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MONTE CARLO CALCULATIONS OF ELECTROMAGNETIC CASCADES WITH ACCOUNT OF THE INFLUENCE OF THE MEDIUM ON BREMSSTRAHLUNG

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Calculations of electromagnetic cascades, initiated by primary electrons with energies of 10^9, 10^{10}, 10^{11}, 5×10^{11}, 10^{12} and 3×10^{12} ev, have been performed for depths up to 2.8 radiation units. The Monte Carlo method was used. Actual cross sections (not asymptotic) for the elementary electromagnetic processes in the nuclear emulsions were employed. Two types of calculations were carried out: those using the Bethe-Heitler formula and those based on formulae that take into account the influence of multiple scattering and polarization of the medium on bremsstrahlung.

1. INTRODUCTION

THE method of irradiating stacks of photoemulsion plates in the stratosphere is used for the experimental investigation of electron-photon showers at high energy. Nuclear emulsions represent a sufficiently dense medium: the radiation unit length (shower) in emulsions being $t_0 = 2.9$ cm. At the present time, individual electron-photon showers, with primary energies up to $\sim 10^{12}$ ev, can be registered with adequate efficiency over several radiation lengths in several liters of emulsion.

Special interest is focused on investigations of electromagnetic cascades at high energies at the beginning of their development (in the first radiation length) where the characteristics of the cascade are determined by the basic processes at energies near to the initial energies of the particles initiating the cascade shower.

One of the basic difficulties in interpreting experimental data, obtained in investigations of this kind, is the absence of theoretical calculations of cascades with allowance for the peculiarities of nuclear emulsions. A number of workers[1-3] have carried out calculations of cascades analogous to nuclear emulsion experiments. However, these calculations cannot be considered satisfactory because they were carried out in the so-called A-approximation (see reference 4), which neglects electron collisions, except for the formation of pairs and bremsstrahlung, making use of asymptotic formulae.

Such assumptions are not justified at small depths near the vertex of the shower, particularly in the domain of low energy electron cascades. Besides, in these calculations not all elementary processes are taken into account, which can have a noticeable influence on the characteristics of the cascade.

Recently, there have appeared several papers devoted to improving the calculations of cascades and, in particular, to take into account some of the peculiarities of the emulsion method. Gardner[5] considered the influence of immediate formation of electron pairs on the different cascade processes. Several papers are devoted to the derivation[6,7] and solution[3,8] of the diffusion equations in order to obtain the energy spectrum of electrons formed in all ways (the number at small depths[3]) and taking into account ionization loss.[7] However, all these works only touch upon the separate question of developing cascades and usually do not solve the problem in view of the complexity of the calculations.

There are other difficulties which excuse the abscence of calculations on the fluctuations in cascades at small depths.

It is clear that the majority of the indicated difficulties in the calculation can be circumvented by applying the Monte Carlo method.

Of course, for a complete solution of the problem it is necessary to take into account the spatial description of the cascade. However, at high energies, primary particles develop showers in a very narrow cone. As a result, the track of the electron is effectively registered in the emulsion and it is possible to reconstruct the spatial description of the shower, at any rate, over several centimeters near the vertex of the shower. This confines calculations of the problem to small depths.

In the present work, calculations of longitudinal electromagnetic cascades were carried out down to depths of 2.8 t_0. The calculations were carried out by the Monte Carlo method. The conditions of the calculation were chosen in accordance with the experimental peculiarities of the photoemulsion method. (Preliminary results were reported in references 9 and 10.)

2. INFLUENCE OF THE MEDIUM ON BREMSSTRAHLUNG IN NUCLEAR EMULSIONS

Landau and Pomeranchuk[11] have shown that the Bethe-Heitler theory of bremsstrahlung breaks down in condensed media at sufficiently high energies. As a consequence of multiple scattering in the medium, the bremsstrahlung process is disturbed, leading to a decreased probability for soft photon emission (particularly in the domain of small frequencies).

Ter-Mikaelyan,[12] in turn, showed that there was an additional weakening of the intensity of the soft bremsstrahlung due to the polarization of the medium.

Rigorous formulae, taking both effects into account, and valid for electron energies $E \gg mc^2$, were obtained by Migdal[13] (for a survey, see reference 14).

The graphs of the intensity of electron bremsstrahlung in nuclear emulsions according to the Bethe-Heitler formula (total screening) and according to Migdal's formula, taking into account both effects of the medium for electrons with energies from 10^{11} to 10^{13} ev, are presented in Fig. 1. For comparison, the dotted curve gives the intensity only taking the polarization effect into account. It is clear that at $E = 10^{12}$ ev the influence of the medium on radiation quanta with energy $\hbar\omega \leq 10^9$ ev is considerable.

To determine the influence of these effects on the energy spectra of particles in electromagnetic cascades and to clarify the possibility of experimentally revealing these effects, it was decided to carry out two types of calculations: using the Bethe-Heitler formula and applying Migdal's formula, taking into account both influences of the medium on the bremsstrahlung. Since the influence of the medium manifests itself more strongly the smaller the energies of the radiated quanta, it was decided to calculate spectra of cascade electrons and pairs up to energies $\sim 10^6$ ev, valid at small depths t in the emulsion. This required the actual (not asymptotic) cross sections for the elementary processes.

3. ELEMENTARY CROSS SECTIONS OF THE ELECTROMAGNETIC PROCESSES, USED IN THE CALCULATION

The following electromagnetic processes in the fields of the nuclear and electronic components of the emulsion were taken into account in the calculation: bremsstrahlung, pair production by photons, pair production by electrons, the Compton effect, photonuclear absorption and ionization deceleration of the electron. The spatial distribution of the particles was neglected in the calculation. We therefore used for all the elementary processes differential cross sections integrated over the angle, and dependent only on the energy of the particles. That is, cross sections of the type $d\sigma = f(\alpha, \beta)\,d\alpha$, where β is the energy of the primary particle and α is the energy of one of the formed particles.

The Bethe-Heitler formula in the Born approximation, taking screening into account by using the Thomas-Fermi model, was used for the differential cross section for bremsstrahlung and pair formation in the field of the nucleus. The screening function was approximated by an analytic expression to within 1 to 2 per cent. The Thomas-Fermi approximation is considered sufficient for the problem under consideration, because the electromagnetic processes in emulsions take place principally in the heavy nuclei Ag and Br.

We took into account the corrections necessi-

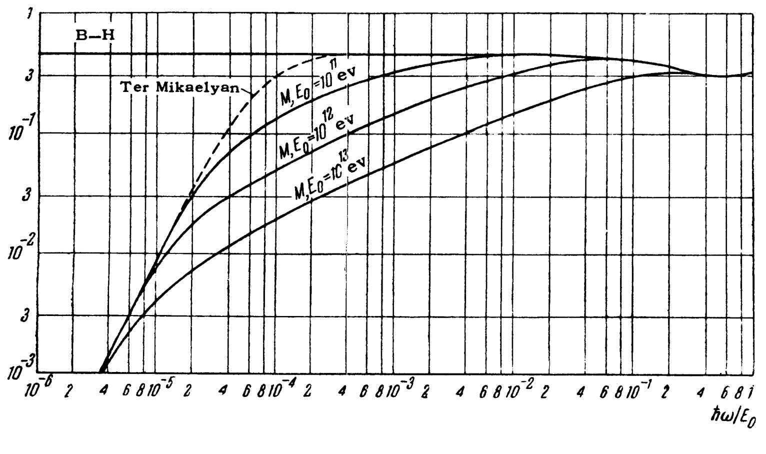

FIG. 1. Energy dependence of electron bremsstrahlung in emulsions, in accordance with the Bethe-Heitler formula (B-H) and Migdal's formula (M) for electrons with energies 10^{11} to 10^{13} ev. Ordinate: intensity of the radiation $(h\omega/E_0)/\lambda(\omega)$, where $\lambda(\omega)$ is the free passage of electrons in emulsions for radiation of quanta with frequency between ω and $\omega + d\omega$. E_0 is the energy of the electron.

tated by failure of the Born approximation. In the expression for the differential cross section for pair formation, the corrections introduced by Davies et al.[15] are independent of the screening:

$$\Delta\sigma = -\frac{28}{9}\frac{Z^2 r_0^2}{137} f(Z), \qquad f(Z) = a^2 \sum_{\nu=1}^{\infty} \frac{1}{\nu(\nu^2 + a^2)},$$

where $a = Ze^2/\hbar c$ and Z is the nuclear charge. For the heavy elements of emulsions $f(Z) = 0.59 \times 10^{-4} Z^2$. In accordance with Olsen and Brown,[16] analogous corrections were introduced into the expressions for bremsstrahlung.

The total cross section σ for pair production by photons with energy $\hbar\omega \leq 10$ Mev was calculated using the known numerical data,[17] by means of analytic integration of the cross section, $d\sigma$, without taking into account the screening in the Born approximation. Then corrections to the Born approximation, σ_b, were introduced in accordance with the experimental results of Dayton.[18]

$$\Delta\sigma = +1.59\cdot 10^{-4} Z^2 \sigma_B \quad \text{for } \hbar\omega = 1.33 \text{ Mev},$$

$$\Delta\sigma = +3.39\cdot 10^{-5} Z^2 \sigma_B \quad \text{for } \hbar\omega = 2.62 \text{ Mev}.$$

Also, in accordance with reference 19, $\Delta\sigma = 0$ at $\hbar\omega = 5$ Mev and $\Delta\sigma = -0.04\,\sigma_B$ at $\hbar\omega = 10$ Mev.

The total cross section for pair production by photons at $\hbar\omega > 10$ Mev in the nuclear field was obtained from numerical integration of the Bethe-Heitler formula, taking screening and the corrections due to Davies[15] into account. The integration was performed on an electronic computer for separate groups of elements with charges Z = 47.35, 7, and 1.

The cross section for bremsstrahlung was calculated in an analogous way: numerical integration of the expression $d\sigma$, with the corrections.[15]

In determining the total cross sections for elements Z_i, pair production and bremsstrahlung in the electron field were also taken into account. The formula of Wheeler and Lamb[20] was used, taking screening into account correctly, generally speaking, for nuclei with charge Z = 1. For pair production, we follow Bethe and Ashkin[21] obtained by means of numerical integration of the cross section subtracted from the different total cross sections in a nuclear field Z = 1 and in the electron field, determined from the results of Borsellino.[21]

The so-called absorption coefficient, τ, is defined as $\tau = \sum n_i\sigma_i$, where σ_i is the total cross section for the interactions under consideration in atoms Z_i, while n_i is the number of atoms Z_i per cm^3 of medium. The value of n_i is determined in accordance with the composition of the emulsion Ilford G-5 at 58 per cent humidity.[22] The composition of the R-NIKFI emulsion is nearly the same.[23]

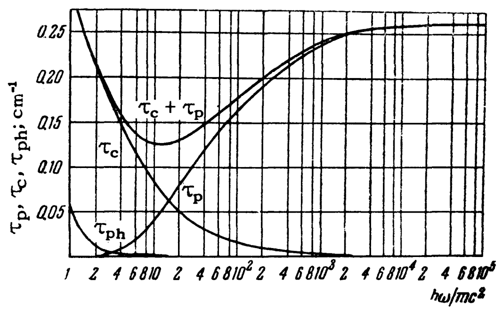

FIG. 2. Absorption coefficients for photons in emulsions for pair production (τ_p), Compton effect (τ_c) and photo-electric effect (τ_{ph}).

The calculated values of the absorption coefficients of γ quanta in emulsions after counting pair production (in the fields of the nuclear and electronic components of the emulsion) and Compton effect are presented in Fig. 2. For comparison, the value of τ_{ph} for the usual photo-effect is presented.

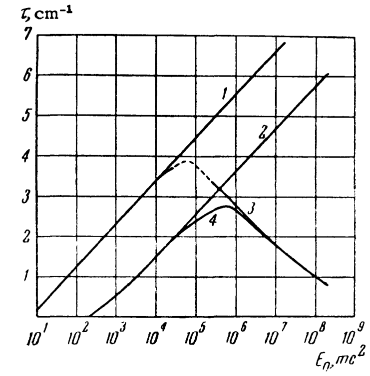

FIG. 3. The coefficient $\tau = 1/\lambda$, for electrons with energy E_0 for the radiation of quanta with energy $\hbar\omega \to \varepsilon$ calculated for emulsions according to the Bethe-Heitler (B – H) and Migdal's (M) formulae. 1) B – H, $\varepsilon = 1.5 \times 10^6$ ev, 2) B – H, $\varepsilon = 10^8$ ev, 3) M, $\varepsilon = 1.5 \times 10^6$ ev, 4) M, $\varepsilon = 10^8$ ev.

The τ coefficients for radiation quanta with energies exceeding the minimum value ϵ (1.5×10^6 ev and 10^8 ev) are given in Fig. 3. Curves 3 and 4 were obtained by numerically integrating Migdal's[13] formula for the differential cross section. The composition of the medium, in this case, is allowed for in the following way. Since the probability of radiation depends on the medium through the quantity $Z^2 \ln(190\, Z^{-1/3})$. We determine for emulsions an effective value $Z_{eff} = 20.54$, and then the parameter $s_1 = (Z_{eff}^{1/3}/190)^2$, which enters into Migdal's formula. The effect of polarization was accounted for by introducing into the formula the parameter $q = s\gamma$ instead of s and dividing the expression for the probability of radiation by γ, where $\gamma = 1 + 5.1 \times 10^{-9} (\hbar\omega)^{-2} E^2$, a parameter that takes into account the deviation of the dielectric constant of the emulsion from unity.

The value of the coefficient τ for electron pair production by electrons was determined in the following way: in the electron energy range $E < 10^7$ ev, the formula of Bhabha,[24] neglecting shielding, was used; in the range $10^7 \text{ ev} \leq E \leq 10^9$ ev we used the modified Bhabha formula,[25] and in the range $E > 10^9$ ev we used the Racah[26] formula, taking shielding into account (see reference 25).

Ionization loss of the electrons was considered constant and equal to 7 Mev/cm (the loss on the ionization plateau[27]). The loss of electron energy in radiating soft quanta with energies $\hbar\omega < \epsilon$ was not considered in the calculation of the cascade. As shown in the actual calculations, these losses can be considered constant to an accuracy of 1 or 2 per cent for electrons with energies $> 10\,\epsilon$. The usual energy loss for electrons was considered to be equal to 7.6 Mev/cm in the case of $\epsilon = 1.5 \times 10^6$ ev and 50 Mev/cm in the case $\epsilon = 10^8$ ev.

4. CONDITIONS OF THE CASCADE CALCULATION

The primary particles initiating the cascade were taken to be electrons with initial energies E_0 of 10^9, 10^{10}, 10^{11}, 5×10^{11}, 10^{12}, and 3×10^{12} ev.

Several problems were calculated simultaneously. Along the t axis of the shower, all secondary particles (electrons and photons) having a total energy greater than some value ϵ (1.5×10^6 or 10^8 ev) were investigated. Results were obtained for four values of the cascade depth t (calculated from the vertex of the shower): $t_1 = 1.0\,t_0$, $t_2 = 1.5\,t_0$, $t_3 = 2.1\,t_0$, and $t_4 = 2.8\,t_0$.

We note that the quantity $t_0 = 2.9$ cm is only used as a unit of length. The probability of different processes is calculated according to the elementary cross sections (see the preceding section).

Two of the elementary processes taken into account — bremsstrahlung and γ-quantum pair production — were performed for definite Z_i. The value Z_i corresponding to the four groups of elements in the emulsion: hydrogen ($Z = 1$), the group of light elements ($Z = 7$), bromine ($Z = 35$), and silver ($Z = 47$). The remaining processes were performed with cross sections, averaged according to the composition of the emulsion.

The values of the coefficients τ_j were specified at the following values of the total energies of the particles, expressed in units of the electron rest mass mc^2: 3, 7, 10, 20, 60, 200, 600, 2×10^3, 6×10^3, 2×10^4, 2×10^5, 2×10^6, and 2×10^8. Linear interpolation was used for intermediate values of the energy.

The sequence of selecting the processes for photons was performed in the following way:

1. Select the distance x, which the photon covers prior to an interaction, in accordance with the total absorption coefficient for all the elements of the emulsion.

2. Select the form of the process. Each of the six processes taken into account (pair production at a fixed Z_i, absorption, and Compton effect) was given its statistical weight.

3. In the case of pair production, the energy of the pair and the depth t at the place of production were noted in the results. Then the relative energy of the components of the pair were selected according to the formula for the differential cross section at a fixed Z_i.

4. For the Compton effect, the Klein-Nishina formula was used to determine the scattering angle of the photon, the angle of the ejected electron and the energies of the photon and electron. We chose for further investigation photons whose scattering angle did not exceed 0.01 radians and electrons with a scattering angle < 30°. The electron yield, independent of the scattering angle, was noted in the appropriate energy intervals.

The scheme for choosing the electron processes was the following:

1. Select the distance x, which the electron covers prior to one of the processes, with the help of the total coefficient τ, taking into account bremsstrahlung with energy $> \epsilon$ and electron pair production.

2. Determine the energy of the electron at the point of interaction, taking into account energy losses by ionization and soft bremsstrahlung with energy $< \epsilon$, along the path x.

3. As the electrons passed through depths t_i ($i = 1, 2, 3, 4$), determine the energies at these depths and record the results in appropriate energy intervals.

4. Select the nature of the process (five possibilities) in accordance with their statistical weights τ_i/τ.

5. Select the radiation energy according to the

TABLE I. Differential energy spectrum of electrons and electron pairs in cascade showers initiated by electrons with energy E_0

Energy interval, ev	Number of pairs down to depths				Number of electrons to depths			
	t_1	t_2	t_3	t_4	t_1	t_2	t_3	t_4
			$E_0 = 10^9$ ev					
$1.5\cdot10^6$—$3.5\cdot10^6$	0.06	0.09	0.16	0.24	0.18	0.34	0.68	0.65
$3.5\cdot10^6$—10^7	0.11	0.30	0.70	1.21	0.30	0.61	0.84	1.16
10^7—$3\cdot10^7$	0.21	0.52	1.05	1.85	0.41	0.88	1.37	1.42
$3\cdot10^7$—10^8	0.27	0.62	1.15	1.74	0.60	1.05	1.06	1.23
10^8—$3\cdot10^8$	0.27	0.52	0.76	1.12	0.45	0.56	0.64	0.62
$3\cdot10^8$—10^9	0.21	0.28	0.46	0.62	0.68	0.45	0.27	0.14
			$E_0 = 10^{10}$ ev					
$1.5\cdot10^6$—$3.5\cdot10^6$	0	0.08	0.16	0.40	0.18	0.46	1.13	2.05
$3.5\cdot10^6$—10^7	0.13	0.41	0.94	2.31	0.28	0.96	1.90	3.59
10^7—$3\cdot10^7$	0.22	0.63	1.72	3.76	0.71	1.24	2.62	4.69
$3\cdot10^7$—10^8	0.37	0.96	2.35	4.71	0.60	1.33	3.14	5.13
10^8—$3\cdot10^8$	0.30	0.70	1.59	3.24	0.54	1.14	2.33	3.47
$3\cdot10^8$—10^9	0.33	0.78	1.63	2.82	0.49	1.14	1.46	1.99
10^9—$3\cdot10^9$	0.19	0.36	0.73	1.20	0.56	0.61	0.68	0.65
$3\cdot10^9$—10^{10}	0.23	0.41	0.55	0.63	0.63	0.49	0.41	0.20

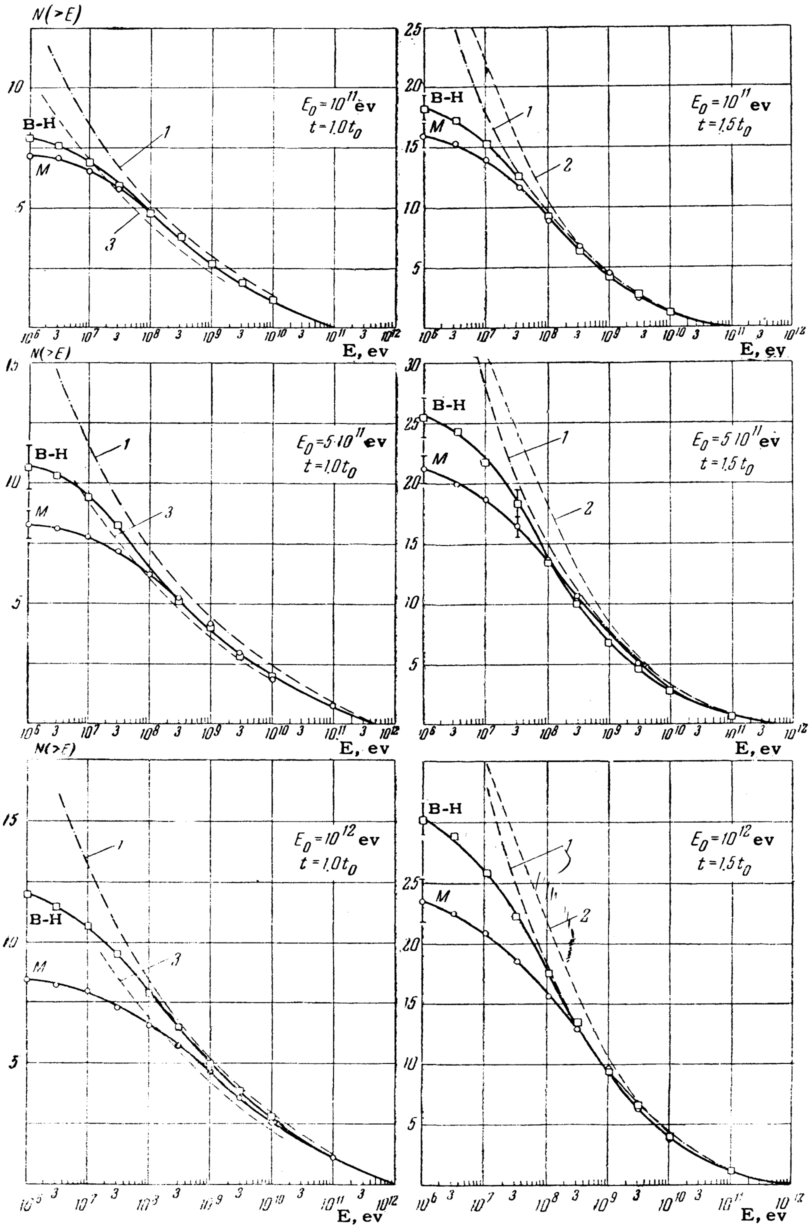

FIG. 4. Integrated energy spectrum of N(> E) electrons to depths t in the cascade shower, initiated by electrons with energy E_0. B–H) calculated according to Bethe-Heitler for emulsions, M) calculated according to Migdal's formula, 1) results of Arley,[1] 2) results of Janossy and Messel,[2] 3) results of Srinivasan et al.[3]

differential cross section, taking screening into account.

6. Select the pair energy and select the energies of the components of the pair according to Rossi's formula[28] (using the average Z of the emulsion).

In all cases, the selection must be performed with respect to a complicated formula, which does not allow a simple converse transformation. Therefore, we used the following method of selection with a double series of random numbers:[29]

The first random number, ξ_1, fixes the value u_1 of the argument of the differential probability $f(u)$. The value $f(u_1)$ is then compared with a second random number ξ_2. If $f(u_1) < \xi_2$, the operation is repeated. We generate the following pair of random numbers ξ_3 and ξ_4 which determines u_3 and carry out the comparison of $f(u_3)$ with ξ_4 and so on, until the condition $f(u_k) > \xi_{k+1}$ is satisfied. The values u_k are used as the generated argument.

As the results of the data are distributed about a number of particles in twelve energy intervals, to depths t_1, t_2, t_3, and t_4. The limits of the intervals coincide with the values of the energies, in which is assigned the numerical value τ_i (see above). The calculation of the cascade was carried out on the "Strela" electronic computer.

5. RESULTS

The purpose of the calculation was to obtain the following characteristics of electromagnetic cascades in their first stages of development in nuclear emulsions:

1. Energy spectrum of cascade electrons reaching depths t_1, t_2, t_3, and t_4.
2. Energy spectrum of electron-positron pairs produced by photons down to depths t_1, t_2, t_3, and t_4.
3. Energy spectrum of the Compton electron yield to depths down to t_1 and t_2.
4. The number of pairs formed by electrons (tridents) with energy corresponding more or less to 3×10^7 ev in depths $\le t_2$.

Besides this, the fact that the cascade calculations covered all particles and not merely a separate single branch of the cascade tree has allowed us to obtain information on fluctuations in the cascade.

A total of more than 1000 cascade trees were followed. At $E_0 = 10^{12}$ ev, 179 cascade trees were calculated using the Bethe-Heitler formula, in the remaining cases, an average of 80 to 100 trees were followed.

The results of the calculations for primary energies $E_0 = 10^9$ and 10^{10} ev are presented in Table I. The influence of the medium is absent at so small E_0. The data can be useful for work with electrons obtained in accelerators.

Figure 4 shows the integrated energy spectrum for cascade electrons at depths t_1 and t_2 at primary energies $E_0 \ge 10^{11}$ ev. It is seen that the total number of electrons in the Bethe-Heitler formula (B-H) is 10 to 40 per cent higher than that calculated by Migdal's formula (M), if the influ-

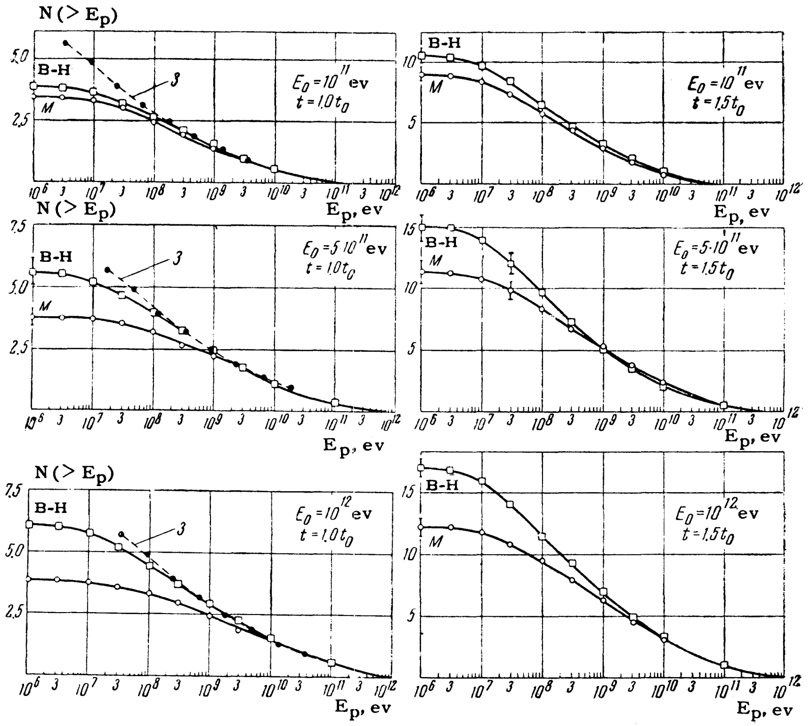

FIG. 5. Integrated energy spectrum $N(>E_p)$ of electron-positron pairs to depths t in cascade showers, formed by electrons with energy E_0 [Notation is the same as in Fig. (4)].

ence of the medium is taken into account. From a comparison of the graphs it is possible to see how much this difference depends on the primary energy E_0 and the considered depth t. For comparison, the cascade curves of Arley,[1] Janossy and Messel,[2] and Srinivasan et al.[3] are also presented. It is seen that the essential difference between our results and the others occurs for low energy electrons.

The errors indicated in Fig. 4 (as in all subsequent figures) are statistical. They are determined from the results of calculations on the distribution of tree cascades with respect to the number of electrons in the cascade, N(>E).

The integrated energy spectrum of electron-positron pairs N(>E) is presented in Fig. 5, formed in depths down to t_1 and t_2 at different E_0. The difference between the Bethe-Heitler and Migdal variants in this case is from 10 to 60 per cent, i.e., visibly greater than in the electron spectrum at high E_0. If we take into consideration that the effectiveness of counting a pair is greater than for electrons only, it becomes clear to what extent it is advantageous to measure electron-positron pairs in order to investigate the influence of the medium.

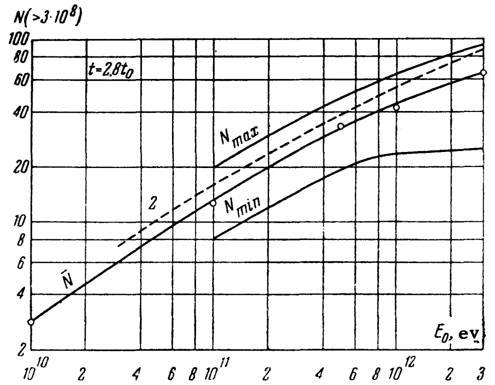

FIG. 6. Number of electrons N, depth t_4 with energy $E > 3 \times 10^8$ ev as a function of the energy, E_0, of the electron initiating the shower. N, N_{max} and N_{min} are the average, the maximum and the minimum number of electrons using Migdal's formula. 2) The results of Janossy and Messel.[2]

Curve 3 in Fig. 5 is drawn from the results of Srinivasan et al.[3] It is seen that it is little different from our results for high energy pairs using the Bethe-Heitler formulation, particularly at $E_0 = 10^{11}$ ev, but very different for low-energy pairs.

In Fig. 6 we present the curves for the number of electrons at depth t_4 with energy $> 3 \times 10^8$ ev as a function of the primary energy E_0. Curves N_{max} and N_{min} tend to the limiting value N in different cascades. The probability that N lies between the limits N_{max} to N_{min} is 0.7. See reference 10 for the determination of N_{max} and N_{min}.

Table II gives the results of the calculation for the number of pairs formed by electrons (tridents) at depths $t \leq 1.55\,t_0$.

In Fig. 7 we present a histogram of the distribution of cascade electrons, depth t_2 with energy $E > 1.5 \times 10^6$ ev ($E_0 = 10^{12}$ ev, B-H variant).

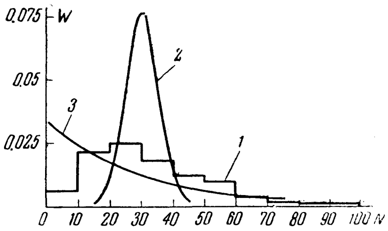

FIG. 7. Probability, W, of finding N electrons with energy $E > 3 \times 10^8$ ev at depth t_2 in a cascade shower, formed by electrons with energy $E_0 = 10^{12}$ ev. 1) Results of the calculation using the Bethe-Heitler formula. 2 and 3) The Poisson and Furry[30] distributions, respectively, at average value $\overline{N}$, determined in the calculation.

For comparison, we also show the curves for Poisson and Furry[30] distributions for the corresponding average $\overline{N}$. It is seen that the Furry curve agrees better with the distribution than the Poisson curve.

Taking the fluctuations in the cascade into account, it is possible to say that in order to detect, experimentally, the influence of the medium on bremsstrahlung from the energy spectrum of

TABLE II. Number of electron-positron pairs with total energy $\geq 1.5 \times 10^6$ ev, produced immediately by electrons in cascade showers, initiated by electrons with energy E_0, to depths $t \leq 1.55\ t_0$. The number of pairs with energies 1.5×10^6 to 3×10^7 ev is indicated in parentheses

E_0, ev	B–H	M.	E_0, ev	B–H	M.
10^9	0.04 (0.02)		$5 \cdot 10^{11}$	0.52 (0.20)	0.34 (0.05)
10^{10}	0.14 (0.08)		10^{12}	0.56 (0.18)	0.51 (0.15)
10^{11}	0.28 (0.07)	0.27 (0.11)			

pairs in electron-photon showers, it is necessary to have not less than 8 to 10 showers with energy $\sim 10^{12}$ ev.

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