SLIPPING AND TEMPERATURE DISCONTINUITY AT THE BOUNDARY OF A GAS MIXTURE

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Boundary conditions have been obtained for the hydrodynamical transport equations for a gas mixture. The conditions take into account slipping and the temperature jump at the boundary with a solid surface.

1. As is known, corrections due to the finite mean free path appear in the boundary conditions for hydrodynamic transport equations before they appear in the equations themselves.[1] Therefore, when making the transition from hydrodynamic flow to molecular flow, there exists a region of values $l/a < 1$ (l is the mean free path and a is the characteristic dimension of the problem), where these equations can still be used, but the boundary-condition corrections linear in l/a cannot be neglected. Furthermore, these corrections may prove to be substantial even at high pressures, if they lead to the appearance of noticeable additions to the convective terms in the transport equations (see, for example, reference 2).

Corrections to boundary conditions for a one-component gas were investigated by many authors.[3,4] The most thorough method for obtaining them was developed by Grad.[5] In this paper we shall generalize this method to find corresponding boundary conditions for a binary inhomogeneous gas mixture on a non-absorbing solid surface.

2. We describe a binary mixture of gases by distribution functions $f_\alpha(\mathbf{r}, \boldsymbol{\xi}, t)$, normalized for the density of the molecules of the α component. Here $\mathbf{r}$ is the radius vector of the molecule, $\boldsymbol{\xi}$ its velocity, and t the time. The internal degrees of freedom of the molecules are neglected.

The principal macroscopic quantities will be introduced with the aid of the equalities:

$$n_\alpha(\mathbf{r},t) = \int f_\alpha(\mathbf{r}, \boldsymbol{\xi}, t)\, d\boldsymbol{\xi}, \qquad \rho_\alpha(\mathbf{r}, t) = m_\alpha n_\alpha, \qquad \rho = \sum_\alpha \rho_\alpha,$$

$$\mathbf{u}_\alpha(\mathbf{r}, t) = \frac{1}{n_\alpha} \int \boldsymbol{\xi} f_\alpha(\mathbf{r}, \boldsymbol{\xi}, t)\, d\boldsymbol{\xi}, \qquad \mathbf{u}(\mathbf{r}, t) = \frac{1}{\rho} \sum_\alpha \rho_\alpha \mathbf{u}_\alpha, \qquad (1)$$

where n_α, ρ_α, and $\mathbf{u}_\alpha$ are respectively the density of the molecules, the mass density, the local velocity of the α component; ρ and $\mathbf{u}$ are the mass density and velocity of the local center of intertia of the mixture, and m_α is the mass of a molecule of type α.

Introducing the relative velocity $\mathbf{c} = \boldsymbol{\xi} - \mathbf{u}$, we form the moments

$$R_{\alpha, i} = m_\alpha \int c_i f_\alpha\, d\boldsymbol{\xi}, \qquad P_{\alpha, ij} = m_\alpha \int c_i c_j f_\alpha\, d\boldsymbol{\xi},$$

$$S_{\alpha, i} = (m_\alpha/2) \int c^2 c_i f_\alpha\, d\boldsymbol{\xi}. \qquad (2)$$

The physical meaning of these moments is obvious.

The partial pressure p_α, the mixture pressure p, the stress tensor of the mixture p_{ij}, and the mixture temperature T are

$$p_\alpha = \tfrac{1}{3} P_{\alpha, ii}, \qquad p = \sum_\alpha p_\alpha, \qquad p_{\alpha, ij} = P_{\alpha, ij} - p_\alpha \delta_{ij},$$

$$p_{ij} = \sum_\alpha P_{\alpha, ij} - p\, \delta_{ij}, \qquad p_\alpha = n_\alpha kT. \qquad (3)$$

Repeated Latin indices will henceforth be summed.

3. We expand $f_\alpha(\mathbf{r}, \boldsymbol{\xi}, t)$ in a series of Hermite vector polynomials. Confining ourselves to the third approximation, we obtain after transformations analogous to those used by Grad[5]

$$f_\alpha(\mathbf{r}, \boldsymbol{\xi}, t) = f_{\alpha 0} \left[1 + \frac{R_{\alpha, i}}{p_\alpha} c_i + \frac{m_\alpha}{2kT} \frac{p_{\alpha, ik}}{p_\alpha} c_i c_k \right.$$

$$\left. + \frac{1}{10} \left(\frac{2m_\alpha}{kT} S_{\alpha, i} - 5 R_{\alpha, i} \right) \left(\frac{m_\alpha}{kT} c^2 - 5 \right) \frac{c_i}{p_\alpha} \right], \qquad (4)$$

where

$$f_{\alpha 0} = n_\alpha (m_\alpha/2\pi kT)^{3/2} \exp(-m_\alpha c^2/2kT). \qquad (5)$$

4. We consider two-dimensional flow of gas near the surface $x = 0$ (the x axis is directed along the outward normal to the surface, the y axis in the flow direction). If the gas is in equilibrium with the wall, then, in the narrow boundary layer whose thickness is on the order of the length of the mean free path of the molecule, the distribution of the molecules that collide with the wall differs from the distribution of the molecules that are directed toward it. It is natural to introduce in this layer two distribution functions $f_\alpha^+(\mathbf{r}, \boldsymbol{\xi}, t)$ for molecules with $\xi_x > 0$, and $f_\alpha^-(\mathbf{r}, \boldsymbol{\xi}, t)$ for molecules with $\xi_x < 0$. These "parts" of the dis-

tribution function will be related to the general distribution function by the equation

$$f_\alpha(\mathbf{r}, \xi, t) = f_\alpha^+(\mathbf{r}, \xi, t) + f_\alpha^-(\mathbf{r}, \xi, t),$$

$$f_\alpha^+(\mathbf{r}, \xi, t) = 0 \quad \text{for } \xi_x < 0,$$

$$f_\alpha^-(\mathbf{r}, \xi, t) = 0 \quad \text{for } \xi_x > 0. \tag{6}$$

By way of a kinetic boundary condition we assume that some of the molecules incident on the wall are specularly reflected, while the others are absorbed on the wall and are then emitted with a Maxwellian distribution corresponding to a wall temperature T_0, i.e.,

$$f_\alpha^+(\xi_x, \xi_y, \xi_z) = \beta_\alpha f_\alpha^-(-\xi_x, \xi_y, \xi_z)$$

$$+ \varkappa_\alpha \exp(-m_\alpha \xi^2/2kT_0) \quad \text{for } x = 0. \tag{7}$$

The parameters β_α and κ_α are not independent, since they are related by the condition that no molecules accumulate on the wall.

Supplementing (7) with the condition $\kappa_\alpha = 0$ at $\xi_x < 0$, we can expand (7) in a series of Hermite vector polynomials. Equating the coefficients of the corresponding polynomials, we obtain an infinite number of boundary conditions for the moments of distribution function. However, in view of the fact that the distribution function is approximated by a finite number of terms in the expansion in Hermite polynomials, all these conditions cannot be satisfied simultaneously.

From simple physical considerations it is clear that it is necessary to retain only the conditions connected with the mass, energy, and tangential momentum fluxes that are normal to the wall, since the wall "deforms" precisely these fluxes.

The first of these boundary conditions expresses the fact that the solid surface absorbs no molecules,

$$u_{\alpha x}\Big|_{x=0} = 0. \tag{8}$$

We insert the distribution function (4) into the kinetic boundary condition (7). We then multiply (7) successively by c_x, $c_x c_y$, $c^2 c_x$, and integrate in velocity space, thus obtaining the second and third boundary conditions:

$$\tfrac{1}{2} p_{\alpha,xy}(1 + \beta_\alpha) + (1 - \beta_\alpha)(S_{\alpha y}/5 \sqrt{\pi} \, q_\alpha$$

$$+ R_{\alpha y} q_\alpha/4 \sqrt{\pi}) + \tfrac{1}{2} \pi \varkappa_\alpha u_y q_{\alpha 0}^4 = 0; \tag{9}$$

$$(1 + \beta_\alpha) S_{\alpha,x} + (1 - \beta_\alpha)(4 p_\alpha$$

$$+ 3 p_{\alpha,xx}) q_\alpha/2\sqrt{\pi} - \pi \varkappa_\alpha q_{\alpha 0}^6 = 0, \tag{10}$$

where

$$\varkappa_\alpha = (2/\pi^{3/2} q_{\alpha 0}^4 q_\alpha)(1 - \beta_\alpha)(p_\alpha + {}^1/_2 p_{\alpha,xx}),$$

$$q_{\alpha 0}^2 = 2kT_0/m_\alpha, \qquad q_\alpha^2 = 2kT(0)/m_\alpha, \tag{11}$$

$T(0)$ is the temperature of the mixture at the boundary.

It is more convenient, however, to have boundary conditions for the entire mixture and one of the components, instead of boundary conditions that pertain to each component separately. To simplify the notation, we shall confine ourselves to the case when the gas molecules are only diffusely scattered by the wall, i.e., $\beta_\alpha = 0$. We obtain from (8)

$$u_x\Big|_{x=0} = 0. \tag{12}$$

Summing (9) and (10) over α and neglecting small terms of second order, we obtain

$$u_y\Big|_{x=0} = -\frac{\pi^{1/2}(p_{1,xy} + p_{2,xy}) + {}^2/_5(S_{1y}/q_1 + S_{2y}/q_2) + {}^1/_2(R_{1y} q_1 + R_{2y} q_2)}{2(p_1/q_1 + p_2/q_2) + (p_{1,xx}/q_1 + p_{2,xx}/q_2)}; \tag{13}$$

$$\frac{T_0 - T(0)}{T(0)} = \frac{(2\pi)^{1/2}}{4(kT_0)^{3/2}} \frac{S_x + (kT_0/2\pi)^{1/2}(p_{1,xx}/m_1^{1/2} + p_{2,xx}/m_2^{1/2})}{n_1 m_1^{1/2} + n_2 m_2^{1/2}}. \tag{14}$$

These formulas determine the speed and temperature of the gas mixture at a boundary with a solid surface. We use these to obtain boundary conditions for the equations of hydrodynamics. For this purpose we assume that the hydrodynamic approximation, which is correct only for $x \gg l$, is applicable all the way to the boundary, and we express the fluxes contained in (13) and (14) in terms of the corresponding gradients. Using the Chapman and Enskog formulas[6] and neglecting terms that contain the small quantities $p_{\alpha,xx}$, we find

$$\frac{T_0 - T(0)}{T(0)} = \frac{(2\pi)^{1/2} \lambda}{4(kT)^{3/2}(n_1 m_1^{1/2} + n_2 m_2^{1/2})} \frac{dT}{dx}\Big|_{x=0}, \tag{15}$$

$$u_y\Big|_{x=0} = \frac{\eta\sqrt{\pi/2kT}}{n_1 m_1^{1/2} + n_2 m_2^{1/2}} \frac{du_y}{dx}\Big|_{x=0} + \frac{\lambda}{5p} \frac{dT}{dy}\Big|_{x=0}$$

$$- \left[\frac{k_T}{5} + \frac{n_1 n_2 m_1 m_2 (m_1^{1/2} - m_2^{1/2})}{p(n_1 m_1^{1/2} + n_2 m_2^{1/2})}\right](c_{1y} - c_{2y})\Big|_{x=0}$$

$$+ \frac{\sqrt{m_2} + \sqrt{m_1}}{2p(n_1 m_1^{1/2} + n_2 m_2^{1/2})}[X_{1y} n_2 - X_{2y} n_1], \tag{16}$$

where η is the coefficient of viscosity of the mixture, k_T is the thermal diffusion ratio, λ the coefficient of heat conduction of the mixture,

$$X_{\alpha y} = {}^2/_5 S_{\alpha y} - {}^1/_2 R_{\alpha y}\, q_\alpha^2 ,$$

$$c_{1y} - c_{2y} = \frac{n^2}{n_1 n_2} D_{12} \left\{ \frac{\partial \gamma}{\partial y} + \frac{n_1 n_2 (m_2 - m_1)}{n\, p} \frac{\partial \ln p}{\partial y} \right.$$

$$\left. + \frac{p_1 p_2}{p p} (F_{1y} - F_{2y}) + k_T \frac{\partial \ln T}{\partial y} \right\} ,$$

D_{12} is the coefficient of mutual diffusion, F_α is the force acting per unit mass of the α component, $n = n_1 + n_2$, and $\gamma = n_1/n$.

Formula (15) determines the temperature discontinuity at the boundary of the gas mixture and the solid body, while Eq. (16) determines the slipping speed.

The first term in (16) describes Maxwellian viscous slipping for a boundary mixture, while the third describes diffuse slipping, due to the presence of concentration, pressure, and temperature gradients in the mixture and also due to the presence of an external field.

The second and fourth terms are due to thermal slipping. The fluxes $X_{\alpha y}$ contained in the fourth term cannot be expressed in terms of known kinetic coefficients. They can be calculated, for example, by a variational method.[7]

It is interesting to note that, owing to diffuse slipping, a body placed in a mixture of gases of inhomogeneous concentration should be set in motion. This phenomenon is to some extent analogous with the radiometric effect.

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