MOTION OF A PLASMA LOOP IN AN AXIALLY SYMMETRIC MAGNETIC FIELD

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Problems related to the dynamics of a plasma loop in an inhomogeneous axially symmetric magnetic field are considered.

OSOVETS[1] and Shafranov[2] have considered the conditions of equilibrium of a plasma loop carrying a current in a magnetic field. In the present work, assuming stability over a small radius b, we consider the motion of such a loop as a whole in an axially symmetric magnetic field and find the relation which describes the variation of its large radius as a function of time.

1. BASIC EQUATIONS

We consider an axially symmetric magnetic field $\mathbf{H}(r, z, t)$ which is specified by the vector potential $\mathbf{A} = \{0, A_\varphi, 0\}$. It will be assumed that the axis of the loop coincides with the symmetry axis of the field so that the configuration of the loop is given completely by the z coordinate of its center and the quantity r, its large radius. The equations of motion can then be written in the form*

$$\frac{d^2\xi}{dt^2} = \omega_H^2 i \frac{\partial a}{\partial \xi}, \tag{1}$$

$$\frac{d^2x}{dt^2} = \omega_H^2 \frac{i}{x^2}\left[x\frac{\partial}{\partial x}(ax) + i\left(\alpha x + \frac{1}{l} + \frac{1}{2}\right)\right], \tag{2}$$

$$i = -\tau \frac{d}{dt}[ax + (1+\alpha x)i], \tag{3}$$

where

$$\omega_H^2 = 2\pi R_0 H_0^2 / Ml, \quad \alpha = 2\pi R_0 / N l r_0, \quad i = Jlx / R_0 c H_0,$$
$$\tau = \sigma s l / c^2, \quad a = A_\varphi / R_0 H_0,$$
$$l = L / 2\pi r = 2[\ln(8r/b) - 2], \quad \xi = z / R_0, \quad x = r / R_0,$$

M is the total mass of the loop, N is the number of electrons, $r_0 = e^2/mc^2$ is the classical electron radius, L is the inductance of the loop, σ is the conductivity of the loop, $s = \pi b^2$ is the cross section, and R_0 and H_0 are constants which have the dimensions of length and field respectively.*

As follows from Eq. (3), the effective real loss is characterized by the quantity $\gamma \approx T/2\pi\tau(1+\alpha)$, where T is some characteristic time during which the current changes. For reasons which will be made clear below we shall be interested in the case $\gamma \ll 1$, i.e., the case in which the reactive component is much larger than the real component.

At the outset we consider the case characterized by $\gamma = 0$; we then introduce corrections for the real losses.

As can be easily shown, Eqs. (1) — (3) can be written in the form

$$d^2\xi/dt^2 = -\partial U/\partial\xi, \quad d^2x/dt^2 = -\partial U/\partial x, \tag{4}$$

where

$$U(x, \xi, t) = \omega_H^2 (ax - \varphi_0)^2 / 2x(1 + \alpha x), \tag{5}$$

and $\varphi_0 = [ax + (1+\alpha x)i]_{t=t_0}$ is a constant which is determined from the initial conditions (for $t = t_0$ we have $x = x_0$, $\xi = \xi_0$, $i = i_0$ and so on).

If it is assumed that $\partial U/\partial t \ll dU/dt$, the system of equations in (4) yields the approximate energy integral (exact when $\partial U/\partial t = 0$)

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{d\xi}{dt}\right)^2 = \left(\frac{dx}{dt}\right)_0^2 + \left(\frac{d\xi}{dt}\right)_0^2 + 2[U_0 - U(x, \xi, t)]. \tag{6}$$

Whence it follows that if $U(x, \xi, t) < U_0 = U(x_0, \xi_0, t_0)$, then if the loop leaves such a field it acquires an energy increment $\Delta E = MR_0^2[U_0 - U]$. This effect lies at the basis of acceleration of a plasma in inhomogeneous magnetic fields.[3,4]

Without making a detailed analysis of Eq. (4), we may note certain features of the behavior of the large radius, assuming for simplicity that $(dx/dt)_0 = (d\xi/dt)_0 = 0$.

*Generally speaking, the right side of Eq. (2) should contain an additional term which describes the force due to the material pressure inside the loop. However, when $b/r \ll 1$ the introduction of this term in Eqs. (1) and (2) would only lead to a small correction and for this reason is neglected.

*Since the quantities which appear in Eqs. (1), (2), and (3) depend on b logarithmically, the dependence of the small radius of the ring on time can be neglected in the first approximation.

1) In the case in which

$$\partial U/\partial x|_{x_0,\,\xi_0} < 0, \quad U(x_0, \xi) > U(x, \xi) \tag{7}$$

for all $x > x_0$ and $\xi > \xi_0$, the radius of the loop increases without limit in the acceleration process.

2) If, however,

$$\partial U/\partial x|_{x_0,\,\xi_0} > 0, \quad U(x_0, \xi) > U(x, \xi) \tag{8}$$

for all $x < x_0$ and $\xi > \xi_0$, the radius of the loop approaches zero (this can happen only when $\varphi_0 = 0$).

3) In the remaining cases the radius of the loop can oscillate about one of the stationary radius values x_{st} determined from the conditions

$$\partial U/\partial x = 0, \quad \partial^2 U/\partial x^2 > 0, \tag{9}$$

where the oscillation amplitude generally increases monotonically during the acceleration process. We note, however, that for a strong dependence of x_{st} on ξ [and also when $(dx/dt)_0 \neq 0$, $(d\xi/dt)_0 \neq 0$] it may turn out that the loop moves from a stability region into a region of instability during the acceleration process [as determined by Eq. (7) or (8)]; conversely, if there are also values of ξ for which the second of the conditions of (7) or (8) is violated, a loop which is in a region of instability may be "captured" in a stability region. In conclusion we may note that if ax and φ_0 are of the same sign, acceleration will be effective only when $x_0 > x_m$, where x_m is the first root of Eq. (9) for which $(\partial^2 U/\partial x^2)\,x_m < 0$.

2. MAGNETIC MIRRORS*

We consider the problem of reflection of a loop from an inhomogeneous magnetic field which remains constant in time.

First of all we may note that when $\gamma \gg 1$ it is impossible to achieve effective reflection. Actually, since the current in this case is approximately $d(ax)/dt$, when the ring comes to rest the restoring forces vanish; the kinetic energy of the ring is converted entirely into heat. Thus, the efficiency of reflection increases as γ is reduced. At the outset we consider the case in which $\gamma = 0$. It follows immediately from simple physical considerations that in order for reflection to take place it is necessary that the energy of the reciprocatory motion of the loop be converted into magnetic energy and not into energy associated with radial motion. Furthermore, in most cases it is desirable that the loop receive no radial velocity as a result of the reflection. Obviously, these requirements will be satisfied if the initial radius of the loop x_0 is made equal to $x_{st}(\xi_0)$ under the condition that x_{st} depends weakly on ξ. Carrying out the operations indicated in Eq. (9) we have

$$h_z \equiv \frac{1}{x}\frac{\partial ax}{\partial x} = \left(\bar{h} - \frac{2\varphi_0}{x^2}\right)\frac{\beta}{2}, \quad \beta = \frac{l + 2(1 + \alpha x l)}{2l(1 + \alpha x)}, \tag{10}$$

where $\bar{h} = 2a/x$ is the mean field at the radius x. For the case $\alpha = 0$ this condition has already been obtained by Osovets.[1]

We consider the case in which the fields are such that the stationary radius does not depend on ξ. This will be the case, obviously, if the vector potential is of the form $a(x, \xi) = R(x)\,z(\xi)$ and φ_0 is set equal to zero.

Assuming at the outset that the orbit is stable, assuming that departures from the orbit $\rho = x - 1$, are small,* and limiting ourselves to the first nonvanishing terms, we have

$$d^2\xi/dt^2 + \frac{\partial}{\partial \xi} U(1, \xi) = 0, \tag{11}$$

$$d^2\rho/dt^2 + 2\beta\mu U(1, \xi)\,\rho = 0, \tag{12}$$

where

$$\mu = [2 - \beta + \partial \ln h_z/\partial \ln x - \partial \ln \beta/\partial \ln x]|_{x=1}. \tag{13}$$

If we now require that $z^2(\xi)$ be an even function of ξ, we obtain "oscillatory" (generally speaking, nonlinear) motion for the ξ-motion; consequently the ρ-motion is described by the Hill equation which, with an appropriate choice of parameters, allows of stable solutions.

Thus we see that a loop which receives an initial velocity $(d\xi/dt)_0$ and is elastically scattered from regions occupied by the field will execute oscillations about the plane $\xi = 0$ with a period

$$T = 2\sqrt{2}\int_0^{\xi_2} [U(1, \xi_1) - U(1, \xi)]^{-1/2} d\xi, \tag{14}$$

where ξ_1 is the maximum excursion from the plane $\xi = 0$, determined from the relation

$$U(1, \xi_1) = U(1, \xi_0) + {}^1/_2 (d\xi/dt)_0^2. \tag{15}$$

If we denote the boundary of the magnetic mirror by ξ_b, the maximum energy of the loop E_m for which reflection takes place is

$$E_m = MR_0^2 U(1, \xi_b). \tag{16}$$

If one now takes account of the real losses, certain correction terms appear in Eqs. (11) and (12) (these terms are of order γ); these terms cause a weak damping of the amplitude of the ξ-motion and an increase in the radial oscillations.

*This problem has been considered in connection with a proposal by Veksler for using inhomogeneous magnetic fields for coherent cyclic acceleration.[5]

*The constant R_0 is chosen to make $x_{st} = 1$.

As an example we consider the motion of a loop in a field described by

$$a(x,\xi)=\begin{cases}\xi x/2 & \text{for } x\leqslant x_1,\\ {}^1/_2 a_1\xi\,[x+(\Delta a/a_1)\,x_1^2/x] & \text{for } x\geqslant x_1,\end{cases} \tag{17}$$

where $\Delta a = 1 - a_1$. In this case

$$T=2\pi(1+\alpha)^{1/2}\beta/a_1\omega_H, \tag{18}$$

$$E_m=a_1^2\omega_H^2 MR_0^2\xi_b^2/2\beta^2(1+\alpha), \tag{19}$$

where the equilibrium radius R_0 is found from the relations

$$R_0^2=r_1^2\frac{\Delta a}{a_1}\frac{\beta}{2-\beta}\Big|_{x=1},\qquad r_1=R_0x$$

We now introduce the effect of the real losses, assuming that $\gamma \ll 1$. Taking account of Eq. (17), we substitute the expression for the current i from Eq. (3) in Eqs. (1) and (2) and expand the right sides of the equations in a series in ρ. Assuming now that when $t = 0$, $\xi = 0$, $i = 0$, and $d\xi/dt = \xi_0$, and limiting ourselves to the first non-vanishing terms in the expansion in ρ and first-order terms in γ, we obtain

$$\frac{d^2\xi}{dt^2}+\Omega^2\xi=\frac{\Omega^2}{\tau'}e^{-t/\tau'}\int_0^t e^{t'/\tau'}\xi(t')\,dt', \tag{20}$$

$$\frac{d^2\rho}{dt^2}+\mu\beta\Omega^2\xi^2(t)\,\rho=-\frac{\Omega^2}{\tau'}\beta\xi(t)\,e^{-t/\tau'}\int_0^t e^{t'/\tau'}\xi(t')\,dt', \tag{21}$$

where

$$\Omega^2=\omega_H^2a_1^2/\beta^2(1+\alpha),\qquad \tau'=\tau(1+\alpha).$$

Whence, to first-order in $\gamma = 1/\Omega\tau'$,

$$\xi(t)=\frac{\dot\xi_0}{\Omega}\,[e^{-t/2\tau'}\sin\Omega t+\gamma(1-e^{-t/2\tau'}\cos\Omega t)]. \tag{22}$$

Now, substituting Eq. (22) in Eq. (21) we have, to the same accuracy,

$$d^2\rho/dt^2+\mu\beta\dot\xi_0^2e^{-t/\tau'}\sin^2\Omega t\cdot\rho = -\gamma\beta\dot\xi_0^2\,[1-e^{-t/2\tau'}\cos\Omega t]\,e^{-t/2\tau'}\sin\Omega t. \tag{23}$$

With an appropriate choice of parameters this equation can have oscillatory solutions, the amplitudes of which increase slowly in time. Thus, if the loop does not receive additional energy the amplitude of its oscillations along the z axis decay approximately as $\exp(-t/2\tau')$, whereas the radial oscillations grow. If, however, the loop receives energy in such a way that its oscillation amplitude along the z axis remains constant the equation for the ρ-motion assumes the form

$$d^2\rho/dt^2+\mu\beta\dot\xi_0^2\sin^2\Omega t\cdot\rho = \gamma\dot\xi_0^2\beta\,[\cos\Omega t-e^{-t/\tau'}]\sin\Omega t, \tag{24}$$

i.e., the amplitude of the radial oscillations remains bounded. In the case in which the energy of the loop increases the radial oscillations first decay,* and then, when $E \gtrsim MR_0^2\Omega^2/2\beta\mu$, increase sharply, i.e., the loop becomes unstable. In the present case this is what actually determines the maximum energy and thus there is no reason for making $\xi_b > (\beta\mu)^{-1/2}$.

3. STABILITY OF A LOOP IN A HIGH-FREQUENCY FIELD

We consider the case in which the field is a harmonic function of time, i.e.,

$$a(x,\xi,t)=\tilde a(x,\xi)\sin\omega t.$$

We shall be interested in a stationary loop which executes small oscillations about the plane $z = 0$ and some equilibrium orbit R_0. Assuming that the field is "barrel-shaped" i.e., $a(-\xi) = a(\xi)$, it is easy to see that the stationary radius, determined from Eq. (10), will be independent of ξ (to first-order terms in ξ). Assuming that $i = 0$ when $t = 0$, we find that in the absence of real losses ($\gamma = 0$) x_{st} is independent of the time. When the real resistance is taken into account the equilibrium radius no longer remains constant but becomes a function of time. It can be shown that for sufficiently large values of γ the presence of real losses leads either to breakup of the beam in the inward direction in the first quarter cycle (cf. reference 6) or to breakup in the outward direction in the first few cycles.

To achieve stable operation it is necessary to make the relative magnitude of the real resistance small.

As before, assuming a stable radius, we expand the right sides of Eqs. (2) and (3) in a series about $x = x_{st} = 1$, $\xi = 0$. Then, limiting ourselves to first-order terms in $\rho = x - 1$, ξ, and γ we have

$$d^2\xi/d\theta^2+np(1-\cos2\theta)=0, \tag{25}$$

$$d^2\rho/d\theta^2+\mu\rho(1-\cos2\theta)=2\gamma' p\,[\cos\theta-e^{-\gamma\theta}]\sin\theta, \tag{26}$$

where

$$\theta=\omega t,\qquad \gamma'=[\omega\tau(1+\alpha)]^{-1},\qquad n=-\frac{\partial\ln h_z}{\partial\ln x}\Big|_{x=1},$$

$$p=\frac{\pi R_0H^2(R_0)}{M\omega^2l\beta(1+\alpha)}.$$

It follows from Eqs. (25) and (26) that taking account of the real resistance has no effect on the ξ-motion in the first approximation. In the equation for ρ, however, the real losses lead to the

*We bear in mind the fact that the parameters are chosen for operation in the first stability region, i.e. $\xi_0^2\mu\beta < \Omega^2$.

appearance of a small external force (of order γ), which does not represent any danger as far as stability is concerned. Although Eqs. (25) and (26) allow a whole series of stability regions, because of their relative narrowness, only the first region is of practical interest. For this region the parameters n, μ, and p must be chosen in such a way that the quantities np and μp lie in the interval $(0, \sim 0.5)$. Assuming that $n \sim \mu \sim \frac{1}{2}$, we find that the loop will be stable if its parameters, and the amplitude and frequency of the magnetic field are chosen in such a way as to satisfy the condition $p < 1$. From the expression for p it follows that for a given total number of particles in the loop and a given magnetic field this condition can be satisfied by making the field frequency sufficiently high.

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