VAN DER WAALS FORCES IN AN INHOMOGENEOUS DIELECTRIC

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The non-additive part of the free energy of an inhomogeneous dielectric, related to the presence of long wavelength fluctuations of the electromagnetic field in it, is calculated with the aid of the diagram technique. The corresponding part of the stress tensor (the van der Waals force stress tensor) is also calculated. In addition, formulas for the Green's functions of an electromagnetic field in an absorbing medium are adduced.

INTRODUCTION

THIS work is devoted to the calculation of additional forces arising in an inhomogeneous dielectric* due to the presence of a long-wave fluctuating the electromagnetic field in it. These forces may be called van der Waals forces since they have the same character as the van der Waals forces of attraction between molecules at large distances.

The contribution of the long wave fluctuations to the free energy is small compared with the total free energy of the body. However, they lead to a qualitatively new effect: the non-additivity of the free energy of the body. Indeed, each change of the density and, consequently, in the dielectric constant of the body in a certain region leads to a change of the fluctuating electromagnetic field within this region by virtue of the Maxwell equations. Thus the part of the free energy under consideration is not determined by the properties of the substance at the given point alone. Therefore, for example, the chemical potential of a thin film of liquid on the surface of a solid depends on the thickness of the film. (A rough calculation based on the assumption that the interaction between the liquid molecules and the underlying substance obeys the van der Waals law leads to the dependence[1] $\mu = \text{const} - a/d^3$). These same effects lead to the appearance of forces of mutual attraction between solids, i.e., to a dependence of the free energy of bodies on the distance between them. Such forces were calculated rigorously by E. M. Lifshitz[2] (see also reference 3). Here, evidently, fluctuations with wavelengths of the order of the dimensions of the inhomogeneities are significant (for example, of the order of the film thickness or of the distance between the attracting solids). This makes possible a macroscopic analysis for macroscopic bodies and to express the results in terms of ϵ, the dielectric constant of the body.

Of course, the corrections to the free energy, necessitated by the fluctuations of the electromagnetic field, cannot be calculated by means of simple average expressions for the energy of the electromagnetic field in the medium — for variable fields in an absorbing medium such a concept is generally meaningless in view of the presence of dissipation. It is therefore necessary to determine directly the correction to the free energy of the body, necessitated by the interaction of the particles of the body with the long-wavelength electromagnetic radiation. We shall calculate here the corrections to the free energy by the Matsubara technique[4] as modified by Abrikosov, Gor'kov and Dzyaloshinskiĭ, and Fradkin.[5] We remark at once that this method makes it possible to recast the whole of the quantum theory of fluctuations in a considerably more convenient form. In the following sections we shall describe the properties of certain functions encountered in the application of the indicated methods to the electromagnetic field in an inhomogeneous absorbing medium. In this connection we shall make wide use of the results of reference 5.

PROPERTIES OF THE GREEN'S FUNCTIONS OF AN ELECTROMAGNETIC FIELD IN AN ABSORBING MEDIUM

A basic role will be played in our work by the temperature or "Matsubara" Green's functions for

*To avoid misunderstandings we emphasize that in the present paper inhomogeneous bodies are understood, in particular, to include bounded bodies even though these are homogeneous over their entire volume.

the electromagnetic field, defined by the formula*

$$\mathfrak{D}_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2, \tau_1-\tau_2) = \begin{cases} -\mathrm{Sp}\,\{e^{(F-\hat{H})/T} e^{\hat{H}(\tau_1-\tau_2)} \hat{A}_\alpha(\mathbf{r}_1) e^{-\hat{H}(\tau_1-\tau_2)} \hat{A}_\beta(\mathbf{r}_2)\}, & \tau_1 > \tau_2, \\ -\mathrm{Sp}\,\{e^{(F-\hat{H})/T} e^{-\hat{H}(\tau_1-\tau_2)} \hat{A}_\beta(\mathbf{r}_2) e^{\hat{H}(\tau_1-\tau_2)} \hat{A}_\alpha(\mathbf{r}_1)\}, & \tau_1 < \tau_2. \end{cases} \quad (1)$$

Here $\hat{A}_\alpha(\mathbf{r})$ is the 4-vector potential operator in the Schrödinger representation, $\hat{H}$ is the system Hamiltonian, and $\hat{F}$ is the free energy of the body.

By virtue of the reality of the electromagnetic field, the operator $\hat{A}_\alpha(\mathbf{r})$ is Hermitian. This permits the components of the expansion of $\mathfrak{D}_{\alpha\beta}$ in a Fourier series in $\tau_1-\tau_2$ to be determined from the equation

$$\mathfrak{D}_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2, \omega_n) = \frac{1}{2}\int_{-1/T}^{1/T} e^{i\omega_n\tau}\,\mathfrak{D}_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2, \tau)\,d\tau, \quad (2)$$

which satisfies the relation

$$\mathfrak{D}_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2, \omega_n) = \mathfrak{D}^*_{\beta\alpha}(\mathbf{r}_2, \mathbf{r}_1, -\omega_n). \quad (3)$$

(The asterisk denotes the complex conjugate quantity.) The temperature Green's function $\mathfrak{D}_{\alpha\beta}$ is connected by a simple relation with the so-called "retarded" Green's function $D^R_{\alpha\beta}$, defined by the formula

$$D^R_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2, t_1-t_2) = \begin{cases} -i\,\mathrm{Sp}\,\{e^{(F-\hat{H})/T}[\hat{A}_\alpha(\mathbf{r}_1, t_1)\hat{A}_\beta(\mathbf{r}_2, t_2) - \hat{A}_\beta(\mathbf{r}_2, t_2)\hat{A}_\alpha(\mathbf{r}_1, t_1)]\}, & t_1 > t_2, \\ 0, & t_1 < t_2. \end{cases} \quad (4)$$

Arguments analogous to those brought forth in reference 5 for the homogeneous medium case lead to the conclusion that $\mathfrak{D}_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2, \omega)$ can be expressed in terms of the Fourier components of $D^R_{\alpha\beta}$. Indeed, if we define $D^R_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2, \omega)$ by

$$D^R_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2, \omega) = \int_{-\infty}^{\infty} e^{i\omega t}\, D^R_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2, t)\,dt,$$

then $\mathfrak{D}_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2, \omega_n)$ and $D^R_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2, \omega)$ are connected for $\omega_n > 0$ by the relation

$$\mathfrak{D}_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2, \omega_n) = D^R_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2, i\omega_n). \quad (5)$$

We remark that it follows from the very definition (3) that $D^R_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2, \omega)$ is an analytic function of ω, having no singularities in the upper half-plane.

The tensor $D^R_{\alpha\beta}$ (or $\mathfrak{D}_{\alpha\beta}$) has ten independent components. However, we have at our disposal a considerable arbitrariness connected with the gauge invariance. Indeed, it is not the quantities $D^R_{\alpha\beta}$, made up of the components of the vector potential $\hat{A}_\alpha$, that have a physical significance, but only the six corresponding quantities made up of the components of the electric field intensity $\hat{E}_i$. Thus, only six physical conditions are imposed on the ten quantities. The remaining arbitrariness may be made use of in order to set the components D^R_{00} and D^R_{0i} equal to zero. Obviously, this corresponds to a gauge with a scalar potential equal to zero. We shall henceforth assume such a gauge (except for the Appendix), since it is the most convenient for the case of a spatially inhomogeneous medium. With this gauge, the Heisenberg operators $\hat{\mathbf{E}}$ and $\hat{\mathbf{H}}$ are related to $\hat{\mathbf{A}}$ by the formulas

$$\hat{\mathbf{E}} = -\partial\hat{\mathbf{A}}/\partial t, \qquad \hat{\mathbf{H}} = \mathrm{curl}\,\hat{\mathbf{A}}, \quad (6)$$

and the operator $\hat{\mathbf{A}}(\mathbf{r}, t)$ itself, in the case of long wavelengths when a macroscopic description of the medium can be used, satisfies the equation*

$$\frac{\partial}{\partial t}\left(\varepsilon\frac{\partial\hat{\mathbf{A}}}{\partial t}\right) + \mathrm{curl\,curl}\,\hat{\mathbf{A}} = 0. \quad (7)$$

(As always, the dielectric constant ϵ is a time-dependent linear operator in the presence of dispersion. We neglect the magnetic properties of matter, since they are completely inessential in the frequency region of interest.)

Differentiating (4) with respect to time, taking into account Eq. (7) and the commutation rule for the vector potential, which has the form

$$\left[\frac{\partial\hat{A}_i(\mathbf{r}, t)}{\partial t}\hat{A}_k(\mathbf{r}', t) - \hat{A}_k(\mathbf{r}', t)\frac{\partial\hat{A}_i(\mathbf{r}, t)}{\partial t}\right] = -4\pi i\delta(\mathbf{r}-\mathbf{r}')\,\delta_{ik}, \quad (8)$$

and going over to the Fourier components in terms of t, we obtain an equation for D^R_{ik}:

$$\{\varepsilon(\mathbf{r}, \omega)\,\omega^2\delta_{il} + \delta_{il}\partial^2/\partial x_m^2 - \partial^2/\partial x_i\partial x_l\}\, D^R_{lk}(\mathbf{r}_1, \mathbf{r}_2, \omega) = 4\pi\delta(\mathbf{r}-\mathbf{r}')\,\delta_{ik}. \quad (9)$$

Substituting $i\omega_n$ for ω in (9), we find that the function $\mathfrak{D}_{ik}$ satisfies the following equation for $\omega_n > 0$:

*Here and below Greek indices $\alpha, \beta, \ldots = 0, 1, 2, 3$ denote the components of 4-vectors and tensors, and Latin ones, $i, k, \ldots = 1, 2, 3$, denote the components of 3-vectors and tensors.

*We use a system of units in which $\hbar = c = 1$. For the sake of simplicity, we assume everywhere that the dielectric is a fluid and consequently isotropic. All formulas are easily generalized to the case of a solid anisotropic dielectric.

FIG. 1

$$\{\varepsilon(\mathbf{r}, i\omega_n)\,\omega_n^2\delta_{ik} - \delta_{il}\partial^2/\partial x_m^2 + \partial^2/\partial x_i\partial x_l\}\,\mathfrak{D}_{lk}(\mathbf{r}_1, \mathbf{r}_2, \omega_n)$$

$$= -4\pi\delta(\mathbf{r}_1 - \mathbf{r}_2)\,\delta_{ik}. \qquad (10)$$

We note that since $\epsilon(i\omega_n)$ is, as is well known, a real quantity, the operator acting on $\mathfrak{D}_{ik}$ on the left-hand side of (10) is Hermitian even in an absorbing medium. Equation (10) defines $\mathfrak{D}_{ik}(\mathbf{r}, \mathbf{r}', \omega_n)$ for $\omega_n > 0$. $\mathfrak{D}_{ik}$ for $\omega_n < 0$ can then be found with the aid of relation (3). It is necessary to bear in mind here that the function $\mathfrak{D}_{ik}$ satisfies a real equation and is therefore real.

Thus, solving Eq. (10) for the given body, we find $\mathfrak{D}_{ik}$.

On the other hand, $\mathfrak{D}_{ik}$ can also be calculated not from (10), but according to the general rules of the Matsubara technique. We separate from the complete system Hamiltonian the part corresponding to the interaction of the particles of the body with the long-wave electromagnetic radiation:

$$\hat{H} = \hat{H}_0 + \hat{H}_{int} = \hat{H}_0 + \int \hat{A}_\alpha(\mathbf{r})\,\hat{j}_\alpha(\mathbf{r})\,d^3\mathbf{r}. \qquad (11)$$

Here $\hat{H}_0$ is the sum of the Hamiltonian of the free electromagnetic field in vacuum and the Hamiltonian of the particles of the solid, i.e., we consider that all interactions between particles are included in $\hat{H}_0$, except for the interactions with the long-wave radiation. $\hat{j}_\alpha(\mathbf{r})$ is the system total current operator. We can now calculate $\mathfrak{D}_{\alpha\beta}$ by the usual diagram technique.[5] Here, however, it is necessary to bear in mind that since we include in $\hat{H}_{int}$ only the interaction with the long-wave oscillations, we must consider that all integrals over the momenta of the long-wave photons are cut off at a momentum k_0 that is much smaller than the reciprocal of the interatomic distance, $1/a$. (We shall not introduce this cutoff explicitly, since the final answer will be formulated in a manner independent of the nature of the cutoff.) Obviously, each long-wave photon line, along which an integration is carried out, will therefore bring in an additional smallness $\sim k_0 a$ and we must restrict ourselves in $D_{\alpha\beta}$ to only those graphs in which no integration is carried out along the long-wave photon lines, i.e., to summing the following series (see Fig. 1)

$$\begin{aligned}\mathfrak{D}_{ik}(\mathbf{r}, \mathbf{r}', \omega_n) &= \mathfrak{D}^0_{ik}(\mathbf{r} - \mathbf{r}', \omega_n) \\ &+ \int \mathfrak{D}^0_{il_1}(\mathbf{r} - \mathbf{r}_1, \omega_n)\,\Pi_{l_1l_2}(\mathbf{r}_1, \mathbf{r}_2, \omega_n) \\ &\times \mathfrak{D}^0_{l_2k}(\mathbf{r}_2 - \mathbf{r}', \omega_n)\,d^3\mathbf{r}_1 d^3\mathbf{r}_2 \\ &+ \int \mathfrak{D}^0_{il_1}(\mathbf{r} - \mathbf{r}_1, \omega_n)\,\Pi_{l_1l_2}(\mathbf{r}_1, \mathbf{r}_1, \omega_n) \\ &\times \mathfrak{D}^0_{l_2l_3}(\mathbf{r}_2 - \mathbf{r}_3, \omega_n)\,\Pi_{l_3l_4}(\mathbf{r}_3, \mathbf{r}_4, \omega_n) \\ &\times \mathfrak{D}^0_{l_4k}(\mathbf{r}_4 - \mathbf{r}', \omega_n)\,a^3\mathbf{r}_1\,d^3\mathbf{r}_2\,d^3\mathbf{r}_3\,d^3\mathbf{r}_4 + \ldots. \end{aligned} \qquad (12)$$

Here $\Pi_{ik}(\mathbf{r}, \mathbf{r}', \omega_n)$ is the Fourier component of the system polarization operator, denoted in the graph by a crosshatched loop. It is understood that all graphs not containing long-wave photon lines must be included; the inclusion of graphs with these lines would exceed the accuracy.

Summing the series (12), we obtain an equation for $\mathfrak{D}_{ik}$:*

$$\mathfrak{D}_{ik}(\mathbf{r}, \mathbf{r}', \omega_n) = \mathfrak{D}^0_{ik}(\mathbf{r} - \mathbf{r}', \omega_n) + \int \mathfrak{D}_{il_1}(\mathbf{r}, \mathbf{r}_1, \omega_n)$$

$$\times\,\Pi_{l_1l_2}(\mathbf{r}_1, \mathbf{r}_2, \omega_n)\,\mathfrak{D}^0_{l_2k}(\mathbf{r}_2 - \mathbf{r}', \omega)\,d^3\mathbf{r}_1\,d^3\mathbf{r}_2. \qquad (13)$$

We apply to Eq. (13) by the operators

$$\{\varepsilon(\mathbf{r}, i\omega_n)\,\omega_n^2\delta_{il} - \delta_{il}\partial^2/\partial x_m^2 + \partial^2/\partial x_i\partial x_l\},$$
$$\{\omega_n^2\delta_{qk} - \delta_{qk}\partial^2/\partial x'^2_m + \partial^2/\partial x'_k\partial x'_q\}. \qquad (14)$$

Taking into account that $\mathfrak{D}_{ik}$ satisfies Eq. (10), and $\mathfrak{D}^0_{ik}$ satisfies the same equation with $\epsilon = 1$, we find, taking Eq. (3) into account,

$$\Pi_{ik}(\mathbf{r}, \mathbf{r}', \omega_n) = \frac{\varepsilon(\mathbf{r}, i\,|\,\omega_n\,|) - 1}{4\pi}\,\omega_n^2\delta_{ik}\delta(\mathbf{r} - \mathbf{r}'). \qquad (15)$$

The fact that $\Pi_{ik}(\mathbf{r}, \mathbf{r}', \omega_n)$ turns out to be proportional to $\delta(\mathbf{r} - \mathbf{r}')$ is connected with the neglect in the macroscopic theory of correlation effects, i.e., of the spatial dispersion of ϵ. It is seen from (15) that to calculate ϵ by the Matsubara technique it is sufficient to calculate the system polarization operator. (Such a method of calculating ϵ was first used by Abrikosov and Gor'kov in their work on the theory of superconducting alloys.[6])

CALCULATION OF THE STRESS TENSOR

We now calculate the corrections to the system free energy, necessitated by the interactions with

*Equation (13) is, evidently, the Dyson equation for the function $\mathfrak{D}_{ik}$

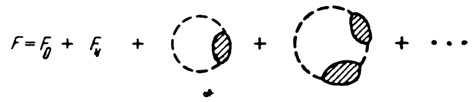

FIG. 2

the long-wave electromagnetic field. The calculations may be performed by summing the Matsubara diagrams for F, i.e., by summing the series of diagrams shown in Fig. 2, where the polarization operator $\Pi_{ik}(\mathbf{r}, \mathbf{r}', \omega_n)$ can be taken to the same approximation as in the calculation of $\mathfrak{D}_{ik}$. Making use of the general rules for writing the diagrams and taking expression (15) for the polarization operator into account, we find that the free energy is given by the series

$$F = F_0 + F_b$$
$$- \frac{T}{2} \sum_{n=-\infty}^{\infty} \left\{ \int \frac{\varepsilon(\mathbf{r}, i|\omega_n|) - 1}{4\pi} \omega_n^2 \mathfrak{D}_{ii}^0 (\mathbf{r}, \mathbf{r}, \omega_n)\, d^3\mathbf{r} \right.$$
$$+ \frac{1}{2} \iint \frac{\varepsilon(\mathbf{r}, i|\omega_n|) - 1}{4\pi} \omega_n^2 \mathfrak{D}_{il}^0 (\mathbf{r} - \mathbf{r}_1, \omega_n)$$
$$\times \frac{\varepsilon(\mathbf{r}_1, i|\omega_n|) - 1}{4\pi}$$
$$\left. \times \omega_n^2 \mathfrak{D}_{li}^0 (\mathbf{r}_1 - \mathbf{r}, \omega_n)\, d^3\mathbf{r}\, d^3\mathbf{r}_1 + \ldots \right\} . \quad (16)$$

Here F_0 is the free energy without allowance for the long-wave electromagnetic field, and F_b is the free energy of black-body radiation in vacuum. The appearance of this term is due to the fact that the energy of the electromagnetic field in vacuum enters into $\hat{H}_0$. The summation is taken over the values of the imaginary frequency $\omega_n = 2\pi n T$.

The series (16) is not summed directly, but to calculate the forces we need not the quantity F, but its variational derivative with respect to ϵ. Making the variation and comparing it with expression (12) for $\mathfrak{D}_{ik}$, we find that

$$\delta F = - \frac{T}{8\pi} \sum_{n=-\infty}^{\infty} \omega_n^2 \int \mathfrak{D}_{ii} (\mathbf{r}, \mathbf{r}, \omega_n)\, \delta\varepsilon (\mathbf{r}, i|\omega_n|)\, d^3\mathbf{r} . \quad (17)$$

Making use of Eq. (3) we can go over to summation over n from 0 to ∞. Here all terms except the zero term are doubled. The prime on the summation sign will henceforth denote that the zero term is taken with a halved weight:*

$$\delta F = - \frac{T}{4\pi} \sum_{n=0}^{\infty}{}' \omega_n^2 \int \mathfrak{D}_{ii} (\mathbf{r}, \mathbf{r}, \omega_n)\, \delta\varepsilon (\mathbf{r}, i\omega_n)\, d^3\mathbf{r} . \quad (18)$$

Equation (18) is completely analogous to the known formula for the variation of the free energy in a dispersionless medium with specified field sources [see, for example reference 3, Eq. (14.1)]:

$$\delta F = - \int \frac{E^2(\mathbf{r})}{8\pi} \delta\varepsilon(\mathbf{r})\, d^3\mathbf{r} . \quad (19)$$

Making use of the Maxwell equations, it is possible to obtain from (19), as is well known, an expression for the stress tensor in a nondispersive medium:

$$\sigma_{ik} = - p_0 \delta_{ik} - \frac{E^2}{8\pi} \left[\varepsilon - \rho \left(\frac{\partial \varepsilon}{\partial \rho} \right)_T \right] \delta_{ik}$$
$$+ \frac{\varepsilon E_i E_k}{4\pi} - \frac{H^2}{8\pi} \delta_{ik} + \frac{H_i H_k}{4\pi} . \quad (20)$$

In our case the function $\mathfrak{D}_{ik}(\mathbf{r}, \mathbf{r}', \omega_n)$ satisfies Eq. (10) with respect to each of its variables. Equation (10) is analogous to the usual equation for the vector potential, but is inhomogeneous, for its right-hand side is proportional to a delta-function. This analogy permits finding an expression for the tensor of the additional stresses by the same transformations as used in the ordinary case. This expression turns out to be formally similar to the usual expression (20). The terms arising from the presence of the right-hand half of Eq. (10) are found to cancel. We omit these simple, although rather cumbersome, calculations and give directly the expression for the additional part of the stress tensor. Here we introduce the notation

$$\mathfrak{D}_{ik}^E (\mathbf{r}, \mathbf{r}', \omega_n) = - \omega_n^2 \mathfrak{D}_{ik} (\mathbf{r}, \mathbf{r}', \omega_n) ,$$
$$\mathfrak{D}_{ik}^H (\mathbf{r}, \mathbf{r}', \omega_n) = e_{ilp} e_{kmq} \frac{\partial^2}{\partial x_l \partial x'_m} \mathfrak{D}_{pq} (\mathbf{r}, \mathbf{r}', \omega_n) , \quad (21)$$

e_{ikl} is a unit tensor, antisymmetric in all its indices. The quantities $\mathfrak{D}_{ik}^E$ and $\mathfrak{D}_{ik}^H$ evidently represent the "Matsubara" averages of certain quadratic combinations of the vectors **E** and **H**, respectively.

In this notation the expression for the stress tensor becomes

$$\sigma_{ik}(\mathbf{r}) = - 2T \sum_{n=0}^{\infty}{}' \left\{ - \frac{\delta_{ik}}{8\pi} \left[\varepsilon(\mathbf{r}, i\omega_n) - \rho \frac{\partial \varepsilon(\mathbf{r}, i\omega_n)}{\partial \rho} \right] \right.$$
$$\times \mathfrak{D}_{ll}^E (\mathbf{r}, \mathbf{r}, \omega_n)$$
$$+ \frac{\varepsilon(\mathbf{r}, i\omega_n)}{4\pi} \mathfrak{D}_{ik}^E (\mathbf{r}, \mathbf{r}, \omega_n)$$
$$\left. - \frac{\delta_{ik}}{8\pi} \mathfrak{D}_{ll}^H (\mathbf{r}, \mathbf{r}, \omega_n) + \frac{\mathfrak{D}_{ik}^H (\mathbf{r}, \mathbf{r}, \omega_n)}{4\pi} \right\} . \quad (22)$$

Equation (22), however, cannot have a direct physical significance, since the quantities $\mathfrak{D}_{ik}^E$ and $\mathfrak{D}_{ik}^H$ entering into it are infinite when $\mathbf{r} = \mathbf{r}'$. This is

*As shown by E. M. Lifshitz,[2] a sum of the same kind as that in (18) arises also when the forces of attraction between solid bodies is calculated to the usual scheme of the quantum theory of fluctuations.

connected with the fact that (unless a corresponding cutoff is explicitly introduced) σ_{ik} receives an infinite contribution from short-wave fluctuations that have no relation to the inhomogeneities of the body in the sense that their contribution to the pressure at a given point is the same in a homogeneous body as in an inhomogeneous body having the same dielectric constant at that point. To recast Eq. (22) in a form independent of the nature of the cutoff, we therefore must subtract from $\sigma_{ik}(\mathbf{r})$ the analogous expression for the homogeneous solid. These "homogeneous" terms can always be included in that part of the pressure which is defined by the properties of the body only at the given point.

Mathematically this subtraction can be carried out more simply by introducing a Green's function $\overline{\mathfrak{D}}_{ik}(\mathbf{r}-\mathbf{r}', \omega_n; \mathbf{r}_0)$ for a homogeneous infinite body having the same dielectric constant at the point $\mathbf{r}_0$ as the inhomogeneous solid under consideration. Such a function obviously satisfies the equation

$$\{\varepsilon(\mathbf{r}_0, i|\omega_n|)\,\omega_n^2\delta_{il} - \delta_{il}\partial^2/\partial x_m^2 + \partial^2/\partial x_i\partial x_l\} \times \overline{\mathfrak{D}}_{lk}(\mathbf{r}-\mathbf{r}', \omega_n; \mathbf{r}_0) = -4\pi\delta(\mathbf{r}-\mathbf{r}')\,\delta_{ik}. \tag{23}$$

This equation, as is well known, can be integrated explicitly (for example, by means of resolving it into a Fourier integral in $\mathbf{r}-\mathbf{r}'$, see the Appendix). After expressing in terms of $\overline{\mathfrak{D}}_{ik}$ the corresponding functions $\overline{\mathfrak{D}}_{ik}^{E}$ and $\overline{\mathfrak{D}}_{ik}^{H}$, by means of Eq. (31) (where all differentiations must, of course, be carried out at constant $\mathbf{r}_0$), and subtracting from σ_{ik} the analogous expression expressed in terms of $\overline{\mathfrak{D}}_{ik}^{E}$ and $\overline{\mathfrak{D}}_{ik}^{H}$, we finally obtain for the stress tensor an expression containing no divergences:

$$\sigma_{ik}(\mathbf{r}) = -2T\sum_{n=0}^{\infty}\lim_{\mathbf{r}\to\mathbf{r}'}\Big\{-\frac{\delta_{ik}}{8\pi}\Big[\varepsilon(\mathbf{r}, i\omega_n) - \rho\frac{\partial\varepsilon(\mathbf{r}, i\omega_n)}{\partial\rho}\Big] \times\Big[\mathfrak{D}_{ll}^{E}(\mathbf{r}, \mathbf{r}', \omega_n) - \overline{\mathfrak{D}}_{ll}^{E}(\mathbf{r}-\mathbf{r}', \omega_n; \mathbf{r})\Big] + \frac{\varepsilon(\mathbf{r}, i\omega_n)}{4\pi}\Big[\mathfrak{D}_{ik}^{E}(\mathbf{r}, \mathbf{r}', \omega_n) - \overline{\mathfrak{D}}_{ik}^{E}(\mathbf{r}-\mathbf{r}', \omega_n; \mathbf{r})\Big] - \frac{\delta_{ik}}{8\pi}\big[\mathfrak{D}_{ll}^{H}(\mathbf{r}, \mathbf{r}', \omega_n) - \overline{\mathfrak{D}}_{ll}^{H}(\mathbf{r}-\mathbf{r}', \omega_n; \mathbf{r})\big] + \frac{1}{4\pi}\big[\mathfrak{D}_{ik}^{H}(\mathbf{r}, \mathbf{r}', \omega_n) - \overline{\mathfrak{D}}_{ik}^{H}(\mathbf{r}-\mathbf{r}', \omega_n; \mathbf{r})\big]\Big\}. \tag{24}$$

We note that at imaginary frequencies the value of the dielectric constant in (24) is related to the imaginary part of $\epsilon(\omega)$ by the relation[3]

$$\varepsilon(i\xi) - 1 = \frac{2}{\pi}\int_0^{\infty}\frac{\omega\varepsilon''(\omega)\,d\omega}{\xi^2+\omega^2}.$$

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APPENDIX

We derive here expressions for the Green's functions of the electromagnetic field in a homogeneous absorbing medium with complex dielectric constant $\epsilon(\omega)$.

Equation (9) obtained above is satisfied by the retarded function D^R if a special gauge $\varphi = 0$ is chosen. Inasmuch as in an homogeneous space D^R depends only on the difference of the coordinates $\mathbf{r}-\mathbf{r}'$, we have, by going over to the Fourier components of this equation,

$$[(\omega^2\varepsilon - k^2)\,\delta_{il} + k_ik_l]\,D_{lk}^R(k, \omega) = 4\pi\delta_{ik}. \tag{A1}$$

Now we determine the function $D_{\alpha\beta}^R$ for an arbitrary gauge of the vector potential. To do this we multiply both halves of (A1) by ω^2 and introduce the function $D_{ik}^E = \omega^2 D_{ik}^R$. (Here D_{ik}^R is the retarded function corresponding to the gauge $\varphi = 0$.) The quantity thus obtained is already gauge-invariant (it differs from the retarded function made up of the components of the electric field intensity by the constant term $4\pi\delta_{ik}$) and satisfies the equation

$$[(\omega^2\varepsilon - k^2)\,\delta_{il} + k_ik_l]\,D_{lk}^E(k, \omega) = 4\pi\omega^2\delta_{ik}. \tag{A2}$$

D_{ik}^E is related to $D_{\alpha\beta}^R$ by the obvious equation

$$D_{ik}^E = \omega^2 D_{ik}^R - \omega k_i D_{0k}^R - \omega k_k D_{i0}^R + k_ik_k D_{00}^R. \tag{A3}$$

From considerations of symmetry, D_{i0}^R can be directed only along the vector $\mathbf{k}$:

$$D_{0i}^R = D_{i0}^R = dk_i. \tag{A4}$$

By the same token, D_{ik}^R can be rewritten as

$$D_{ik}^R = a\delta_{ik} + bk_ik_k. \tag{A5}$$

Substituting (A3) and (A4) into (A2), we obtain two equations for D_{00}^R, a, b, and d:

$$a(\varepsilon\omega^2 - k^2) = 4\pi, \qquad a + \varepsilon(\omega^2 b + D_{00}^R - 2\omega d) = 0. \tag{A6}$$

Thus we see that $D_{\alpha\beta}^R$ is defined to within two arbitrary functions. We shall now derive formulas for certain particular cases. Let d = b = 0. Then

$$D^R_{ik} = 4\pi\delta_{ik} / (\varepsilon\omega^2 - k^2),$$

$$D^R_{00} = -4\pi / \varepsilon(\varepsilon\omega^2 - k^2), \qquad D^R_{i0} = 0. \qquad \text{(A7a)}$$

The case $\varphi = 0$ corresponds to $D_{00} = d = 0$ and

$$D^R_{ik} = \frac{4\pi}{\varepsilon\omega^2 - k^2}\left(\delta_{ik} - \frac{k_i k_k}{\varepsilon\omega^2}\right). \qquad \text{(A7b)}$$

Finally, for the so-called transverse gauge, div $\mathbf{A} = 0$, we have

$$D^R_{ik} = \frac{4\pi}{\varepsilon\omega^2 - k^2}\left(\delta_{ik} - \frac{k_i k_k}{k^2}\right),$$

$$D^R_{00} = 4\pi / \varepsilon k^2, \qquad D^R_{0i} = 0. \qquad \text{(A7c)}$$

The temperature Green's function $\mathfrak{D}_{\alpha\beta}(k, \omega_n)$ is related to $D^R_{\alpha\beta}(k, \omega)$ by

$$\mathfrak{D}_{\alpha\beta}(k, \omega_n) = D^R_{\alpha\beta}(k, i\omega_n), \qquad \omega_n > 0,$$

$$\mathfrak{D}_{\alpha\beta}(k, \omega_n) = \mathfrak{D}^*_{\beta\alpha}(k, |\omega_n|), \qquad \omega_n < 0,$$

whence we obtain in the above-cited three cases [we recall that $\epsilon(i\omega_n)$ is a real function for $\omega_n > 0$]:

$$\mathfrak{D}_{ik} = -\frac{4\pi}{\varepsilon(i|\omega_n|)\,\omega_n^2 + k^2}\,\delta_{ik}, \qquad \mathfrak{D}_{00} = \frac{4\pi/\varepsilon(i|\omega_n|)}{\varepsilon(i|\omega_n|)\,\omega_n^2 + k^2},$$

$$\mathfrak{D}_{0i} = 0; \qquad \text{(A8a)}$$

$$\mathfrak{D}_{ik} = -\frac{4\pi}{\varepsilon(i|\omega_n|)\,\omega_n^2 + k^2}\left(\delta_{ik} + \frac{k_i k_k}{\varepsilon(i|\omega_n|)\,\omega_n^2}\right),$$

$$\mathfrak{D}_{00} = \mathfrak{D}_{0i} = 0; \qquad \text{(A8b)}$$

$$\mathfrak{D}_{ik} = -\frac{4\pi}{\varepsilon(i|\omega_n|)\,\omega_n^2 + k^2}\left(\delta_{ik} - \frac{k_i k_k}{k^2}\right),$$

$$\mathfrak{D}_{00} = \frac{4\pi}{\varepsilon(i|\omega_n|)\,k^2}, \qquad \mathfrak{D}_{0i} = 0. \qquad \text{(A8c)}$$

We now write expressions for the usual "Feynman" Green's function $D_{\alpha\beta}$. Its real and imaginary parts are related to the retarded function $D^R_{\alpha\beta}$ by the equations (see reference 5 and also reference 7):

$$\operatorname{Re} D_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2, \omega) = \operatorname{Re} D^R_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2, \omega),$$

$$\operatorname{Im} D_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2, \omega) = \coth\frac{\omega}{2T}\operatorname{Im} D^R_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2, \omega). \qquad \text{(A9)}$$

The function D is of particular interest at absolute zero, when the usual techniques of quantum field theory can be used. Going to the limit $T = 0$ in (A9), we obtain (in components of a Fourier expansion in $\mathbf{r} - \mathbf{r}'$)

$$\operatorname{Re} D_{\alpha\beta}(k, \omega) = \operatorname{Re} D^R_{\alpha\beta}(k, \omega),$$

$$\operatorname{Im} D_{\alpha\beta}(k, \omega) = \frac{\omega}{|\omega|}\operatorname{Im} D^R_{\alpha\beta}(k, \omega). \qquad \text{(A10)}$$

Noting that the real part of $\epsilon(\omega)$ is an even function of ω, and that the imaginary part is an odd function, it is easy to verify that to go from D^R to D at $T = 0$ it is necessary to replace ω by $|\omega|$ in (A7). Thus we obtain for the gauge (A8a)

$$D_{ik} = \frac{4\pi}{\varepsilon(|\omega|)\,\omega^2 - k^2}\,\delta_{ik}, \qquad D_{00} = -\frac{4\pi/\varepsilon(|\omega|)}{\varepsilon(|\omega|)\,\omega^2 - k^2},$$

$$D_{0i} = 0. \qquad \text{(A11)}$$

Equation (A11) may turn out to be useful in the calculation of the energy losses of fast particles in matter.[6]

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