sokolov, Arutyunyan, and Muradyan, J. Exptl. Theoret. Phys. (U.S.S.R.) **36**, 594 (1959), Soviet Phys. JETP **9**, 412 (1959).

[2] L. K. Acheson, Jr., Phys. Rev. **82**, 488 (1951).

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301