COHERENT ELECTRON RADIATION IN THE SYNCHROTRON. II.

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The electromagnetic interaction of electrons in a thin relativistic bunch in coherent radiation in a synchrotron is considered. The tangential forces exerted on an individual electron by the bunch are considered for different particle phase distributions.

EARLIER[1] we have considered the coherent radiation spectrum of an electron bunch in a synchrotron. By summing the intensities of the individual harmonics for the entire spectrum it is possible to compute the total power associated with the coherent radiation for a given electron distribution in the bunch. This calculation has been carried out by Schiff for the case of a Gaussian distribution.[2] Schwinger has carried out this calculation for the case of a rectangular distribution and the results are given in reference 3. It is also of interest to compute the effect of the coherent radiation on the bunch. It is clear that when the coherent radiation becomes sizeable it must have an effect on the motion of the particles in the bunch. Coherent radiation arises as a result of the electromagnetic interaction of electrons within a bunch; hence the problem of determining the effect of coherent radiation on a bunch means essentially determining the tangential component of the electromagnetic forces exerted by the bunch on an individual electron. An accurate calculation of these forces for a bunch with an arbitrary distribution of particles, in which the phase and betatron oscillations are taken into account, is not feasible. However, two particular cases have been solved by Tamm[4] and Rytov.[5] These are for bunches which are segments of toroids or ellipsoids which move as a rigid body (common angular velocity) and which have a uniform electron density.

In the present paper it is assumed that all the particles in a bunch move in coaxial circles with a uniform linear velocity, $v \sim c$. This means that we can neglect betatron oscillations and the instantaneous spread in particle energy. This approach is justified because in practice the thickness of a bunch in a synchrotron is much smaller than its length. Hence, in the first approximation, the transverse dimensions do not appear in the expressions for the tangential forces.

We neglect the interaction of the bunch with the chamber walls, the pole pieces of the magnet, and the other constructional elements, assuming that the electrons move in an infinite free space. As long as the longitudinal dimensions of the bunch are small compared with the transverse dimensions of the chamber it is valid to neglect the effect of the wall on the inter-electron interaction. In cases in which the bunches are large compared with the cross section of the chamber the interaction with remote electrons is affected by the shielding of the walls and the present results cannot be applied directly.

The calculation of the tangential forces in coherent radiation has acquired new interest in connection with the experimental work being carried on in many laboratories on the formation and maintenance of electron bunches.[6]

1. INTERACTION BETWEEN TWO CHARGES IN A BUNCH

First we determine the interaction between two charges. We use the coordinate system used in reference 1. Call A the point of observation; at a given time t_A, this is the location of the charge which experiences the interaction with a second charge (radiating charge) located at point P. The effective source point of the field of the second charge, corresponding to the interaction at point A at time t_A, is designated Q; the corresponding time, which precedes t_A, we call t_Q. We use a four-dimensional notation and denote the corresponding world points by x_{Ai}; x_{Pi}; x_{Qi} ($i = 1, 2, 3, 4$). We introduce the 4-vector $R_i = x_{Ai} - x_{Qi}$. Since x_{Ai} and x_{Qi} lie on one light cone, $R_i^2 = 0$. Using the 4-dimensional Liénard-Wiechart potentials $A_i = -eu_i/R_k u_k$, where $u_i = dx_{Qi}/ds_Q$ is the 4-velocity of the charge Q, it is easy to obtain an expression for the Lorentz 4-force acting on the charge A:

$$f_i = -\frac{e^2}{c}\left\{\left[\frac{w_k \nu_k}{(R_k u_k)^2} - \frac{(1 + R_k w_k)(u_k \nu_k)}{(R_k u_k)^3}\right] R_i + \frac{(1 + R_k w_k)(R_k \nu_k)}{(R_k u_k)^3} u_i - \frac{R_k \nu_k}{(R_k u_k)^2} w_i\right\}. \quad (1)$$

Here $w_i = du_i/ds_Q$, while ν_i is the 4-velocity of charge A. We assume further that charges A and Q have the same velocity v and move in coaxial circles characterized by radii r_A and r_Q. In the case in which $r_A = r_Q$, i.e., when the orbits of the electrons lie on the common cylindrical surface, $f_i = -\partial\Phi/\partial x_{Ai}$, where the convection scalar potential

$$\Phi = -\frac{e^2}{c}\frac{1 + R_k w_k}{R_k u_k} = \frac{\gamma}{c} e^2 \frac{1 - \beta^2 (r_A / r_Q)\cos\xi}{R - \beta r_A \sin\xi}, \quad (2)$$

$$\beta = v/c; \quad \gamma = (1 - \beta^2)^{-1/2}; \quad \xi = \psi_A - \psi_Q,$$
$$R = [(z_A - z_Q)^2 + (r_A - r_Q)^2 + 4r_A r_Q \sin^2(\xi/2)]^{1/2}, \quad (3)$$

and R is the distance between points A and Q. Equation (2) is similar to the potential derived by Tamm for motion with a common angular velocity.[4]

When $r_A \neq r_Q$ the force given by Eq. (1) is not deriveable from a potential. However one is easily convinced that the projection of (1) on the tangent to the orbit at A coincides in direction with the derivative of (2). The usual Lorentz force is given by $\mathbf{f} = (c/\gamma) f_\alpha$ ($\alpha = 1, 2, 3$). It follows from the above that the tangential component of this force is

$$f_\tau = -\frac{e^2}{r_A}\frac{\partial}{\partial\psi_A}\frac{1 - (\beta^2 r_A / r_Q)\cos\xi}{R - \beta r_A \sin\xi}. \quad (4)$$

Hence, although the force given in (1) cannot be derived from a potential the expression in Eq. (2) may be taken as a "potential for the tangential forces." One is easily convinced that the expression $(e^2/c) u_k \nu_k / R_k u_k$ is the "potential for the axial forces" for (1) in the same sense as (2). The phases at points A, P, and Q are related by the retardation condition

$$\psi_P - \psi_A + \xi = \beta R / r_Q. \quad (5)$$

Taking account of Eqs. (3) and (5) and the fact that $r_Q \equiv r_P$ and $z_Q \equiv z_P$, the force given by (4) may be considered a function of the coordinates of points A and P.

We derive an explicit expression for the force (4) for the simple relativistic case in which both electrons lie on one circle, i.e., $r_A = r_P = a$; $z_A = z_P = 0$ (the case considered by Tamm). Let $\psi_{AP} \equiv \psi_A - \psi_P > 0$, i.e., the charge A, on which the force is exerted, lies in front of the radiating charge P. If $\frac{3}{2}\gamma^3\psi_{AP} \ll 1$,

$$f_\tau \approx \frac{e^2}{\gamma^2 a^2 \psi_{AP}^2} - \frac{4}{3}\frac{e^2}{a^2}\gamma^4 + \frac{11}{3}\frac{e^2}{a^2}\gamma^{10}\psi_{AP}^2 \ldots \quad (6)$$

If $\frac{3}{2}\gamma^3\psi_{AP} \gg 1$, but $\psi_{AP} < 1$,

$$f_\tau \approx 2 \cdot 3^{-1/3} e^2 / a^2 \psi_{AP}^{4/3}. \quad (7)$$

As is well known, $\frac{3}{2}\gamma^3 c/a = \omega_{max}$ is the frequency at which the maximum radiation energy of a single electron is radiated. Hence the condition $\frac{3}{2}\gamma^3\psi_{AP} \gg 1$ is satisfied if points A and P are separated by a distance large compared with $\lambda\!\!\!^{-}_{max} = c/\omega_{max}$; in this case the coherent force (7) is greater than the Coulomb force $e^2/\gamma^2 a^2 \psi_{AP}^2$. When $\psi_{AP} < 0$,

$$f_\tau \approx -\frac{e^2}{\gamma^2 a^2 \psi_{AP}^2} + \frac{e^2}{8a^2} + \frac{e^2}{2^8 a^2}\psi_{AP}^2 \ldots \quad (8)$$

2. ACTION OF THE BUNCH UPON AN INDIVIDUAL ELECTRON

In order to obtain the total tangential force $\overline{f}_\tau$ exerted on charge A by the entire bunch, it is necessary to sum (4) over all electrons in the bunch. At distances much larger than the mean distance between the charges near point A the summation may be replaced by integration over the coordinates of point P.

Let the particle distribution in the bunch be given by the function $F(\psi, r, z)$. For the time being we neglect the region in the immediate proximity of A; thus

$$\overline{f}_\tau(\psi_A, r_A, z_A) = -e^2 \frac{\partial}{\partial\psi_A}\int_0^\infty dr_P \int_{-\infty}^{+\infty} dz_P \times \int_{-\pi}^{+\pi} \frac{r_P / r_A - \beta^2\cos\xi}{R - \beta r_A \sin\xi} F(\psi_P, r_P, z_P)\, d\psi_P. \quad (9)$$

In Eq. (9), carrying out the integration over ξ in place of ψ_P, using Eq. (5) we have

$$\overline{f}_\tau = -e^2 \frac{\partial}{\partial\psi_A}\int_0^\infty dr_P \int_{-\infty}^{\infty} dz_P \times \int_{-\pi}^{+\pi} \frac{r_P / r_A - \beta^2\cos\xi}{R} F\left\{\left[\psi_A - \xi + \beta\frac{R}{r_P}\right], r_P, z_P\right\} d\xi, \quad (10)$$

where R is given by Eq. (3).

We must now make certain assumptions as to the bunch. We assume that $F(\psi, r, z) = N\varphi(\psi) U(\rho, z)$, where $\rho = r - a$ (a is the radius of the stable orbit). We assume that $U(\rho, z)$ is an even function of ρ and z. Then the normalization condition is

$$a \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(\rho, z)\, d\rho\, dz = 1.$$

We assume that the mean radius for the cross section of the bunch is

$$\sigma_0 = a \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{\rho^2 + z^2}\, U(\rho, z)\, d\rho\, dz \ll a.$$

The effective angular dimensions of the bunch are characterized by the quantity $\vartheta_0 = 2 \int_0^{\pi} \psi\varphi(\psi)\, d\psi$. From reference 1 it follows that in the case of small oscillations $\vartheta_0 = 2\Phi_0/\pi$, where Φ_0 is the mean amplitude of the phase oscillations.

We consider an extended bunch, i.e., the case in which $a\vartheta_0 \gg \sigma_0$; we will be interested in points A close to the stable orbit, where $\rho_A \sim z_A \lesssim \sigma_0$. We assume that $\varphi(\psi)$ is a smooth function so that it does not vary significantly at distances $a\Delta\psi \sim \sigma_0$. Then, in Eq. (10) we change the argument:

$$[\psi_A - \xi + \beta R / r_p] \rightarrow [\psi_A - \xi + 2\beta\, |\sin(\xi/2)|\,].$$

We now expand (10) in a series in powers of $\rho_A/a \sim z_A/a \lesssim \sigma_0/a \ll 1$. We limit our consideration to the first and linear terms in the expansion. Because $U(\rho, z)$ is even, (10) is an even function of z_A. Hence in the expansion we eliminate the term which is linear in z_A. The linear term in ρ_A remains. First we consider the first term of the expansion

$$\bar{f}_\tau(\psi_A) = -Ne^2 \frac{d}{d\psi_A} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\rho\, dz U(\rho, z) \times \int_{-\pi}^{\pi} \frac{1 - \beta^2 \cos\xi}{\sqrt{\rho^2 + z^2 + 4a^2 \sin^2(\xi/2)}}\, \varphi\left[\psi_A - \xi + 2\beta \left|\sin\frac{\xi}{2}\right|\right] d\xi. \tag{11}$$

For simplicity we assume that the bunch is circular in cross section with a radius σ and has a density which is constant over the cross section. Then we can carry out the integration over σ and z in Eq. (11). For values of ϑ_0 which are not too large, we replace the trigonometric functions in the first term by their series expansions. We have

$$\bar{f}_\tau = -\frac{Ne^2}{a^2} 2\left(\frac{a}{\sigma}\right)^2 \frac{d}{d\psi_A} \int_{-\pi}^{\pi} \left(\frac{1}{\gamma^2} + \beta^2 \frac{\xi^2}{2}\right) \left\{\sqrt{\xi^2 + \left(\frac{\sigma}{a}\right)^2} - |\xi|\right\} \times \varphi\left[\psi_A - \xi + \beta\left|\xi - \frac{\xi^3}{24}\right|\right] d\xi. \tag{12}$$

The term with γ^{-2} gives the Coulomb force; the term with $\beta^2\xi^2/2$ gives the coherent force. First we consider the latter: the quantity in the curly brackets in Eq. (12) can be replaced by its limiting value for $\sigma \rightarrow 0$: $\frac{1}{2}(\sigma/a)^2 |\xi|^{-1}$. In the ultrarelativistic case $\gamma \gg 1$, the argument $\varphi(\psi)$ in (12) (with $\xi > 0$) assumes the form

$$\psi_A - \xi/2\gamma^2 - \xi^3/24 \approx \psi_A - \xi^3/24,$$

if $\xi \gg 1/\gamma$ and, if $\xi < 0$ it assumes the form $\psi_A - 2\xi$. Hence, for bunches characterized by $\gamma^{-3} \ll \vartheta_0 < 1$ the coherent force is

$$\bar{f}_\tau = -\frac{2}{3^{1/3}} \frac{Ne^2}{a^2} \frac{d}{d\psi_A} \int_0^{\infty} \varphi(\psi_A - \xi) \frac{d\xi}{\xi^{1/3}} - \frac{1}{8} \frac{Ne^2}{a^2} \frac{d}{d\psi_A} \int_0^{\infty} \varphi(\psi_A + \xi)\, \xi d\xi. \tag{13}$$

The first term in Eq. (13) gives the contribution due to the interaction of charges lying behind charge A while the second term gives the contribution from charges lying in front of it. Since $\varphi(\psi) \sim 1/\vartheta_0$, in Eq. (13) the first term is of order $Ne^2/a^2\vartheta_0^{4/3}$ while the second term is approximately Ne^2/a^2. Consequently, when $\vartheta_0 \ll 1$ the second term can be neglected. Strictly speaking, directly at the end and behind of the bunch this procedure is not justified; however, for the main part of the bunch and the space in front of it, when $\gamma^{-3} \ll \vartheta_0 \ll 1$ we may assume

$$\bar{f}_\tau = -\frac{2}{3^{1/3}} \frac{Ne^2}{a^2} \frac{d}{d\psi_A} \int_0^{\infty} \varphi(\psi_A - \xi) \frac{d\xi}{\xi^{1/3}}. \tag{14}$$

For $\psi_A \gg \vartheta_0$, i.e., far in front of the bunch, from Eq. (14) we obtain the asymptotic expression for $\bar{f}_\tau$, which coincides with (7); behind the bunch we have $f_\tau \approx 0$. The Coulomb term in Eq. (12) is of order $(Ne^2/\gamma^2 a^2 \vartheta_0^2) \ln(a\vartheta_0/\sigma)$. Since the logarithm is always relatively small, the smallness of the Coulomb forces as compared with (14) gives the condition $(\vartheta_0\gamma^3)^{2/3} \gg 1$. This is almost the same as the condition $\vartheta_0 \gg \gamma^{-3}$.

Calculation of the term which is linear in ρ_A in the expansion in (10) shows that it makes a contribution of order $(\rho_A/a) Ne^2/a^2\vartheta_0^2$. As long as $\rho_A \ll a\vartheta_0$ this term can be neglected in comparison with (14). Consequently, (14) is the main component of the force being sought.

Summing the tangential forces which act on the individual electrons for the entire bunch we obtain the total reaction force on the bunch for the coherent radiation. Multiplying this force by the velocity of the bunch we find the total power of the coherent radiation. In particular, in the case $\gamma^{-3} \ll \vartheta_0 \ll 1$, using Eq. (14) we have

$$W_{\text{coher}} = -\frac{2}{3^{1/3}} \frac{Ne^2 c}{a^2} \int_0^{\infty} \frac{d\xi}{\xi^{1/3}} \frac{d}{d\xi} \int_{-\infty}^{\infty} \varphi(\psi_A)\, \varphi(\psi_A - \xi)\, d\psi_A. \tag{15}$$

It is easier to compute the total power of the coherent radiation from Eq. (15) than by summing the intensities of the spectral harmonics. Furthermore, making the calculation by means of Eq. (15) serves as a good check on the validity of the expression for the tangential forces. For example, for a Gaussian distribution

$$\varphi(\psi) = \frac{1}{\pi\vartheta_0}\exp\left\{-\frac{1}{\pi}\left(\frac{\psi}{\vartheta_0}\right)^2\right\},$$

we have

$$W_{\text{coher}} = \frac{2^{5/3}\,\Gamma(5/6)\,Ne^2c}{3^{1/6}\sqrt{\pi}\,a^2(2\sqrt{\pi}\,\vartheta_0)^{4/3}},$$

which coincides with the result obtained by Shiff.[2] For a rectangular distribution we also get exact agreement with the results obtained by Tamm[4] and Schwinger (cf. reference 3).

3. CALCULATION OF THE FORCES FOR CERTAIN MODELS

We now use Eq. (14) with various phase distributions, assuming that $\gamma^{-3} \ll \vartheta_0 \ll 1$.

For a rectangular distribution

$$\varphi(\psi) = 1/4\,\vartheta_0, \quad |\psi| \leqslant 2\vartheta_0; \quad \varphi(\psi) = 0, \quad |\psi| > 2\vartheta_0,$$

and, using the notation $\psi_A/2\vartheta_0 = p_1$, we find

$$\overline{f}_\tau = -\frac{Ne^2}{3^{1/3}a^2(2\vartheta_0)^{4/3}}\chi(p_1);$$

$$\chi(p_1) = \begin{cases} 0; & p_1 < -1 \\ (p_1+1)^{-1/3}; & -1 < p_1 < 1. \\ -[(p_1-1)^{-1/3}-(p_1+1)^{-1/3}]; & p_1 > 1 \end{cases} \qquad (16)$$

The discontinuities at the points $\psi_A = \pm 2\vartheta_0$ are due to the fact that the assumption concerning the slow variation of the bunch are not satisfied at these points and Eq. (14) cannot be used. It is this distribution which was considered by Tamm and Rytov and they have shown that the region close to the ends of the bunch does not make an important change in the basic expression for the forces (16); at the discontinuities there are sharp maxima.

We consider a triangular bunch:

$$\varphi(\psi) = \frac{1}{3\vartheta_0}\left[1-\frac{\psi}{3\vartheta_0}\right],$$

$$0 < \psi < 3\vartheta_0, \quad \varphi(\psi) = 0, \quad \psi > 3\vartheta_0, \qquad \varphi(-\psi) = \varphi(\psi).$$

Using the notation $p_2 = \psi_A/3\vartheta_0$, we have

$$\overline{f}_\tau = -\frac{3^{2/3}Ne^2}{a^2(3\vartheta_0)^{4/3}}\chi(p_2),$$

$$\chi(p_2) = \begin{cases} 0 & p_2 < -1 \\ (p_2+1)^{2/3} & -1 < p_2 < 0 \\ [(p_2+1)^{2/3}-2p_2^{2/3}] & 0 < p_2 < 1 \\ [(p_2+1)^{2/3}-2p_2^{2/3}+(p_2-1)^{2/3}] & p_2 > 1 \end{cases} \qquad (17)$$

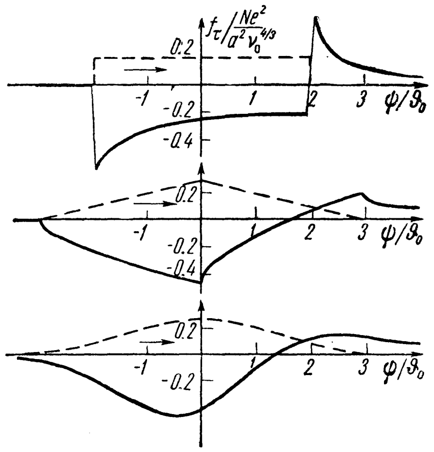
For a Gaussian distribution

$$\overline{f}_\tau = -\frac{2}{3^{1/3}\sqrt{\pi}}\frac{Ne^2}{a^2(\sqrt{\pi}\,\vartheta_0)^{4/3}}\chi(p_3), \qquad p_3 = \frac{\psi_A}{\sqrt{\pi}\,\vartheta_0},$$

$$\chi(p_3) = \frac{d}{dp_3}\int_0^\infty \exp\{-(p_3-\xi)^2\}\frac{d\xi}{\xi^{1/3}}$$

$$= \frac{e^{-p_3^2}}{6}\sum_{n=0}^{\infty}\frac{\Gamma(n/2-1/6)}{n!}(2p_3)^n. \qquad (18)$$

The results obtained in (16) — (18) are shown in the three graphs. In each of these the dashed curves represent the distribution over phase for the particles in the bunch and ϑ_0 is identical in all cases. The force scale is the same throughout. The arrow indicates the direction of motion (to the right). It is apparent from these curves that the main losses in coherent radiation occur at the rear of the bunch. On the other hand, the electrons in the front part of the bunch may obtain additional energy by virtue of the interaction.



4. INTERACTION AT CLOSE DISTANCES

In conclusion we shall concern ourselves with calculating the interaction at close distances. Strictly speaking we can obtain the total force acting on the charge A if, in Eq. (9), we exclude from the region of integration a sphere with center at A and radius much larger than the mean distance between electrons δ and add the force due to the interaction of A with the remaining charges within the sphere. The latter force can be computed by means of Eq. (4). Carrying out this operation we find that in addition to the force indicated by (9) the electron at A experiences a force $f_{\delta T}$, directed along its line of motion. With $\delta \gg a/\gamma^3$: $f_{\delta T} \sim (e^2/a^2)(a/\delta)^{4/3}$ and when $\delta \ll a/\gamma^3$, $\gamma \gg 1$ this force approaches the value $(2e^2/3a^2)\gamma^4$, i.e., it compensates for the dissipation force due to radiation.

For a bunch characterized by $\vartheta_0 \ll 1$ the force $f_{\delta T}$ can be neglected compared with that given in (14) if $N^{5/9} \gg (a\vartheta_0/\sigma_0)^{8/9}$. On the other hand, for a circular current uniformly filling the entire circle, (9) vanishes and $f_{\delta T}$ determines the total force exerted on the electron at A by the bunch. Since there is no coherent radiation in this case, with $\delta \ll a/\gamma^3$ the radiation will be determined completely by the fluctuations in the density of electrons in the bunch. For present-day electron synchrotrons, in which $\gamma \sim 10^3$ and $a \sim 1$ m, the case $\delta \ll a/\gamma^3$ corresponds to a particle density of approximately 10^{21} cm^{-3}; this is far beyond what can be achieved. However, at smaller energies ($\gamma \sim 10$) and very small cross sections it is possible that $\delta \ll a/\gamma^3$. In this case (very small electron densities) there should be correlation in the distribution of the particles in the bunch and the fluctuation in the density should be smaller than for non-interacting particles. Hence the average losses due to radiation for one electron should be smaller than $(2e^2c/3a^2)\gamma^4$. However this problem requires further analysis.

The authors are indebted to Academician I. E. Tamm for making us acquainted with the results of his unpublished work.[4]

[1] L. V. Iogansen and M. S. Rabinovich, J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 1013 (1958), Soviet Phys. JETP **8**, 708 (1959).

[2] L. Schiff, Rev. Sci. Instr. **17**, 6 (1946).

[3] J. S. Nodvick and D. S. Saxon, Phys. Rev., **96**, 180 (1954).

[4] I. E. Tamm, Отчет ФИАН (Report of the Institute of Physics, Acad. of Sciences, U.S.S.R.), 1948.

[5] S. M. Rytov, Отчет ФИАН (Report of the Institute of Physics, Acad. Sci., U.S.S.R.), 1949.

[6] T. Ohkawa, Rev. Sci. Instr. **29**, 108 (1958).

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