NUCLEON-NUCLEON SCATTERING IN HIGH ANGULAR MOMENTUM STATES

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The properties of the one-meson approximation to elastic nucleon-nucleon scattering are considered in detail, and the scattering phase shifts are calculated. A new method of phase-shift analysis is proposed, which should lessen the ambiguity of the solution and make it possible to get more accurate values of the experimental phase shifts.

OKUN' and Pomeranchuk[1] have shown that, since peripheral interactions of a particle involve effectively the minimum possible number of π mesons, processes involving large orbital angular momenta can be calculated on the basis of the present meson theory.

Let us examine nucleon-nucleon scattering in greater detail. For the part of the amplitude that corresponds to sufficiently large orbital angular momenta l there are contributions only from diagrams in which a single π meson is exchanged.* In the one-meson approximation with the symmetric pseudoscalar meson-nucleon interaction the nucleon-nucleon scattering is described in the center-of-mass system by the amplitude ($\hbar = c = 1$)

$$\langle \mathbf{k}', s_1', s_2', \tau_1', \tau_2' | \mathbf{M} | \mathbf{k}, s_1, s_2, \tau_1, \tau_2 \rangle = \frac{g^2}{4m\gamma}\left\{\frac{\langle 1', 2' | (\boldsymbol{\tau}_1\boldsymbol{\tau}_2)(\boldsymbol{\sigma}_1\mathbf{q})(\boldsymbol{\sigma}_2\mathbf{q}) | 1, 2\rangle}{q^2+\mu^2} - \frac{\langle 2', 1' | (\boldsymbol{\tau}_1\boldsymbol{\tau}_2)(\boldsymbol{\sigma}_1\mathbf{p})(\boldsymbol{\sigma}_2\mathbf{p}) | 1, 2\rangle}{p^2+\mu^2}\right\},$$

$$\mathbf{q} = \mathbf{k} - \mathbf{k}', \quad q = 2k\sin\theta/2, \quad \mathbf{p} = \mathbf{k} + \mathbf{k}',$$

$$p = 2k\cos\theta/2, \quad \gamma = \sqrt{m^2 + k^2/m}, \qquad (1)$$

The absolute square of this amplitude gives the differential cross-section for the transition between states prescribed by the momentum of one of the nucleons, the components of the spins of both nucleons along the initial momentum, and the isotopic spin components of both nucleons.

Here the notations are: g is the the renormalized meson-nucleon interaction constant ($g^2 \approx 15$), m the mass of the nucleon, μ the mass of the π meson, $\boldsymbol{\tau}_1$ and $\boldsymbol{\sigma}_1$ the Pauli matrices in the isotopic and spin spaces, respectively, of the first nucleon, $\boldsymbol{\tau}_2$ and $\boldsymbol{\sigma}_2$ the analogous matrices for the second nucleon, and θ the scattering angle. Strictly speaking, the amplitude (1), is only the main part of the exact one-meson amplitude, since the only one of the effects caused by the use of the exact vertex part and the exact Green's function that has been taken into account is the renormalization of the interaction constant. Besides this the exact one-meson amplitude contains a contribution of the same type as is given by many-meson diagrams, which need not be used (this is proved in reference 1). In nonrelativistic approximation the amplitude (1) corresponds to the well known one-meson interaction of nucleons, which is obtained in the lowest order of perturbation theory (cf. e.g., references 3* and 4). The part of the exact one-meson amplitude that has been omitted corresponds to an interaction that vanishes at infinity like $\exp(-3\mu r)$.

To make the change to the representation of the total spin and total isotopic spin ($\mathbf{k}$, S, S_z, T, T_3) we must reconstruct the scattering operator M, for which the amplitude (1) is the matrix element in the representation $\mathbf{k}$, s_1, s_2, τ_1, τ_2. It is not hard to verify that the scattering operator has the form

$$\mathbf{M} = \frac{g^2}{4m\gamma}\frac{3-(\boldsymbol{\tau}_1\boldsymbol{\tau}_2)}{4}\frac{p^2}{p^2+\mu^2}\{-1+(\boldsymbol{\sigma}_1\mathbf{n})(\boldsymbol{\sigma}_2\mathbf{n}) - (\boldsymbol{\sigma}_1\mathbf{l})(\boldsymbol{\sigma}_2\mathbf{l})\} + \frac{g^2}{4m\gamma}\left\{(\boldsymbol{\tau}_1\boldsymbol{\tau}_2)\frac{q^2}{q^2+\mu^2} + \frac{3-(\boldsymbol{\tau}_1\boldsymbol{\tau}_2)}{4}\frac{p^2}{p^2+\mu^2}\right\}(\boldsymbol{\sigma}_1\mathbf{m})(\boldsymbol{\sigma}_2\mathbf{m}), \quad (2)$$

where $\mathbf{n} = [\mathbf{k}\times\mathbf{k}']/k^2$, $\mathbf{l} = \mathbf{p}/p$, $\mathbf{m} = \mathbf{q}/q$. We note

*A calculation of the contribution from diagrams with two mesons exchanged, made by Galanin, Grashin, Ioffe, and Pomeranchuk, provides an estimate of the accuracy of the one-meson approximation for various values of l. It turns out that for nonrelativistic energies (up to 100 – 300 Mev) the one-meson approximation makes the main contribution even for the relatively small values $l = 3, 4$. The results of an analysis of the experimental data[2] for 90 and 150 Mev agree with these estimates.

*We take occasion to point out that Eqs. (47.10) and (47.13) of reference 3 contain a superfluous factor 3 in the central part of the potential.

the following fact. In the amplitude (1) the second (exchange) term exists only for identical particles. For different particles, keeping the former interaction of the two particles with the meson, we should have simply

$$\mathbf{M} = \frac{g^2}{4m\gamma}(\boldsymbol{\tau}_1\boldsymbol{\tau}_2)\frac{q^2}{q^2+\mu^2}(\sigma_1\mathbf{m})(\sigma_2\mathbf{m}).$$

Comparison with Eq. (2) shows that inclusion of the symmetry changes the interaction. This is indeed not surprising, since inclusion of the symmetry in fact means the introduction of a new (exchange) interaction, to describe which additional terms in the scattering operator are required.

The scattering operator is diagonal in the total spin and the total isotopic spin, and therefore we shall denote its matrix elements by the symbol M (S, β, α, T), where α and β are the initial and final components of the total spin along the initial momentum. Calculation gives the following scattering amplitudes:

$$\left.\begin{array}{l} M(1,\beta,\alpha,1) = M_{\beta\alpha}(\theta,\varphi) - M_{\beta\alpha}(\pi-\theta,\varphi+\pi) \\ M(0,0,0,1) = M(\theta) + M(\pi-\theta) \end{array}\right\} \quad T=1,$$

$$\left.\begin{array}{l} M(1,\beta,\alpha,0) = -3\,[M_{\beta\alpha}(\theta,\varphi) + M_{\beta\alpha}(\pi-\theta,\varphi+\pi)] \\ M(0,0,0,0) = -3\,[M(\theta) - M(\pi-\theta)] \end{array}\right\} T=0, \tag{3a}$$

where the matrices $M_{\beta\alpha}(\theta, \varphi)$ and $M(\theta)$, combined into a single four-row matrix, have the form

$$M_{\beta\alpha}(\theta,\varphi) = -M(\theta)\,L_{\beta\alpha}(\theta,\varphi) \quad (\alpha,\beta = 1, 0, -1, 0); \quad M(\theta) = \frac{1}{\mu}G\frac{\cos\theta - 1}{x-\cos\theta}, \quad G = \frac{g^2\mu}{4m\gamma}, \quad x = 1 + \frac{1}{2}\left(\frac{\mu}{k}\right)^2;$$

$$L_{\beta\alpha}(\theta,\varphi) = \begin{pmatrix} (1-\cos\theta)/2 & -\sin\theta e^{-i\varphi}/\sqrt{2} & (1+\cos\theta)e^{-2i\varphi}/2 & 0 \\ -\sin\theta e^{i\varphi}/\sqrt{2} & \cos\theta & \sin\theta e^{-i\varphi}/\sqrt{2} & 0 \\ (1+\cos\theta)e^{2i\varphi}/2 & \sin\theta e^{i\varphi}/\sqrt{2} & (1-\cos\theta)/2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3b}$$

Here φ is the azimuthal angle of the scattering, and the fourth value of α and β corresponds to the singlet.

Since for our purpose it is important to separate the large orbital angular momenta l, which are well described by the one-meson amplitude, from the small angular momenta, for which many-meson diagrams must be included, we shall go over to the representation J, m_J, l, S, T, T_3 (J and m_J are the total angular momentum and its component along the initial momentum). In the representation of the total angular momentum the scattering is described by a matrix $S^{JT}_{l',l}$, diagonal in J, m_J, S, T, T_3 and independent of m_J and T_3 (cf., e.g., reference 5). The index for the total spin is omitted, since the singlet is distinguished from the triplet by the use of the notation S^T_l. The scattering amplitude in the representation **k**, S, S_z, T, T_3 is connected with the matrix $S^{JT}_{l',l}$ by the following general relation:*

$$M(S,\beta,\alpha,T) = \frac{1}{ik}\sum_{l'}^{\infty}\sum_{J}^{\infty}\sum_{l}^{\infty}\sqrt{4\pi(2l+1)}\,i^{l-l'}\,Y^{(\alpha-\beta)}_{l'}(\theta,\varphi)$$
$$\times (l, S, 0, \alpha \mid l, S, J, \alpha)$$
$$\times (l', S, \alpha-\beta, \beta \mid l', S, J, \alpha)\,(S^{JT}_{l',l} - \delta_{l',l}), \tag{4}$$

where $(l, S, m_l, m_S \mid l, S, J, m_J)$ are Clebsch-Gordan coefficients in the notation of Condon and Shortley.[6] This relation enables us to express the scattering matrix and the scattering phase shifts in terms of the coefficients of the expansion of the amplitudes M (S, β, α, T) in associated Legendre polynomials:

$$M(S,\beta,\alpha,T) = \frac{2}{k}e^{i(\alpha-\beta)\varphi}\sum_{l=|\alpha-\beta|}^{\infty}(2l+1)\,P^{(\alpha-\beta)}_l$$
$$\times (\cos\theta)\,a_l\,(S,\beta,\alpha,T). \tag{5}$$

Let us introduce the following notations for the nonvanishing elements of the scattering matrix:

$$\left.\begin{array}{l} S^{JT}_{l,l} - 1 = 2i\eta^{JT}_l \\ S^{J\ \ T}_{J-1,\,J+1} = S^{J\ \ T}_{J+1,\,J-1} = 2i\xi^T_J \end{array}\right. \quad \text{for the triplet,}$$

$$S^T_l - 1 = 2i\eta^T_l \qquad \text{for the singlet.}$$

For T = 1 only the matrix elements with odd l in the triplet and even l in the singlet are different from zero, and the reverse is the case for T = 0; therefore the index for the total isotopic spin can be omitted (in the triplet coefficients of the expansion of the amplitudes we shall also omit the index for the total spin, and in the singlet coefficients, all spin indices). In the case in which $|\eta^J_l|$, $|\xi_J|$, $|\eta_l| \ll 1$, i.e., for large orbital angular momenta, the real parts of these quantities are respectively the same as the matrix phases $\bar{\delta}_{l,J}$, $\bar{\epsilon}_J$, $\bar{\delta}_l$ (bar phase shifts) introduced by Stapp and others.[7] If we parametrize the scattering matrix by means of the proper phase shifts[5] δ^J_l, ϵ_J, δ_l, then for large orbital angular momenta

*We call attention to the fact that the right member of Eq. (4) is doubled as compared with the analogous expression for nonidentical particles. The right side of Eq. (5) is also doubled.

$$\mathrm{Re}\,\eta_l = \delta_l \equiv \bar{\delta}_l, \quad \mathrm{Re}\,\eta_J^J = \delta_J^J \equiv \bar{\delta}_{J,J},$$

$$\mathrm{Re}\,\eta_{J\pm1}^J = \cos^2\varepsilon_J \cdot \delta_{J\pm1}^J + \sin^2\varepsilon_J \cdot \delta_{J\mp1}^J,$$

$$\mathrm{Re}\,\xi_J = \tfrac{1}{2}\sin 2\varepsilon_J\,(\delta_{J-1}^J - \delta_{J+1}^J).$$

We note that the mixing parameters ϵ_J are usually large, so that the quantities $\eta^J_{J\pm1}$ can be decidedly different from the corresponding proper phase shifts $\delta^J_{J\pm1}$.

Solving Eqs. (4) and (5) for η and ξ, we get for the quantities different from zero

$$\begin{aligned}(2l-1)\,\eta_l^{l-1} &= (l-1)\,a_l(1,1) + \sqrt{2}(l+1)\,a_l(1,0)\\ &+ (l-1)(l+1)(l+2)\,a_l(-1,1) + l a_l(0,0)\\ &+ (l-1)(l+1)\,[a_l(1,0) + a_l(0,1)]\,\sqrt{2},\\ \eta_l^l &= a_l(1,1) - \sqrt{2}\,a_l(1,0) - (l-1)(l+2)\,a_l(-1,1)\\ &+ [a_l(1,0) + a_l(0,1)]\,\sqrt{2},\\ (2l+3)\,\eta_l^{l+1} &= (l+2)\,a_l(1,1)\\ &+ \sqrt{2}\,l a_l(1,0) + (l-1)\,l\,(l+2)\,a_l(-1,1)\\ &+ (l+1)\,a_l(0,0) - l(l+2)\,[a_l(1,0) + a_l(0,1)]\sqrt{2},\\ \xi_J &= \frac{\sqrt{J(J+1)}}{2J+1}\{-a_{J-1}(1,1) - 2\sqrt{2}(J-1)\,a_{J-1}(1,0)\\ &- (J-2)(J-1)\,a_{J-1}(-1,1) + a_{J-1}(0,0)\\ &+ (J-1)\,[a_{J-1}(1,0) + a_{J-1}(0,1)]\sqrt{2}\}\\ &= \frac{\sqrt{J(J+1)}}{2J+1}\{-a_{J+1}(1,1) + 2\sqrt{2}(J+2)\,a_{J+1}(1,0)\\ &- (J+2)(J+3)\,a_{J+1}(-1,1) + a_{J+1}(0,0)\\ &- (J+2)\,[a_{J+1}(1,0) + a_{J+1}(0,1)]\sqrt{2}\},\\ \eta_l &= a_l. \end{aligned} \tag{6}$$

We emphasize that the relations (6) are of a general character, i.e., are valid in any approximation as to the number of mesons, since no specific properties of the scattering amplitude were used in deriving them. In the one-meson approximation $a_l(1,0) + a_l(0,1) = 0$, which is equivalent to the equation $e^{-i\varphi}M(1,0,1,T) = e^{i\varphi}M(1,1,0,T)$; this leads to some simplification in the equations (6) (the last terms vanish), and also to a linear relation between the triplet elements with the same l but different J:

$$(l+1)(2l-1)\,\eta_l^{l-1} + (2l+1)\,\eta_l^l - l(2l+3)\,\eta_l^{l+1} = 0. \tag{7}$$

Substituting in Eq. (6) the expansion coefficients of the amplitudes (3)

$$\begin{aligned} a_l(1,1) &= \tfrac{1}{2} G\frac{k}{\mu}\left\{(2-x)\,\delta_{l0} - \tfrac{1}{3}\delta_{l1} + (x-1)^2 Q_l\right\} E_l,\\ a_l(1,0) &= \frac{Gk/\mu}{\sqrt{2l(l+1)}}\left\{\tfrac{2}{3}\delta_{l1} + l(x-1)\,[xQ_l - Q_{l-1}]\right\} E_l,\\ a_l(-1,1) &= \frac{Gk/\mu}{2(l-1)(l+1)(l+2)}\{[(l-1)(x^2-1)-2]\\ &\times Q_l + 2xQ_{l-1}\}\,E_l,\\ a_l(0,0) &= G\frac{k}{\mu}\left\{(x-1)\,\delta_{l0} + \tfrac{1}{3}\delta_{l1} - x(x-1)\,Q_l\right\} E_l,\\ a_l &= G\frac{k}{\mu}\{(x-1)\,Q_l - \delta_{l0}\}\,E_l, \end{aligned} \tag{8}$$

where the projection function

$$E_l \equiv E_l(S,T) = -\tfrac{3}{2} + 2T + (1-2S)(^3/_2 - T)(-1)^l$$

takes the values 1, 0, −3, depending on the spin and parity, we get

$$\begin{aligned} \eta_l^{l-1} &= \frac{Gk/\mu}{2l-1}\{Q_{l-1} - Q_l\}\,E_l,\\ \eta_l^l &= -\frac{Gk/\mu}{l+1}\{[l(x-1)-1]\,Q_l + Q_{l-1}\}\,E_l, \quad \text{for } l \geqslant 2;\\ \eta_l^{l+1} &= \frac{Gk/\mu}{2l+3}\{Q_l - Q_{l+1}\}\,E_l;\\ \xi_J &= \frac{Gk/\mu}{2J+1}\sqrt{\frac{J}{J+1}}\{[1-(2J+1)(x-1)]\,Q_J - Q_{J-1}\}\,E_{J-1},\\ \eta_l &= G\frac{k}{\mu}\{(x-1)\,Q_l - \delta_{l0}\}\,E_l. \end{aligned} \tag{9}$$

For the ^{3}S and ^{3}P states we have

$$\begin{aligned} \eta_1^0 &= G\frac{k}{\mu}[1-(x-1)\,Q_0],\\ \eta_1^1 &= -\tfrac{1}{2}G\frac{k}{\mu}[1-(x-1)(Q_0 - Q_1)], \quad \text{for } T=1;\\ \eta_1^2 &= 0.1\,G\frac{k}{\mu}[1-(x-1)(Q_0 + 3Q_1)];\\ \eta_0^1 &= -G\frac{k}{\mu}[1-(x-1)\,Q_0] \quad \text{for } T=0. \end{aligned} \tag{10}$$

In Eqs. (8) – (10) the following notations have been used: δ_{l0} and δ_{l1} are Kronecker symbols, and $Q_l \equiv Q_l(x)$ is the Legendre function of the second kind. For small energies ($k/\mu \ll 1$, which corresponds to laboratory energies $E \ll 40$ Mev)

$$Q_l \approx [l!\,2^{l+1}/(2l+1)!!\,]\,(k/\mu)^{2l+2}.$$

In determining the accuracy of the one-meson approximation for the various quantities we must take into account the following properties of these quantities. The off-diagonal element ξ_J can be expressed in terms of the (J + 1)st coefficients of the expansion of the amplitudes [cf. Eq. (6)]. These same coefficients occur in the expressions for the diagonal elements corresponding to the orbital angular momentum $l = J + 1$, and thus, generally speaking, the errors incurred in all

TABLE I. Values of η_l, η_l^J, ξ_J for $T = 1$ (the usual spectroscopic notations are used for the diagonal elements η_l, η_l^J)

E, Mev	10	17	40	100	200	300	400	670
x	3	2.165	1.5	1.2	1.1	1.0675	1.05	1.03
$^1S_0$	−4.8	−8.6	−19	−37	−57	−70	−82	−100
$^1D_2$	0.17	0.38	0.99	1.9	2.4	2.5	2.6	2.5
$^1G_4$	0.004	0.018	0.11	0.41	0.75	0.94	1.1	1.2
$^3P_0$	4.8	8.6	19	37	57	70	82	100
$^3P_1$	−3.0	−5.3	−11	−21	−31	−37	−43	−54
$^3P_2$	0.11	0.28	0.89	2.4	4.3	5.7	6.9	9.2
ξ_2	−0.23	−0.53	−1.5	−3.5	−5.4	−6.7	−7.7	−9.5
$^3F_2$	0.015	0.052	0.27	1.0	2.1	3.0	3.8	5.3
$^3F_3$	−0.037	−0.13	−0.58	−1.9	−3.6	−4.9	−5.9	−7.9
$^3F_4$	0.001	0.006	0.048	0.25	0.62	0.94	1.2	1.9
ξ_4	−0.004	−0.021	−0.14	−0.59	−1.2	−1.7	−2.1	−2.7
$^3H_4$	$2\cdot10^{-4}$	0.001	0.016	0.12	0.35	0.57	0.78	1.3
$^3H_5$	$-8\cdot10^{-5}$	−0.005	−0.056	−0.35	−0.88	−1.4	−1.8	−2.6
$^3H_6$	$2\cdot10^{-5}$	$2\cdot10^{-4}$	0.004	0.039	0.14	0.23	0.35	0.60

these quantities (ξ_J, $\eta_{J+1}^{\cdot\cdot}$, a_{J+1}) owing to the use of the one-meson approximation are of the same order of magnitude. For small energies $|\xi_J| \gg |\eta_{J+1}^{\cdot\cdot}|$; evidently a similar relation will also hold in this range of energies for the corresponding corrections to the one-meson approximation. Furthermore, for a very wide range of energies in the one-meson approximation $|\eta_l^{l+1}| \ll |\eta_l^{l-1}|$, $|\eta_l^l|$ (we note that for small energies η_l^{l-1}, $\eta_l^l \sim k^{2l+1}$, and $\eta_l^{l+1} \sim k^{2l+3}$), and therefore the fractional accuracy of the quantity η_l^{l+1} must be much lower than those of η_l^{l-1} and η_l^l. For small energies ($k/\mu \lesssim 1$) the one-meson approximation can be quite useless for the determination of η_l^{l+1}. This must be particularly noticeable for small l ($^3P_2$, $^3D_3$). On going over to the proper phases we get δ_{J+1}^J and ϵ_J in the one-meson approximation with the errors characteristic of the $(J-1)$st coefficients, since these phases depend on η_{J-1}^J. In order not to introduce these artificial errors into the calculation it is better to use the Stapp parametrization or to use the coefficients (8) directly.

These remarks show that the accuracy of calculations in the one-meson approximation of the various measurable quantities corresponding to a given orbital angular momentum l is determined by the accuracy of the elements η_l^{l-1}, η_l^l, ξ_{l-1}, η_l, except in those special cases in which through cancellations a measurable quantity will actually depend only on η_l^{l+1}. In principle other types of cancellations are possible, for example if the triplet elements occur only in the form of the linear combination (7), which vanishes in the one-meson approximation. In these cases there is a corresponding decrease in the accuracy of the calculations. Besides this, in the one-meson approximation all the quantities are real, and therefore they cannot be used in cases in which the imaginary part of the scattering amplitude cannot be neglected.

Table I shows the values in degrees of η and ξ for $T = 1$, as calculated from Eqs. (9), (10) with $G = 0.55/\gamma$ for a number of energies E (in the laboratory system), and Table II shows the values of η and ξ for $T = 0$. The tables begin with $l = 0$, although it is obvious that for small values of l the one-meson approximation can be quite useless.

An estimate of the accuracy of the one-meson approximation found by calculation of the two-meson phase shifts shows that with good accuracy (of the order of 20 percent) one can use the one-meson F and G phase shifts for $E \lesssim 200$ Mev; their accuracy improves with decrease of the energy. For a fixed energy the accuracy of the various phase shifts increases exponentially with increase of l. The D phase shifts are given with good accuracy by the one-meson approximation for $E \lesssim 50$ Mev. To determine the goodness of the one-meson S and P phase shifts one would have to study many-meson diagrams,* and this has not been done so far. There is evidently no region of applicability of the one-meson S phase shifts, since for small energies these phases are $\delta_1 = \delta_0^1 \sim k^3$ and cannot provide the main contribution. An analogous situation exists for the $^3P_2$ phase shifts.

For a comparison of the one-meson phase shifts with the experimental values one needs a rather precise phase-shift analysis of the experi-

*We emphasize that it is necessary to study the exact many-meson diagrams; calculations in low orders of perturbation theory (cf., e.g., reference 4) cannot serve as a reliable basis for estimating the corrections to the one-meson phase shifts.

TABLE II. Values of η_l, η_l^J, ξ_J for T = 0

E, Mev	10	17	40	100	200	300	400	670
x	3	2.165	1.5	1.2	1.1	1.0675	1.05	1.03
$^1P_1$	—3.8	—5.9	—9.7	—13	—14	—13	—13	—12
$^1F_3$	—0.08	—0.24	—0.98	—2.6	—3.9	—4.6	—4.9	—5.1
$^1H_5$	—0.002	—0.012	—0.12	—0.60	—1.3	—1.8	—2.1	—2.6
$^3S_1$	—4.8	—8.6	—19	—37	—57	—70	—82	—100
ξ_1	6.0	10	20	35	50	59	67	81
$^3D_1$	—0.54	—1.4	—4.5	—12	—22	—29	—34	—46
$^3D_2$	0.88	2.1	6.5	16	26	33	40	51
$^3D_3$	—0.032	—0.11	—0.57	—2.1	—4.5	—6.4	—8.0	—11
ξ_3	—0.093	0.31	1.3	4.1	7.3	9.4	11	14
$^3G_3$	—0.005	—0.024	—0.18	—0.96	—2.4	—3.6	—4.8	—7.3
$^3G_4$	0.016	0.075	0.53	2.3	5.2	7.3	9.3	13
$^3G_5$	$—4\cdot10^{-4}$	—0.003	—0.039	—0.28	—0.85	—1.4	—1.9	—3.1

mental data, including phase shifts for high angular momenta. The usual procedure in phase-shift analysis is as follows: all the measured quantities are written as functions of the phase shifts with $l \le l_{\max}$, and all the phase shifts for $l > l_{\max}$ are assumed to be zero. The nonvanishing phase shifts are then varied to get the best agreement with the experimental data. The main shortcoming of such a procedure as applied to the existing data is the ambiguity of the solution, caused by the large number of variable parameters and the incompleteness of the set of measurements. Thus in the analysis of the data on p-p scattering at 310 Mev with $l_{\max} = 5$ (14 parameters)[7] eight acceptable solutions are obtained, more precisely eight more or less broad many-dimensional regions (of ellipsoidal type) centered on the solutions. In analyzing the data for 150 Mev Ohnuma and Feldman[8] obtained 11 such regions, some of them connecting continuously with each other. The analysis for 40 Mev[9] leaves so much arbitrariness in the allowed solutions that it is practically impossible to get any sort of quantitative estimate of the phase shifts.

Besides this difficulty, which in principle can be removed by an increase of the amount of experimental data, there remains the problem of the stability of the solutions with respect to the inclusion of the phases that have been dropped (with $l > l_{\max}$). It is obvious that the least stable values (we are speaking in terms of fractional stability) will be the last of the included phase shifts with $l \approx l_{\max}$, which are those of interest to us here. Therefore, for example, for a comparison of the phase shifts with $l = 4$ we need an analysis with at least $l_{\max} = 5$, and to test the stability of these phase shifts it is desirable to continue the analysis to $l_{\max} = 6$. It is obvious that the analysis then becomes cumbersome and practically not feasible.

It seems reasonable to us to propose a different method of phase-shift analysis, which would take into account our knowledge of the phases with large l: to use the phase shifts for $l \le l_1$ as unknown parameters and take for all the others (with $l > l_1$) the values in the one-meson approximation. The simplest way to do this is to write the measured quantities as functions of the "corrected" amplitudes

$$M'(S,\ \beta,\ \alpha,\ T) = M(S,\ \beta,\ \alpha,\ T)$$

$$-\frac{2}{k}e^{i(\alpha-\beta)\varphi}\sum_{l=|\alpha-\beta|}^{l_1}(2l+1)P_l^{(\alpha-\beta)}(\cos\theta)\,a_l(S,\ \beta,\ \alpha,\ T)$$

$$+\frac{1}{ik}\sum_{l'=0}^{l_1}\sum_{J=0}^{l_1+1}\sum_{l=0}^{l_1+2}\sqrt{4\pi(2l+1)}\,i^{l-l'}Y_{l'}^{(\alpha-\beta)}(\theta,\ \varphi)$$

$$\times(l,\ S,\ 0,\ \alpha|\,l,\ S,\ J,\ \alpha)(l',\ S,\ \alpha-\beta,\ \beta\,|\,l',\ S,\ J,\ \alpha)$$

$$\times(S_{l',l}^{JT}-\delta_{l',l}), \qquad (11)$$

where M(S, β, α, T) and a_l(S, β, α, T) are respectively the amplitudes (3) and their expansion coefficients (8), and the rest of the right member is the usual expression for the matrix elements* in terms of a finite number of phase shifts ($l_{\max} = l_1$). The off-diagonal elements of the scattering matrix, corresponding to $J \ge l_1$, which will be involved in this part, must be included in the one-meson approximation. For example one can begin the analysis for 310 Mev with $l_1 = 2$, in view of the satisfactory accuracy of the F phase shifts. There are then in all 5 variable parameters (instead of the 14 parameters of reference 7); the analysis is much easier and, what is more important, the number of solutions must be smaller. To test the stability of the solutions with respect to errors in the phase shifts with $l > l_1$, which have been included in the one-meson approximation,

*Expressions for the various measurable quantities in terms of the amplitudes M′, and the parts of the amplitudes (11) required for the "correction," with effects of the Coulomb interaction included, can be found in reference 7.

and to improve the solution, one must repeat the analysis with $l_1 = 3$ (9 parameters) and examine the displacement of the solutions found previously. If new solutions (isolated from the former ones) appear, they must be rejected. It may turn out that the value $l_1 = 2$ is too low for the case in question, and that the solution found in the first step is not accurate enough. Nevertheless successive stages of the analysis with $l_1 = 3$ and $l_1 = 4$ must give better results than those obtained by Stapp and others[7] by the usual method. Furthermore the analysis is less cumbersome and is a natural physical scheme for breaking up the analysis into several stages and thus reducing the volume of the calculations. Even from the experimental data available at present the proposed procedure of phase-shift analysis should give more definite and accurate information about the phase shifts.*

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*From the example of the phase-shift analysis for $E < 40$ Mev with $l_{max} = 2$ carried out by MacGregor[10] we can see how an inspection of the one-meson approximation to the higher phase shifts (in this case $^3P_0$, $^3P_1$, $^1D_2$) can reduce the number of solutions. On the other hand, the analysis[9] for 40 Mev shows what a large effect can come from the neglect of the phase shifts with $l > l_{max}$ and what difficulties arise from the inclusion of additional phase shifts as arbitrary parameters.

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