SOVIET PHYSICS JETP VOLUME 37(10), NUMBER 1 JANUARY, 1960

THEORY OF WEAK SOLUTIONS OF He^4 *IN LIQUID* He^3

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Submitted to JETP editor January 23, 1959

J. Exptl. Theoret. Phys. (U.S.S.R.) **37**, 143-153 (July, 1959)

A weak solution of He^4 in liquid He^3 is considered as a Boltzmann gas of impurity excitations (associated with the He^4 atoms) in a Fermi liquid. The spectrum of the impurity excitations and the thermodynamics of the solution are examined. Kinetic equations for Fermi and impurity excitations of the solution are derived. The dependences of the diffusion, thermal diffusion, viscosity and thermal conductivity coefficients on temperature and concentration are determined.

PURE liquid He^3 near absolute zero is a Fermi quantum liquid, for which Pomeranchuk[1] was the first to develop a qualitative theory that yielded the temperature dependence of both thermodynamic quantities (specific heat and entropy) and kinetic coefficients (viscosity and thermal conductivity). A theory constructed recently by Landau[2] provides a quantitative description of liquid He^3 as a system of Fermi "quasi-particles," the number of which equals that of the atoms in a unit of volume. Physically a quasi-particle is a He^3 atom in the self-consistent field of the surrounding atoms in the liquid He^3. At temperatures much below the degeneracy temperature most of the quasi-particles are condensed into a Fermi sphere [or into a Fermi layer or "bubble" in the case of the roton spectrum (3)] and only a negligible fraction is in the diffuse Fermi zone. In this temperature region the theory permits determination of the thermodynamic and kinetic properties of liquid He^3.[1,3,4]

An essential difference between the Landau theory of a Fermi liquid and the Fermi-gas theory is that the energy of a quasi-particle is a functional of the distribution function. At temperatures close to $T = 0$ this relation is given by[2,4]

$$\varepsilon = \varepsilon(\mathbf{p}) + \int f(\mathbf{p}, \mathbf{p}')\,\nu\,d\tau' \qquad d\tau = 2d\mathbf{p}/(2\pi\hbar)^3, \tag{1}$$

where ν is the difference between the true distribution function and its value at $T = 0$ (f is Landau's function in reference 5). At sufficiently low temperatures the momentum dependence of the excitation energy may be of two types:

$$\varepsilon(\mathbf{p}) = a + |\mathbf{p} - \mathbf{p}_0|\,p_0/m, \tag{2}$$

where p_0 is the momentum limit, and a and m are constants (this is equivalent to the gas spectrum $\epsilon = p^2/2m$), or

$$\varepsilon(\mathbf{p}) = a + (\mathbf{p} - \mathbf{p}_0)^2/2m. \tag{3}$$

The type of spectrum cannot be determined from presently available experimental data; we note only that spectrum (2) seems more natural.

When liquid He^3 contains a small number of foreign atoms, such as a small admixture of He^4, in the Boltzmann case these atoms will cause excitations with the spectrum

$$E = \Delta + q^2/2M \tag{4}$$

or

$$E = \Delta + (q - q_0)^2/2M, \tag{5}$$

where $\mathbf{q}$ is the momentum of the impurity excitation; Δ is the zero-point energy; q_0 and M are experimentally determined parameters. (4) and (5) may be supported by the same line of reasoning that was used by Landau and Pomeranchuk[6] for the spectrum of impurities in liquid He II.

We shall consider the thermodynamic and kinetic phenomena in weak solutions* of He^4 in liquid He^3. In addition to the coefficients of viscosity η and thermal conductivity κ the new kinetic factors which appear are the diffusion coefficient D and thermal diffusion coefficient Dk_T, where k_T is the thermal diffusion ratio. These coefficients will, as usual,[7] be determined by means of the equations

$$\mathbf{i} = -\rho D\left(\nabla c + \frac{k_T}{T}\nabla T\right) \quad \text{for } \mathbf{g} = \mathbf{g}_i + \mathbf{g}_f = 0, \tag{6}$$

$$\mathbf{Q} = -\varkappa\nabla T \qquad\qquad \text{for } \mathbf{i} = 0, \tag{7}$$

where $\mathbf{i}$ is the impurity current, $\mathbf{Q}$ is the heat flux, $\mathbf{g}$ is the total momentum flux in the solution,

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*We are considering an unstratified region in which the solution is not a superfluid.

consisting of the momentum transported by impurity excitations ($\mathbf{g}_i$) and the momentum transported by Fermi excitations ($\mathbf{g}_f$).

It was shown in reference 8 that the collision integrals due to the scattering of Fermi excitations contain, instead of the usual deviation δn_f of the Fermi-excitation distribution from equilibrium, the effective expression

$$\overline{\delta n}_f = \delta n_f - \frac{\partial n}{\partial \varepsilon} \int \delta n_f f(\mathbf{p}, \mathbf{p}')\, d\tau'. \tag{8}$$

This same expression replaces the usual δn_f in equations for the different fluxes resulting from Fermi excitations. For example, the momentum flux of Fermi excitations, which determines the Fermi portion of the solution's viscosity, is given by

$$\Pi^f_{ik} = \int p_i \frac{\partial \varepsilon}{\partial p_k} \overline{\delta n}_f d\tau. \tag{9}$$

Similarly, the energy flux, which determines the Fermi part of the thermal conductivity, is given by

$$Q_f = \int \varepsilon \frac{\partial \varepsilon}{\partial \mathbf{p}} \overline{\delta n}_f d\tau. \tag{10}$$

Therefore the fact that $\overline{\delta n}_f$ is given by (8) appears nowhere in the calculations.

1. THERMODYNAMICS OF WEAK SOLUTIONS OF He^4 IN LIQUID He^3

We now determine the conditions under which dissolved He^4 atoms can be described by Boltzmann statistics. Departures from classical statistics arise at temperatures where either the quantum degeneracy of the impurity gas or the interaction between impurity excitations is important. The spin of the atoms is zero, so that the degeneracy temperature T_0 of the impurity gas for spectrum (4) is[9,10]

$$T_0 = (\hbar^2 n_0^{2/3} \bar{c}^{2/3} / kM)(3^{2/3}\pi / 2), \tag{11}$$

where $n_0 = \rho/m_3$ is the number of atoms of pure He^3 in 1 cm^3, m_3 is the He^3 atomic mass and $\bar{c}$ is the He^4 concentration. We shall compare (11) with temperatures at which interactions between impurities play a part. The interaction energy of impurity particles is $\sim U\bar{c}$, where U is the characteristic energy of interactions between helium atoms and is of the order of a few degrees. Degeneracy occurs before the interaction becomes prominent[10] when the inequality $(kT_0/U\bar{c})^{1/2} \sim \bar{c}^{-1/3} \gg 1$ is satisfied. When $\bar{c} < 10^{-2}$ we have $T_0 < 0.2°K$. Pomeranchuk[10] has shown for bosons and spectrum (5) that the degeneracy temperature is given by

$$kT_0 \sim 2\bar{c}^2 \pi^4 \hbar^6 n_0^2 / q_0^4 M. \tag{11a}$$

In this case the departure from classical statistics results from the interaction of impurities and appears at temperatures $\sim U\bar{c}/k$. Classical statistics will thus be applicable for $T \gtrsim 0.1°K$ with $\bar{c} \lesssim 10^{-2}$.

The chemical potential of Fermi excitations in a weak solution of He^4 in liquid He^3 has the standard form[11]

$$\mu_f = \mu_0 - kT\bar{c}, \tag{12}$$

where μ_0 is the chemical potential of pure liquid He^3. Khalatnikov and Abrikosov[3] have considered the thermodynamics of pure liquid He^3. The presence of impurities results in additional contributions to the free energy F, entropy S and specific heat C, which are calculated by means of (12).

For a spectrum of type (4) we have

$$F_i = -N_4 kT \ln\left[\frac{eV}{N_4}\left(\frac{MkT}{2\pi\hbar^2}\right)^{3/2} e^{-\Delta/kT}\right],$$

$$S_i = N_4 k\left\{\ln\left[\frac{V}{N_4}\left(\frac{MkT}{2\pi\hbar^2}\right)^{3/2}\right] + \frac{5}{2}\right\}, \quad C_i = \frac{3}{2} N_4 k,$$

where N_4 is the number of impurities in a volume V. When $V = 1$ N_4 is related to the concentrations $c = N_4 m_4/(N_3 m_3 + N_4 m_4)$ and $\bar{c}$ as follows:

$$c = N_4 m_4 / \rho = N_4 m_4 / N_3 m_3 = \bar{c} m_4 / m_3, \quad N_4 / N_3 \ll 1.$$

Here m_3 and m_4 are the masses of He^3 and He^4 atoms; N_3 is the number of He^3 atoms per unit volume. For the spectrum (5) we have

$$F_i = -N_4 kT \ln\left[\frac{eV}{N_4} \frac{q_0^2}{\hbar^3} \sqrt{\frac{MkT}{2\pi^3}}\, e^{-\Delta/kT}\right],$$

$$S_i = N_4 k\left\{\ln\left(\frac{V}{N_4} \frac{q_0^2}{\hbar^3} \sqrt{\frac{MkT}{2\pi^3}}\right) + \frac{3}{2}\right\}, \quad C_i = \frac{1}{2} N_4 k.$$

S_i and C_i are proportional to the concentration and are therefore small at temperatures ~ 1°. On the other hand, at temperatures ~ 0.1° they may have an important influence on the entropy and specific heat of the solution.

2. THE KINETIC EQUATION

The kinetic equation which determines the distribution function n of elementary excitations in a solution of He^4 in liquid He^3 is

$$\frac{\partial n}{\partial t} + \frac{\partial \varepsilon}{\partial \mathbf{p}} \frac{\partial n}{\partial \mathbf{r}} - \frac{\partial \varepsilon}{\partial \mathbf{r}} \frac{\partial n}{\partial \mathbf{p}} = J(n). \tag{13}$$

We regard the solution as having macroscopic gradients of thermodynamic quantities and small gradients of the velocity $\mathbf{u}$. In this case the distribution functions n_f and n_i will differ slightly from their equilibrium values, as follows:

$$n_{\mathrm{f}} = n_{\mathrm{f}0} + \delta n_{\mathrm{f}}, \qquad n_{\mathrm{i}} = n_{\mathrm{i}0} + \delta n_{\mathrm{i}}, \tag{14}$$

where

$$n_{\mathrm{f}0} = [\exp\{(\varepsilon - \mathbf{pu} - \mu)/kT\} + 1]^{-1}, \tag{15}$$

$$n_{\mathrm{i}0} = A(c, T)\exp\{-(E - \mathbf{qu})/kT\},$$

$$A(c, T) = (c\rho/m_4)(2\pi MkT)^{-3/2} \tag{16}$$

for the impurity spectrum (4).

We shall now express the left member of the kinetic equation in terms of gradients of T, c, and u. For this purpose we shall, as is customary,[12,4] substitute n from (14) and (15) into (13) and shall make use of thermodynamic identities and hydrodynamic equations. We shall also assume that at the considered point of the liquid $\mathbf{u} = 0$. The kinetic equation for Fermi excitations therefore becomes

$$-\frac{1}{2}\frac{\partial n_{\mathrm{f}0}}{\partial\varepsilon}\left(p_i\frac{\partial\varepsilon}{\partial p_k} - \frac{1}{3}p_l\frac{\partial\varepsilon}{\partial p_l}\delta_{ik}\right)\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial u_l}{\partial x_l}\right)$$

$$+\frac{\partial n_{\mathrm{f}0}}{\partial\varepsilon}kT\frac{\partial\varepsilon}{\partial\mathbf{p}}\frac{m_3}{m_4}\nabla c$$

$$-\frac{\partial n_{\mathrm{f}0}}{\partial\varepsilon}\frac{\partial\varepsilon}{\partial\mathbf{p}}\left(\frac{\varepsilon - \mu}{T} - s\right)\nabla T = J_{\mathrm{ff}} + J_{\mathrm{fi}} \tag{17}$$

and similarly, the kinetic equation for impurities is

$$\frac{1}{2}\frac{n_{\mathrm{i}0}}{kT}\left(q_i\frac{\partial E}{\partial q_k} - \frac{1}{3}q_l\frac{\partial E}{\partial q_l}\delta_{ik}\right)\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial u_l}{\partial x_l}\right)$$

$$+ n_{\mathrm{i}0}\frac{\partial E}{\partial\mathbf{q}}\frac{\nabla c}{c} + n_{\mathrm{i}0}\frac{\partial E}{\partial\mathbf{q}}\left(\frac{E}{kT} - \frac{3}{2}\right)\frac{\nabla T}{T}$$

$$+ n_{\mathrm{i}0}\frac{\partial E}{\partial\mathbf{q}}\frac{\nabla\rho}{\rho} = J_{\mathrm{if}} + J_{\mathrm{ii}}. \tag{18}$$

In (17) and (18) we have omitted the term resulting in second viscosity, which will not be considered below. Furthermore, without affecting accuracy we may omit the term in (18) that contains $\nabla\rho$, which is relatively small.* J_{ff} and J_{fi} are the collision integrals for the scattering of Fermi excitations on other Fermi excitations and on impurities. J_{if} and J_{ii} have analogous meanings. By means of the substitutions†

$$\delta\bar{n}_{\mathrm{f}} = -n_{\mathrm{f}0}(1 - n_{\mathrm{f}0})\psi, \qquad \delta n_{\mathrm{i}} = -n_{\mathrm{i}0}\varphi \tag{19}$$

we reduce the collision integrals to the standard forms

$$J_{\mathrm{ff}} = \int w_{\mathrm{ff}}\, n_{\mathrm{f}01}n_{\mathrm{f}02}(1 - n'_{\mathrm{f}01})(1 - n'_{\mathrm{f}02})(\psi_1 + \psi_2 - \psi'_1 - \psi'_2)$$

$$\times\,\delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}'_1 - \mathbf{p}'_2)\,\delta(\varepsilon_1 + \varepsilon_2 - \varepsilon'_1 - \varepsilon'_2)\,d\tau_2 d\tau'_2 d\mathbf{p}'_2,$$

$$J_{\mathrm{fi}} = \int w_{\mathrm{fi}}\, n_{\mathrm{f}0}n_{\mathrm{i}0}(1 - n'_{\mathrm{f}0})(\psi + \varphi - \psi' - \varphi')$$

$$\times\,\delta(\mathbf{p} + \mathbf{q} - \mathbf{p}' - \mathbf{q}')\,\delta(\varepsilon + E - \varepsilon' - E')\,d\tau' d\mathbf{q} d\mathbf{q}',$$

$$J_{\mathrm{if}} = \int w_{\mathrm{fi}}\, n_{\mathrm{f}0}n_{\mathrm{i}0}(1 - n'_{\mathrm{f}0})(\psi + \varphi - \psi' - \varphi')$$

$$\times\,\delta(\mathbf{p} + \mathbf{q} - \mathbf{p}' - \mathbf{q}')\,\delta(\varepsilon + E - \varepsilon' - E')\,d\tau d\mathbf{p}' d\mathbf{q}',$$

$$J_{\mathrm{ii}} = \int w_{\mathrm{ii}}\, n_{\mathrm{i}0}n_{\mathrm{i}01}(\varphi + \varphi_1 - \varphi' - \varphi'_1)\,\delta(\mathbf{q} + \mathbf{q}_1 - \mathbf{q}' - \mathbf{q}'_1)$$

$$\times\,\delta(E + E_1 - E' - E'_1)\,d\mathbf{q}_1 d\mathbf{q}' d\mathbf{q}'_1. \tag{20}$$

We also require the cross sections for the scattering of impurities on Fermi excitations and on other impurities. The interaction law of these excitations is, of course, unknown, but just as in the theory[12] of weak solutions of He^3 in He II, it is reasonable for specific estimates to use a δ-function interaction law:

$$V_{\mathrm{fi}} = v_{01}\,\delta(\mathbf{x} - \mathbf{y}), \quad V_{\mathrm{ii}} = v_{02}\,\delta(\mathbf{y}_1 - \mathbf{y}_2),$$

where **x** is the coordinate of a Fermi excitation and **y** is the impurity coordinate. The unknown constants v_{01} and v_{02} are determined by comparing theory with experiment. The cross sections are calculated as in the theory of slow-neutron scattering by atoms,[12,13] and we obtain

$$\sigma_{\mathrm{fi}} = \left(\frac{mM}{m + M}\right)^2\frac{|v_{01}|^2}{\pi\hbar^4}, \tag{21}$$

$$\sigma_{\mathrm{ii}} = M^2|v_{02}|^2/\pi\hbar^4. \tag{22}$$

We also obtain the cross section for the scattering of an impurity on a Fermi particle with the spectrum (3). This is analogous to the scattering of an impurity on a roton and the total cross section is correspondingly[12]

$$\sigma_{\mathrm{fi}} = \frac{|v_{01}|^2 M}{2\pi\hbar^4}\frac{Mm}{m + M}. \tag{21a}$$

3. DIFFUSION

We consider first the diffusion of impurities. The kinetic equations (17) and (18) for Fermi excitations and impurities were derived above; with a non-zero concentration gradient in the system they are

$$kT\frac{\partial n_{\mathrm{f}0}}{\partial\varepsilon}\frac{\partial\varepsilon}{\partial\mathbf{p}}\frac{m_3}{m_4}\nabla c = J_{\mathrm{ff}} + J_{\mathrm{fi}}, \tag{22}$$

$$n_{\mathrm{i}0}\frac{\partial E}{\partial\mathbf{q}}\frac{\nabla c}{c} = J_{\mathrm{if}} + J_{\mathrm{ii}}. \tag{23}$$

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*The independent variables are actually c and T. Therefore $\nabla\rho = (\partial\rho/\partial c)\nabla c + (\partial\rho/\partial T)\nabla T$.

†We shall hereinafter omit the bar above δn_f.

An exact general solution of the problem is hardly possible. We shall therefore consider the two limiting cases; results in the intermediate region can then be obtained by interpolation. The first limiting case is represented by the high-temperature region, where the effective number of excitations in the diffuse Fermi zone is much greater than the number of impurities. In the low-temperature region, on the other hand, the situation is reversed and the effective number of excitations in the diffuse Fermi zone is much smaller than the number of impurities.

It follows from symmetry that the deviations (19) of the distribution functions from equilibrium may be sought in the forms

$$\varphi = a(E)(\mathbf{q}\nabla c), \qquad \psi = b(\varepsilon)(\mathbf{p}\nabla c). \tag{24}$$

It is important here that the functions a and b are independent of angles. Spectrum (2) for Fermi excitations and spectrum (4) for impurity particles will be considered in greatest detail.

The High-Temperature Region. There is a relatively small number of impurity particles; collisions among these are unlikely and insignificant. The concentration region where this occurs is given more precisely by the condition $t_{ff} \ll t_{fi}$, i.e., the effective time for a collision between two Fermi particles is much shorter than the effective time for a collision between a Fermi particle and an impurity. The kinetic equation for impurities is therefore greatly simplified. Simple calculations show that the Fermi distribution function deviates considerably less from equilibrium than the impurity distribution function. The condition $t_{fi} \ll t_{fi}$ is sufficient so that in the mixed collision integrals J_{if} and J_{fi} in (20) we can neglect ψ compared with φ. Furthermore, the momenta of excited Fermi particles are much greater than those of impurities; we may therefore neglect the variation in the absolute value of the relative momentum of colliding particles. From all of these consideration we obtain

$$n_{i0}\frac{\partial E}{\partial \mathbf{q}}\frac{\nabla c}{c} = J_{if} = n_{i0}a(\mathbf{q}\nabla c)\int \sigma^*_{if}\frac{\partial \varepsilon}{\partial p}n_{0f}(1-n_{0f})\,d\tau_f,$$

where σ^*_{if} is the transport cross section for the scattering of an impurity on a Fermi excitation;

$$\sigma^*_{if} = -\frac{1}{2}\int \sigma_{if}(1-\cos\psi)\,d\cos\psi.$$

Here ψ is the angle between the vectors $\mathbf{q}$ and $\mathbf{q}'$. According to (21) σ^*_{if} does not depend on the energy of the colliding particles; we therefore obtain

$$\delta n_i = -\frac{n_{i0}}{c}\left\{\sigma^*_{if}(MkT)\left[\frac{\partial \varepsilon}{\partial p}\frac{d\tau_f}{d\varepsilon}\right]_{\varepsilon=\mu}\right\}^{-1}(\mathbf{q}\nabla c).$$

With the aid of δn_i it is easy to calculate the impurity current

$$\mathbf{i} = \int \mathbf{q}\,\delta n_i\,d\mathbf{q} = -\frac{\rho}{m_4\sigma^*_{if}(\partial\tau_f/\partial p)_{\varepsilon=\mu}}\nabla c. \tag{25}$$

A comparison of (25) with (6) yields the diffusion coefficient in this limiting case:

$$D = \frac{1}{m_4\sigma^*_{if}}\frac{1}{(\partial\tau_f/\partial p)_{\varepsilon=\mu}} = \pi^2\left(\frac{\hbar}{p_0}\right)^2\frac{\hbar}{\sigma^*_{if}m_4} = \frac{2kT}{3m_4}t_{if}, \tag{26}$$

where

$$t_{if}^{-1} = {}^3/_2\,\sigma^*_{if}kTN_3v_f/\mu \tag{27}$$

is the effective time for the scattering of impurities on Fermi particles; $v_f = p_0/m$; $N_f = {}^3/_2 N_3kT/\mu$ is the number of Fermi particles in the diffuse zone. As already noted, the high-temperature region is given by the condition $t_{ff} \ll t_{fi}$, i.e.,

$$\bar{c} \ll (T/T_0)^2, \qquad kT_0 = \mu. \tag{28}$$

According to (26) and (27), when (28) is satisfied the diffusion coefficient for a weak solution of He^4 in liquid He^3 is constant, being independent of both concentration and temperature.

The Low-Temperature Region. We now consider the opposite limit, when the number of impurity excitations is much greater than the effective number of excitations in the diffuse Fermi zone. This condition is represented more exactly by $t_{ii} \ll t_{if}$, where $t_{ii}^{-1} = \sigma^*_{ii}N_4\bar{v}_i$, $t_{if}^{-1} = \sigma^*_{fi}N_fv_f$ are the effective times for the scattering of impurities and Fermi excitations on impurities. The low-temperature region is therefore represented by the inequality

$$\bar{c} \gg (T/T_0)v_f/\bar{v}_i,\ v_f = p_0/m,\ \bar{v}_i \sim 2(2kT/\pi M)^{1/2} \tag{28a}$$

In this case the impurity distribution function departs from equilibrium much less than the Fermi function. Therefore φ may be neglected by comparison with ψ in the collision integrals J_{if} and J_{fi} in (20). To determine ψ we use the kinetic equation for Fermi excitations:

$$\frac{\partial n_{f0}}{\partial \varepsilon}kT\frac{\partial \varepsilon}{\partial \mathbf{p}}\frac{m_3}{m_4}\nabla c = J_{fi} + J_{ff}. \tag{29}$$

The two ways of determining ψ yield identical results: 1) by omitting J_{ff} in (29) as a small quantity because $t_{fi} \ll t_{ff}$; 2) by multiplying (29) by the Fermi-particle momentum $\mathbf{p}$ and integrating with respect to $d\tau_p$. Since $\int \mathbf{p}J_{ff}d\tau_p = 0$

we thus eliminate J_{ff}. We shall use the first method. In virtue of what has been stated above in connection with the solution in the high-temperature region, we now obtain

$$J_{\mathrm{fi}} = -kT \frac{\partial n_{\mathrm{f0}}}{\partial \varepsilon} b\, (\mathbf{p}\nabla c)\, N_4 \sigma^*_{\mathrm{fi}} \frac{\partial \varepsilon}{\partial p}.$$

Hence

$$\delta n_{\mathrm{f}} = -\frac{m_3}{m_4}\frac{kT}{m}\frac{\partial n_{\mathrm{f0}}}{\partial \varepsilon}\frac{1}{\sigma^*_{\mathrm{fi}} v_{\mathrm{f}} N_4}(\mathbf{p}\nabla c). \qquad (30)$$

Using the momentum conservation law $\mathbf{g}_i + \mathbf{g}_f = 0$, we obtain for the impurity current

$$\mathbf{g}_{\mathrm{i}} = -\int \mathbf{p}\,\delta n_{\mathrm{f}}\, d\tau_{\mathrm{f}} = -\rho D \nabla c. \qquad (31)$$

(30) and (31) are used to give the diffusion coefficient

$$D = (km / p_0 \sigma^*_{\mathrm{fi}}\rho)\, T / c = (kT / m_4)\, t_{\mathrm{fi}}, \qquad (32)$$

where

$$t_{\mathrm{fi}}^{-1} = \sigma^*_{\mathrm{if}} N_4 v_{\mathrm{f}}. \qquad (33)$$

It is easily seen from (26) – (28) and (32) — (33) that an interpolation formula for the diffusion coefficient which includes both limiting cases is given by

$$D = tkT / m_4, \qquad \text{where } 1/t = (3/2t_{\mathrm{if}} + 1/t_{\mathrm{fi}}).$$

We may treat analogously the case in which Fermi excitations are described by spectrum (3) and impurities by spectrum (4). The cross section (21a) must now be used; omitting the steps of the calculation we give the results: a) for high temperatures

$$D = \frac{4}{3}\sqrt{\frac{2}{\pi}}\left(\frac{k}{M}\right)^{1/2}\left(\frac{m_3}{\rho\sigma^*_{\mathrm{if}}}\right)\frac{\mu_{\mathrm{lim}}}{k}T^{-1/2}, \qquad (34)$$

b) for low temperatures

$$D = \frac{1}{6}\frac{k}{\mu_{\mathrm{lim}}}\frac{p_0}{\rho\sigma^*_{\mathrm{if}}}\frac{T}{c}. \qquad (35)$$

In (34) and (35) μ_{lim} denotes the Fermi energy limit for spectrum (3):

$$\mu_{\mathrm{lim}} = [\rho(2\pi\hbar)^2 / m_3 16\pi p_0^2 \sqrt{2m}]^2.$$

In liquid He^3, $\mu_{\mathrm{lim}} = 1.05°\mathrm{K}$ when this spectrum is realized.[3]

4. THERMAL DIFFUSION

With non-zero concentration and temperature gradients in the solution at constant pressure, the impurity current **i** is defined by (6). The system of kinetic equations for thermal diffusion is

$$n_{\mathrm{i0}}\frac{\partial E}{\partial \mathbf{q}}\left(\frac{E}{kT} - \frac{3}{2}\right)\frac{\nabla T}{T} = J_{\mathrm{if}} + J_{\mathrm{ii}},$$

$$-\frac{\partial n_{0\mathrm{f}}}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial \mathbf{p}}\left(\frac{\varepsilon - \mu}{T} - s\right)\nabla T = J_{\mathrm{ff}} + J_{\mathrm{fi}}.$$

The thermal diffusion ratio k_T is calculated just like the diffusion coefficient; for spectrum (2) we obtain $k_T = c$. Thus the thermal diffusion coefficient is smaller than the diffusion coefficient by a factor equal to the concentration.

5. VISCOSITY

At sufficiently low temperatures, for which alone the present theory is valid, the viscosity coefficient η of the solution will consist of the coefficient η_f of Fermi viscosity, resulting from momentum transfer by Fermi particles, and the coefficient η_i of impurity viscosity, resulting from momentum transfer by impurity particles:

$$\eta = \eta_{\mathrm{f}} + \eta_{\mathrm{i}}. \qquad (36)$$

a) Fermi Viscosity of the Solution

The kinetic equation for Fermi excitations when the system contains non-zero gradients of the macroscopic velocity **u** is

$$-\frac{1}{2}\frac{\partial n_{0\mathrm{f}}}{\partial \varepsilon}\left(p_i\frac{\partial \varepsilon}{\partial p_k} - \frac{1}{3}p_l\frac{\partial \varepsilon}{\partial p_l}\delta_{ik}\right)$$
$$\times\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\frac{\partial u_l}{\partial x_l}\delta_{ik}\right) = J_{\mathrm{ff}} + J_{\mathrm{fi}}.$$

We choose our z axis in the direction of the velocity and let the velocity gradient be perpendicular to the z axis along the x axis. In spherical coordinates with a polar z axis the kinetic equation then becomes

$$\frac{\partial n_{0\mathrm{f}}}{\partial \varepsilon} p \frac{\partial \varepsilon}{\partial p}\frac{\partial u}{\partial x}\cos\theta\sin\theta\cos\varphi = J_{\mathrm{ff}} + J_{\mathrm{fi}}.$$

In accordance with the symmetry of the problem the solution of the kinetic equation will be sought in the form

$$\psi = p\frac{\partial \varepsilon}{\partial p}\frac{\partial u}{\partial x}\cos\theta\sin\theta\cos\varphi\, b(\varepsilon).$$

Since a general solution is hardly possible, we shall determine the dependence of n_f on temperature and concentration in the two limiting cases. 1) At high temperatures the number of Fermi particles in the diffuse zone is given by $N_f \gg N_4$ (more exactly: $t_{ff} \ll t_{fi}$, $t_{if} \ll t_{ii}$). In this case only the scattering of Fermi particles by each other is important and J_{fi} may be omitted from the kinetic equation. 2) At low temperatures the number of Fermi particles in the diffuse zone is given by $N_f \ll N_4$. Here only collisions between Fermi particles and impurities are important and collisions between Fermi particles may be neglected. The intermediate region can be obtained by interpolating between the two limiting cases.

The High-Temperature Region. The collision integral J_{ff} in (20) is

$$J_{ff} = \int w_{ff}\, n_{f01} n_{f02} (1 - n'_{f01})(1 - n'_{f02})(\psi_1 + \psi_2 - \psi'_1 - \psi'_2) \times \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}'_1 - \mathbf{p}'_2)\,\delta(\varepsilon_1 + \varepsilon_2 - \varepsilon'_1 - \varepsilon'_2)\, d\tau_2 d\tau'_1 d\mathbf{p}'_2. \quad (37)$$

If we assume that the collision probability w_{ff} depends only on the angles between momenta, by transforming to dimensionless variables in the collision integral we find that it can be represented by

$$J_{ff} = \overline{\sigma^*_{ff}}\, n_{f01}(1 - n_{f01}) \times \cos\theta \sin\theta \cos\varphi \frac{\partial u}{\partial x} p \frac{\partial \varepsilon}{\partial p} \int b v_f n_{0f} (1 - n_{0f})\, d\tau, \quad (38)$$

where

$$\overline{\sigma^*_{ff}} = (kT/\mu)\, \overline{\sigma^*_{ff,0}}, \quad (39)$$

in which $\overline{\sigma^*_{ff0}}$ is the viscosity transport cross section.[4][*] We shall, as usual, calculate the z component of the momentum transported in the x direction:

$$\Pi^f_{xz} = -\int p \frac{\partial \varepsilon}{\partial p} \cos\theta \sin\theta \cos\varphi\, \delta n\, d\tau = \eta_f \frac{\partial u}{\partial x}. \quad (40)$$

The result obtained is (see reference 14)

$$\eta_{fh} = \frac{2}{15}\left(p_0 / \overline{\sigma^*_{ff\,0}}\right) \left(\frac{\mu}{k}\right)^2 T^{-2} = \frac{1}{5} N_3 \frac{p_0^2}{m} \bar{t}_{ff}. \quad (41)$$

We have thus obtained the Pomeranchuk temperature law $\eta \sim T^{-2}$, as is not surprising since our initial premises agree with those of Pomeranchuk.

The Low-Temperature Region. Here only the scattering of Fermi particles by impurities is important; we therefore omit J_{ff} in the kinetic equation for Fermi excitations. In calculating J_{fi} we also consider that in this case $\delta n_i \ll \delta n_f$, and that therefore in J_{fi} n_i may be replaced by the equilibrium Maxwellian distribution function n_{i0}. As a result we have

$$J_{fi} = v_f \overline{\sigma^*_{fi}}\, n_{f0} (1 - n_{f0})\, p \frac{\partial \varepsilon}{\partial p} \frac{\partial u}{\partial x} \cos\theta \sin\theta \cos\varphi N_4 b.$$

In this case the effective deviation of the Fermi distribution function from equilibrium will be

$$\delta n_f = -\frac{1}{kT} n_{f0} (1 - n_{f0})\, p \frac{\partial \varepsilon}{\partial p} \frac{\partial u}{\partial x} \bar{t}_{fi} \cos\theta \sin\theta \cos\varphi, \quad (42)$$

where

$$\bar{t}_{fi} = 1 / \overline{\sigma^*_{if}} N_4 v_f. \quad (43)$$

Here $\overline{\sigma^*_{fi}}$ is the viscosity transport cross section. The ordinary transport cross section which is obtained when diffusion and thermal conductivity are considered is denoted by the same symbol without a bar. We use (42) and (40) to determine the Fermi part of the viscosity coefficient at low temperatures:

$$\eta_{fi} = p_0 / 5 \overline{\sigma^*_{fi}}\, \bar{c} = N_3 p_0^2 \bar{t}_{fi} / 5m. \quad (44)$$

We note that at low temperatures η_f is inversely proportional to the concentration.

An interpolation formula for η_f which gives correct values in both limiting cases is

$$\eta_f = N_3 p_0^2 \bar{t}_f / 5m, \quad 1/\bar{t}_f = (1/\bar{t}_{ff} + 1/\bar{t}_{fi}). \quad (45)$$

b) Impurity Viscosity of the Solution

The impurity viscosity of the solution is calculated in exactly the same manner as the Fermi part of the viscosity coefficient. The interpolation formula for η_i which gives correct values at the high-temperature and low-temperature limits is therefore

$$\eta_i = (c\rho / m_4)\, kT \bar{t}_i, \quad \bar{t}_i^{-1} = (\bar{t}_{if}^{-1} + \bar{t}_{ii}^{-1}),$$

where

$$1/\bar{t}_{ii} = \bar{v}_i \overline{\sigma^*_{ii}} N_4, \quad 1/\bar{t}_{if} = v_f \overline{\sigma^*_{if}} N_f,$$

$$N_f = (3T / 2T_0) N_3. \quad (46)$$

In all of the foregoing calculations σ_{ff0}, σ^*_{if} and σ_{ii} [(21 and (22)] were constants, being independent of momentum. In the theory that has been developed here these quantities are unknown and are determined and normalized by comparing the theory with experiment. For a preliminary estimate of the contributions made by different processes to the viscosity of the solution we set

$$\overline{\sigma^*_{ff\,0}} \sim a_1 \cdot 10^{-15}\ \text{cm}^2, \quad \overline{\sigma^*_{if}} \sim a_2 \cdot 10^{-15}\,\text{cm}^2,$$

$$\overline{\sigma^*_{ii}} \sim a_3 \cdot 10^{-15}\ \text{cm}^2,$$

where a_1, a_2, and a_3 are constants of the order of unity.

Furthermore,[3]

$$\mu / k = p_0^2 / 2mk = 3.3^\circ\text{K}, \quad m = 1.43\ m_3,$$

$$p_0 / \hbar = 0.76 \cdot 10^{-8}\ \text{cm}^{-1}, \quad \rho = 0.078\ \text{g/cm}^3.$$

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[*] Unlike the ordinary transport cross section (which we denote by σ^*), the vicosity transport cross section $\overline{\sigma^*}$ is obtained by averaging the corresponding cross sections multiplied by $(1 - \cos\psi)^2 \sin^2\varphi$. This is shown in greater detail in reference 14.

Inserting these numerical values into (36), (45), and (46), we obtain the following expression for the viscosity of the solution:

$$\eta = 1.24\cdot 10^{-4}/a_1T^2\{1+7.2\,(a_2/a_1)\,cT^{-2}\}$$
$$+\,2.1\cdot 10^{-5}/a_2\{1+1.1\,(a_3/a_2)\,cT^{-1/2}\}.$$

6. THERMAL CONDUCTIVITY

The thermal conductivity coefficient κ of the solution, like the viscosity coefficient, includes Fermi and impurity parts:

$$\varkappa = \varkappa_f + \varkappa_i. \qquad (47)$$

The system of kinetic equations required to determine the thermal conductivity is the same as that used in discussing thermal diffusion and is given in Sec. 4. In accordance with the symmetry of the problem we seek a solution in the form

$$\varphi = \left[E/kT - \frac{3}{2}\right]\left(\frac{\partial E}{\partial \mathbf{q}}\frac{\nabla T}{T}\right)a, \ \ \psi = \left\{\frac{\varepsilon-\mu}{T} - s\right\}\left(\frac{\partial \varepsilon}{\partial \mathbf{p}}\nabla T\right)b.$$

Further calculations are entirely analogous to those performed for the viscosity. We first calculate the Fermi part of the thermal conductivity coefficient in the high-temperature and low-temperature limits (κ_{fh} and κ_{fl}, respectively). An interpolation formula for κ_f is then constructed. The impurity part κ_i of the thermal conductivity coefficient is calculated similarly. The results are

$$\varkappa_{fh} = {}^5/_6C_0(p_0/m)^2 t_{ff}, \ \ \varkappa_{fl} = {}^5/_6C_0(p_0/m)^2 t_{fi},$$
$$\varkappa_{ih} = 5\,(\rho k^2/m_4M)\,cTt_{if}, \ \ \varkappa_{il} = 5\,(\rho k^2/m_4M)\,cTt_{ii}.$$

Interpolation formulas for κ_f and κ_i are

$$\varkappa_f = {}^5/_6\,C_0(p_0/m)^2 t_f, \ \ t_f^{-1} = t_{ff}^{-1} + t_{fi}^{-1},$$
$$\varkappa_i = 5\,(\rho k^2/m_4M)\,cTt_i, \ \ t_i^{-1} = t_{if}^{-1} + t_{ii}^{-1},$$

where C_0 is the specific heat of a unit volume of pure liquid He^3 and the other quantities have the same meanings as in (45) and (46) for the viscosity. The absence of bars indicates that we are here using the ordinary transport cross sections.[4] We note that at the high-temperature limit Pomeranchuk's law[1] for the thermal conductivity, $\kappa_f \sim T^{-1}$, is obtained.

In conclusion the authors wish to express their deep gratitude to B. I. Davydov for a valuable discussion.

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Translated by I. Emin
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