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VISCOSITY IN THE HYDRODYNAMIC THEORY OF MULTIPLE PARTICLE PRODUCTION

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The model of a viscous ultrarelativistic fluid is used to describe the dispersion of the meson-nucleon cloud produced in the collision of high-energy nucleons. An asymptotic solution of one-dimensional equations is obtained. It is shown that when viscosity is taken into account the angular distribution of secondary particles is less anisotropic than in the case of an ideal fluid.

1. In Landau's hydrodynamic theory[1,2] of multiple particle production in the collisions of high-energy nucleons the expansion of the meson-nucleon cloud is described as the dispersion of an ultrarelativistic ideal fluid.

Although the theory agrees satisfactorily with experiment in general there are experimental indications that the angular distribution of secondary particles is less anisotropic than is predicted by Landau's theory.[3] It is therefore of interest to investigate the expansion of the system using the model of an ultrarelativistic viscous fluid. The energy dissipation which occurs during the motion of a viscous substance is accompanied by increased entropy of the system. Therefore, in distinction from Landau's theory, additional particles are produced in the expanding meson cloud.

The study of the dissipative processes from the hydrodynamic point of view leads to new parameters — the phenomenological coefficients of viscosity, thermal conductivity etc. The kinematic coefficients could theoretically be calculated from a specific type of interaction in the meson-nucleon cloud; this would involve quantum-statistical averaging of the field operators and a hydrodynamic description of the interacting particles (see reference 4, for example). However, since we do not know the real form of the interaction in the meson cloud at such high energies the indicated procedure is extremely tentative. In the present paper we therefore consider only the kinematic aspect of the problem; we assume the viscosity coefficient to have a given constant value independent of temperature and determine its influence on the characteristics of an elementary act. The viscosity coefficient could be determined experimentally by comparing the theory with experimental data. Two papers of Hamaguchi[5,6] are concerned with the role of viscosity in the hydrodynamic theory of multiple particle production and will be analyzed below.

2. The equations of relativistic hydrodynamics which take dissipative processes (viscosity and thermal conduction) into account are as follows:[7]

$$\partial T_i^k / \partial x^k = 0, \qquad \partial n^i / \partial x^i = 0, \tag{1}$$

$$T_{ik} = w u_i u_k + p g_{ik} + \tau_{ik}, \quad n_i = n u_i + \nu_i. \tag{2}$$

Here u_i is the velocity 4-vector, which satisfies the relation $u^i u_i = -1$; p is pressure; $w = p + \epsilon$ is the enthalpy density; ϵ is the energy density; n is the particle density; n_i is the particle current vector; g_{ik} is the metric tensor ($g_{11} = g_{22} = g_{33} = 1$, $g_{00} = -1$);

$$\tau_{ik} = -\zeta_1 \left(\frac{\partial u_i}{\partial x^k} + \frac{\partial u^k}{\partial x^i} + u_i u^l \frac{\partial u_k}{\partial x^l} + u_k u^l \frac{\partial u_i}{\partial x^l} \right) - \left(\zeta_2 - \frac{2}{3} \zeta_1 \right) \frac{\partial u^l}{\partial x^l} (u_i u_k + g_{ik}); \tag{3}$$

$$\nu_i = -\frac{\varkappa}{c} \left(\frac{nT}{w} \right)^2 \left[\frac{\partial}{\partial x^i} \left(\frac{\mu}{T} \right) + u_i u_k \frac{\partial}{\partial x^k} \left(\frac{\mu}{T} \right) \right]. \tag{4}$$

ζ_1 and ζ_2 are the two (positive) viscosity coefficients; c is the velocity of light (we shall let c = 1); T is the temperature; μ is the chemical potential; κ is the thermal conductivity. The viscosity tensor satisfies the relation[7]

$$\tau_k^i u^k = 0. \tag{5}$$

For the expanding meson-nucleon cloud we assume with Landau that $\mu = 0$. Since then $\nu_i = 0$ thermal conductivity is absent from the Fermi-Landau system. Assuming furthermore $\epsilon = 3p$ for the equation of state, we obtain the equation[5]

$$\tau_i^i = 0. \tag{6}$$

for the tensor τ_{ik}. Equation (6) will always be satisfied if $\zeta_2 = 0$, which we shall assume hereinafter.

We now consider the one-dimensional symmetrical expansion of an infinite plane layer of thickness Δ in a vaccum with viscosity taken into account. The equations in (1) are now represented more conveniently by[3]

$$\frac{1}{3}\frac{\partial\varepsilon}{\partial x^i}+\frac{4}{3}\varepsilon\left[u^k\frac{\partial u_i}{\partial x^k}-\frac{1}{3}u_i\frac{\partial u^k}{\partial x^k}\right]=-\frac{16}{9}\zeta u_i\left(\frac{\partial u^l}{\partial x^l}\right)^2-\frac{\partial\tau_i^k}{\partial x^k}. \tag{7}$$

We transform to the new independent variables t and z by means of

$$x^0=t\cosh z,\qquad x^1=t\sinh z. \tag{8}$$

This is equivalent to a transition to a system of curvilinear coordinates in which the matter is almost at rest. Milekhin[3] has made successful use of this transformation in investigating the motion of an ultrarelativistic ideal fluid. Equation (7) is now written for arbitrary curvilinear coordinates through replacement of the ordinary derivatives by covariant derivatives:[3,7]

$$\frac{1}{3}\frac{\partial\varepsilon}{\partial x^i}+\frac{4}{3}\varepsilon\left[u^k\left(\frac{\partial u_i}{\partial x^k}-\frac{1}{2}u^l\frac{\partial g_{kl}}{\partial x^i}\right)-\frac{1}{3}\frac{u_i}{\sqrt{-g}}\frac{\partial(\sqrt{-g}u^k)}{\partial x^k}\right]=-\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\tau_i^k)}{\partial x^k}+\frac{1}{2}\frac{\partial g_{kl}}{\partial x^i}\tau^{kl}-\frac{16}{9}\zeta u_i\left[\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}u^l)}{\partial x^l}\right]^2. \tag{9}$$

The tensor τ_{ik} must also be given in covariant form.

We assume that in the original coordinate system we have $u^0=\cosh\eta$, $u'=\sinh\eta$. In the system represented by (8) the components of the 4-velocity and metric tensor are[3]

$$u^0=\cosh(\eta-z),\qquad u^1=t^{-1}\sinh(\eta-z),$$
$$g^{00}=-1,\qquad g^{11}=t^{-2},$$
$$g^{22}=g^{33}=1,\qquad g^{ik}=0\ \text{for}\ i\neq k. \tag{10}$$

We assume furthermore that $\eta'=\eta-z$ (η' will be denoted by η for convenience). The components of τ_{ik} in the variables t, z using (10), are

$$\tau_0^0=\frac{4}{3}\zeta\sinh^2\eta\left[\sinh\eta\frac{\partial\eta}{\partial t}+\frac{\cosh\eta}{t}\left(1+\frac{\partial\eta}{\partial z}\right)\right],$$
$$\tau_1^1=-\frac{4}{3}\zeta\cosh^2\eta\left[\sinh\eta\frac{\partial\eta}{\partial t}+\frac{\cosh\eta}{t}\left(1+\frac{\partial\eta}{\partial z}\right)\right],$$
$$\tau_0^1=\frac{4}{3}\zeta\sinh\eta\cosh\eta\left[\frac{\cosh\eta}{t^2}\left(1+\frac{\partial\eta}{\partial z}\right)+\frac{\sinh\eta}{t}\frac{\partial\eta}{\partial t}\right],$$
$$\tau_1^0=-t^2\tau_0^1. \tag{11}$$

After simple but laborious calculations (9) becomes

$$\frac{1}{3}\frac{\partial\varepsilon}{\partial t}-\frac{4}{9}\left[\sinh 2\eta\frac{\partial\eta}{\partial t}+\frac{1}{t}(\cosh 2\eta-2)\left(1+\frac{\partial\eta}{\partial z}\right)\right]\varepsilon=\frac{16}{9}\zeta\cosh\eta\left[\sinh\eta\frac{\partial\eta}{\partial t}+\frac{\cosh\eta}{t}\left(1+\frac{\partial\eta}{\partial z}\right)\right]^2-\frac{\partial\tau_0^0}{\partial t}-\frac{\tau_0^0}{t}-\frac{\partial\tau_0^1}{\partial z}+t\tau^{11},$$

$$\frac{1}{3}\frac{\partial\varepsilon}{\partial z}+\frac{4}{9}\left[t(\cosh 2\eta+2)\frac{\partial\eta}{\partial t}+\sinh 2\eta\left(1+\frac{\partial\eta}{\partial z}\right)\right]\varepsilon=-\frac{16}{9}\zeta t\sinh\eta\left[\sinh\eta\frac{\partial\eta}{\partial t}+\frac{\cosh\eta}{t}\left(1+\frac{\partial\eta}{\partial z}\right)\right]^2-\frac{\partial\tau_1^0}{\partial t}-\frac{\tau_1^0}{t}-\frac{\partial\tau_1^1}{\partial z}. \tag{12}$$

The solution of (12) will be sought for $t\gg\Delta$. We shall also assume $\partial\eta/\partial z\ll 1$, $\partial\eta/\partial t<\eta/t$ for $\eta\ll 1$, which will be justified by the subsequent solution. Equation (12) can then be put into the simple form

$$\frac{3}{4}\frac{\partial\varepsilon}{\partial\tau}+\varepsilon=\frac{\zeta}{\Delta}e^{-\tau},\qquad \frac{3}{4}\frac{\partial\varepsilon}{\partial z}+2\eta\varepsilon=-\frac{\zeta}{\Delta}\eta e^{-\tau}. \tag{13}$$

Here $\tau=\ln(t/\Delta)$. In (13) $\sinh\eta$ and $\cosh\eta$ are expanded in powers of η but we retain only terms that are linear in η. The second equation in (13) shows that ϵ depends slightly on z. Assuming $z/\tau\ll 1$, we therefore obtain a solution for ϵ which satisfies the initial condition $\epsilon\sim\epsilon_0$ for $\tau\sim 1$:

$$\varepsilon=\varepsilon_0\left[\exp\left\{-\frac{4}{3}\left(\tau+\frac{z^2}{2\tau}\right)\right\}+\gamma e^{-\tau}-\gamma\exp\left\{-\frac{4}{3}\left(\tau-\frac{z^2}{2\tau}\right)\right\}\right],\quad \gamma=4\tau/\varepsilon_0\Delta. \tag{14}$$

It must be noted that (14) is an asymptotic solution which is correct only in order of magnitude for $\tau\sim 1$ and $z\sim\tau$.

The second equation of (13) now gives

$$\eta\approx\frac{z}{\tau}\frac{1+\gamma\exp(4z^2/3\tau)}{2+\frac{9}{4}\gamma\exp\{(\tau+2z^2/\tau)/3\}-2\gamma\exp(4z^2/3\tau)}. \tag{15}$$

It is thus evident that the assumptions $\partial\eta/\partial z\sim 1/\tau$; $\partial\eta/\partial\tau<\eta$ are actually satisfied for $z/\tau\ll 1$ and $\tau\gg 1$.

We now perform the new transformation of variables

$$\alpha=\tau+z,\qquad \beta=\tau-z;$$
$$\alpha=\ln\frac{x^0+x^1}{\Delta},\qquad \beta=\ln\frac{x^0-x^1}{\Delta}, \tag{16}$$

which is equivalent to transforming to the customary coordinate system used in references 1, 2, 5, and 6. The criterion for the use of our solution in the new variables will be the condition $\alpha\sim\beta$; this corresponds to the region of maximum particle density in the case of an ideal fluid.

The expression for the energy density becomes

$$\varepsilon = \varepsilon_0 \left\{ \exp\left[-\frac{4}{3}(\alpha + \beta - \sqrt{\alpha\beta}) \right] + \gamma \exp\left[-\frac{\alpha+\beta}{2} \right] - \gamma \exp\left[-\frac{4}{3}\sqrt{\alpha\beta} \right] \right\}. \qquad (17)$$

The 4-velocity will be approximated by

$$u^2 \approx e^{2\eta}(x^0 + x^1)/2(x^0 - x^1). \qquad (18)$$

The first term in (17) is Landau's solution for an ideal fluid; the second and third terms take viscosity into account. It is interesting that in the region $\alpha \sim \beta$ the energy density in the viscous fluid is greater than in the ideal fluid, whereas for the region of the most energetic particles, $\beta \sim 0$, the energy density may be lower than in the ideal case. As a consequence the leading edge of the matter will disintegrate more rapidly (with smaller α for a given value of β), and consequently the fastest particles will possess relatively lower energy, a smaller fraction of the particles will be emitted at small angles and the angular distribution of the produced particles will be less anisotropic.

With very high viscosity, $\gamma \sim 1$, our solution is valid only in the region $\alpha \approx \beta$. For the exact form of secondary-particle angular and energy distributions we would have to solve extremely complicated three-dimensional second order partial differential equations. However, it can be maintained that the nondisintegrating part of the meson-nucleon cloud will be located in the region $\alpha \sim \beta$, and since $u^2 \sim 1$ in this region all secondary particles will possess approximately the same energy and their angular distribution will obviously be isotropic. The number N of mesons produced will be given by the formula of Rozental' and Chernavskiĭ:[8] $N \sim E_c/\mu$, where E_c is the nucleon energy in the center-of-mass system and μ is the mass of a pion.

3. Hamaguchi[5,6] also investigated the influence of viscosity on the principal features of multiple production as described by the hydrodynamic theory. However, his solution of the one-dimensional hydrodynamic equations is incorrect because of a number of mistakes, some of which we shall now discuss.

In reference 5 the sought solution is expanded in powers of the small viscosity coefficient in order to solve the hydrodynamic equations for one-dimensional expansion. The Landau solution is taken as the zeroth approximation. By retaining only linear terms in the viscosity coefficient the author obtains a set of linear partial differential equations with variable coefficients as a basis for the given approximation. For the energy density ϵ_0 in the solution which neglects viscosity he assumes, as a quite rough approximation of Landau's solution, $\epsilon_0 = \epsilon_0^* \exp[-4/3(\alpha - \beta)]$. In addition to the fact that for the region of maximum particle density $\alpha \sim \beta$ the expression for ϵ_0 is incorrect, it must be noted that this approximation seriously neglects terms which are of the same order of magnitude as other terms left in the equations. The original equations contain the terms $\partial\epsilon_0/\partial\xi$ and $\partial\epsilon_0/\partial x^0$ (here $\xi = x^0 \quad x'$); it is easily seen that when $\epsilon_0^* \exp[-4/3(\alpha + \beta - \sqrt{\alpha\beta})]$ (Landau's solution) is replaced by ϵ_0 the terms $\frac{1}{2}\sqrt{\beta/\alpha}$ and $\frac{1}{2}\sqrt{\alpha/\beta}$ are neglected compared with unity although both of these terms are of the order of unity for the important region $\alpha - \beta$.

For the correction of 4-velocity u_1 Hamaguchi seeks a solution in the form $u_1^2 = f_1(x^0/\xi)^N$, where N is a number selected to make f_1 slightly dependent on x^0 and ξ in the solution. However, [see Eq. (24) of reference 5], f_1 is found to be always strongly dependent on its arguments. It can therefore be shown that any number greater than 4 can be taken for N instead of the value 4.541 given by Hamaguchi. Since the value of N strongly affects the solutions the uncertainty of its value makes the solution somewhat arbitrary.

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[1] L. D. Landau, Izv. Akad. Nauk SSSR, Ser. Fiz. **17**, 51 (1953).

[2] S. Z. Belen'kiĭ and L. D. Landau, Usp. Fiz. Nauk **56**, 309 (1955).

[3] G. A. Milekhin, J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 1185 (1958), Soviet Phys. JETP **8**, 829 (1959).

[4] M. Namiki and C. Iso, Progr. Theoret. Phys. **18**, 591 (1957).

[5] M. Hamaguchi, Nuovo cimento **6**, 1243 (1956).

[6] M. Hamaguchi, Nuovo cimento **5**, 1622 (1957).

[7] L. D. Landau and E. M. Lifshitz, Механика сплошных сред, (Mechanics of Continuous Media) 2nd Ed., Moscow, 1954.

[8] I. L. Rozental' and D. S. Chernavskiĭ, Usp. Fiz. Nauk **52**, 185 (1954).

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303