SECOND FORBIDDEN COULOMB β-DECAY TRANSITIONS

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From a classification of the terms appearing in the series expansion of the β-decay interaction Hamiltonian, rules are obtained which make it possible to predict which nuclear matrix elements contribute to β transitions of a given type, and to estimate the values of these elements. It is shown that there exists a simple case of second forbidden Coulomb transitions, $\Delta j = 2$ (no), with properties similar to those of unique transitions. The angular β-ν correlation, the β-γ correlation with circularly polarized γ quantum, and the spectrum are found for this case.

THE study of the angular correlations in the general case of β decay is extremely complicated, since the theoretical formulas contain a large number of unknown nuclear matrix elements. The special cases in which the number of nuclear matrix elements is a minimum are the simplest from the point of view of the interpretation of the experimental results. Many papers have been devoted to the study of the correlations in allowed transitions, in which there are not more than two such unknown matrix elements. In particular, the angular β-ν correlation and the β-γ correlation with circularly polarized γ-ray quantum have been analyzed in a number of papers.[1-4] Cases of interest among the forbidden β-decay transitions are those of unique transitions and of the so-called first-forbidden Coulomb transitions. In the former case, the number of unknown nuclear matrix elements is one, and in the latter case, three, so that in the former case the correlations do not depend on the matrix elements at all, and in the latter case they depend on two ratios of matrix elements. These cases have been treated in a number of papers.[5-9]

In the present paper we derive simple rules that make it possible to predict which nuclear matrix elements will contribute to β-decay transitions of a given type, and to get approximate values of these elements. Second forbidden transitions are studied on the basis of these rules. It is shown that there is a simple case of Coulomb transitions, $\Delta j = 2$ (no), with properties similar to those of the unique transitions. The angular β-ν correlation and the β-γ correlation with circularly polarized γ-ray quantum are found for this case.

1. THE NUCLEAR MATRIX ELEMENTS

Let us take the Hamiltonian of the β-decay interaction in the form proposed by Gell-Mann and Feynman[10]

$$H = G\,[\overline{\Psi}_p \gamma_\mu (1 + \lambda\gamma_5)\Psi_n]\left[\overline{\Phi}_e \gamma_\mu \frac{1}{\sqrt{2}}(1 + \gamma_5)\Phi_\nu\right]. \tag{1}$$

In the Coulomb field of a nucleus of charge Z, the wave function of an electron with momentum $\mathbf{p}$, total energy E, and spin component ξ can be represented as the sum of two bispinors:

$$\Phi_{\mathbf{p},\xi} = 4\pi\sum_{j\mu}\begin{pmatrix}\sqrt{\frac{E+1}{2E}}N_j\left[(A_j+B_j)\,\Omega_{j\mu}^{(l)}\left(\frac{\mathbf{r}}{r}\right)(\Omega_{j\mu}^{(l)}(\mathbf{n}))_\xi^* + (A_j'+B_j)\,\Omega_{j\mu}^{(l')}\left(\frac{\mathbf{r}}{r}\right)(\Omega_{j\mu}^{(l')}(\mathbf{n}))_\xi^*\right]\\ \sqrt{\frac{E-1}{2E}}N_j\left[(A_j-B_j)\,\Omega_{j\mu}^{(l)}\left(\frac{\mathbf{r}}{r}\right)(\Omega_{j\mu}^{(l')}(\mathbf{n}))_\xi^* + (A_j'-B_j)\,\Omega_{j\mu}^{(l')}\left(\frac{\mathbf{r}}{r}\right)(\Omega_{j\mu}^{(l)}(\mathbf{n}))_\xi^*\right]\end{pmatrix}$$

$$l = j - {}^1/_2, \qquad l' = j + {}^1/_2, \qquad \mathbf{n} = \mathbf{p}/p;$$

$$A_j = \gamma_j + i\frac{\alpha ZE}{p} + \frac{\gamma_j - i\alpha ZE/p}{\gamma_j + i\alpha ZE/p}(j + {}^1/_2), \qquad B_j = \frac{\gamma_j - i\alpha ZE/p}{\gamma_j + i\alpha ZE/p}\,i\,\frac{\alpha ZE}{p}, \qquad A_j' = \gamma_j + i\frac{\alpha ZE}{p} - \frac{\gamma_j - i\alpha ZE/p}{\gamma_j + i\alpha ZE/p}(j + {}^1/_2),$$

$$N_j = \frac{2\Gamma(\gamma_j + i\alpha ZE/p)}{\Gamma(2\gamma_j + 1)}e^{\pi\alpha ZE/2p}(2pr)^{\gamma_j - 1}e^{i\pi(\gamma_j - 1)/2}, \qquad \gamma_j = \sqrt{(j + {}^1/_2)^2 - (\alpha Z)^2}, \qquad \alpha = 1/137. \tag{2}$$

The first bispinor contains $\Omega_{j\mu}^{(l)}(\mathbf{r}/r)$ and can be regarded as the term corresponding to the emission of the electron with total angular momentum j and orbital angular momentum $l = j - \frac{1}{2}$. The second contains $\Omega_{j\mu}^{(l')}(\mathbf{r}/r)$ and can be regarded as the term corresponding to the emission of the electron with orbital angular momentum $l' = j + \frac{1}{2}$.

For $Z \to 0$, the terms of the second type go to zero, and the terms of the first type go over into the usual expansion of a plane wave in spherical functions. Similarly, the wave function of a neutrino with momentum $\mathbf{q}$ and spin component η is

$$\Phi_{\mathbf{q},\eta} = 4\pi \sum_{j_\nu \nu} \frac{1}{\sqrt{2}} (2j_\nu + 1) M_{j_\nu} \begin{pmatrix} \Omega_{j_\nu\nu}^{(l_\nu)}(\mathbf{r}/r)\,[\Omega_{j_\nu\nu}^{(l_\nu)}(\mathbf{v})]_\eta^* \\ \Omega_{j_\nu\nu}^{(l_\nu)}(\mathbf{r}/r)\,[\Omega_{j_\nu\nu}^{(l'_\nu)}(\mathbf{v})]_\eta^* \end{pmatrix};$$

$$l_\nu = j_\nu - {}^1\!/_2, \qquad l'_\nu = j_\nu + {}^1\!/_2, \qquad \mathbf{v} = \mathbf{q}/q;$$

$$M_{j_\nu} = \frac{2\Gamma(j_\nu + {}^1\!/_2)}{\Gamma(2j_\nu + 2)} (2qr)^{j_\nu - {}^1\!/_2} \exp\{i\pi(j_\nu - {}^1\!/_2)/2\}. \tag{3}$$

The substitution of Eqs. (2) and (3) in Eq. (1) breaks the Hamiltonian up into a sum of terms; each contains one term from the expansion of the electron wave function and one from the neutrino function, and can be characterized by the two quantum numbers (j, l) of the electron and the two numbers (j_ν, l_ν) for the neutrino. In addition, the terms differ in the part referring to the nucleus. Here they are divided into the nonrelativistic terms with the operators γ_4 and $\boldsymbol{\gamma}\gamma_5$ between the wave functions of the initial and final states of the nucleus, and the relativistic terms, which have the operators $\gamma_4\gamma_5$ and $\boldsymbol{\gamma}$; the latter are of the order $v_{\rm nuc}/c$ compared with the former, where $v_{\rm nuc}$ is the average speed of the nucleons in the nucleus. The parity is $(-1)^{l+l_\nu}$ for the nonrelativistic terms and $(-1)^{l+l_\nu+1}$ for the relativistic terms.

Thus all the terms of the Hamiltonian can be classified in terms of four quantum numbers and the parity (cf. table). Their values can be found

j	l or l'	j_ν	l_ν	Parity	Estimate of value
1/2	0	1/2	0	no	1
3/2	1	1/2	0	yes	pR
1/2	1	1/2	0	yes	αZ
1/2	0	3/2	1	yes	qR
1/2	0	1/2	0	yes	$v_{\rm nuc}/c$
5/2	2	1/2	0	no	$(pR)^2$
3/2	1	3/2	1	no	$pR\,(qR)$
1/2	0	5/2	2	no	$(qR)^2$
3/2	2	1/2	0	no	$\alpha Z\,(pR)$
1/2	1	3/2	1	no	$\alpha Z\,(qR)$
3/2	1	1/2	0	no	$(v_{\rm nuc}/c)\,(pR)$
1/2	0	3/2	1	no	$(v_{\rm nuc}/c)\,(qR)$
1/2	1	1/2	0	no	$(v_{\rm nuc}/c)\,\alpha Z$

approximately with the factor $N_j M_{j_\nu}$. It is easy to see that as to their values, the terms are naturally divided into three groups: the ordinary type, the Coulomb type ($\sim \alpha Z$), and the relativistic terms ($v_{\rm nuc}/c$).

Each term of the expansion of the Hamiltonian can lead to several nuclear matrix elements. This is due to the fact that the classification has been carried out in terms of the quantum numbers of the individual particles (electron and neutrino), whereas the nuclear matrix element is characterized by the set of quantum numbers of the pair as a whole. In fact in the general case, a nuclear matrix element can be written

$$(j_2 \,|\, \overline{\Psi}_{j_2m_2} O_s \Psi_{j_1m_1} r^k Y_{LM}^* \,|\, j_1)_J \equiv \langle O_s r^k Y_{LM}^* \rangle_J. \tag{4}$$

Such "reduced" matrix elements are connected with the usual expressions written as integrals by the relation

$$\int \overline{\Psi}_{j_2m_2}(\mathbf{r})\, O_s \Psi_{j_1m_1}(\mathbf{r})\, r^k Y_{LM}^*\left(\frac{\mathbf{r}}{r}\right) d\mathbf{r} = \frac{1}{\sqrt{4\pi}} \sum_{JN} C_{j_1m_1}^{j_2m_2JN} C_{JN}^{LMS\sigma} \langle O_s r^k Y_{LM} \rangle_J. \tag{5}$$

Here $O_s = \gamma_4$, $\gamma_4\gamma_5$, $\boldsymbol{\gamma}$, and $\boldsymbol{\gamma}\gamma_5$, with the first two of these operators corresponding physically to the emission of the electron-neutrino pair in a singlet state, $S = 0$, and the other two to emission of the pair in a triplet state, $S = 1$. The order L of the spherical harmonic represents the orbital momentum of the pair, and J its total angular momentum, which is formed from S and L by the vector-addition rule $\mathbf{J} = \mathbf{L} + \mathbf{S}$, to which there corresponds the sum over $\mathbf{J}$ in Eq. (5). The parity of the nuclear matrix element is $(-1)^L$ for nonrelativistic elements and $(-1)^{L+1}$ for relativistic elements and, as will be seen shortly, is the same as the parity of the term in the expansion of the Hamiltonian that leads to the matrix element in question. The degree k is the power of R in the estimate given in the table for this term.

Thus any matrix element is fully characterized by the set of quantum numbers J, L, S. This set can be found from the quantum numbers of the electron and neutrino by the rules of vector addition. For this relationship we must require that the triangle rule hold for all rows and columns of the array

$$\begin{pmatrix} l & {}^1\!/_2 & j \\ l_\nu & {}^1\!/_2 & j_\nu \\ L & S & J \end{pmatrix}.$$

Furthermore, the additional condition $l + l_\nu + L$ = (even number), must be satisfied. By using these rules we can find for any term in the table the nuclear matrix elements that correspond to it.

A nuclear matrix element of a definite type contributes to the β-decay transition if two conditions are satisfied:

$$|j_2 - j_1| \leqslant J \leqslant j_2 + j_1,$$

$$\Pi = \begin{cases}(-1)^L & \text{for nonrelativistic nuclear matrix elements,}\\ (-1)^{L+1} & \text{for relativistic elements,}\end{cases}$$

where j_1 is the angular momentum of the initial nucleus, j_2 is that of the final nucleus, and Π is the change of parity in the β decay.

For example, the term of order αZ (qR) (no) leads to four types of nuclear matrix elements; two of them, with $J = 1$, give small corrections to allowed transitions, and the other two, with $J = 2$, contribute to forbidden transitions $\Delta j = 2$ (no).

2. THE SECOND-FORBIDDEN COULOMB TRANSITIONS

As has already been remarked, all the terms of the expansion of the Hamiltonian (1) fall naturally into three groups: terms of the ordinary type, those of the Coulomb type, and relativistic terms. The contribution of terms of the Coulomb type rises rapidly with increase of the charge of the nucleus, so that even for nuclei with $Z \geq 30$, all other terms can be neglected in comparison with Coulomb terms. Such β-decay transitions are called Coulomb transitions. They are realized when $\alpha Z \gg pR$ (or qR) and $\alpha Z \gg v_{nuc}/c$. In this case in the second order of the expansion of the Hamiltonian, there remain two terms: αZ (qR) and αZ (pR). For second-forbidden transitions [$\Delta j = 2, 3$ (no)] these terms lead to two nuclear matrix elements $\langle rY_{2M}\rangle_2$ and $\langle \boldsymbol{\gamma}\gamma_5 rY_{2M}\rangle_2$. They do not contribute to the unique β-decay transitions.

The writer has made a calculation of the angular β-ν correlation and β-γ correlation with circularly polarized γ-ray quantum, neglecting the finite size of the nucleus. It was found that these two matrix elements always enter in the same combination,

$$\sqrt{2}\langle rY_{2M}\rangle_2 + \lambda\langle \boldsymbol{\gamma}\gamma_5 rY_{2M}\rangle_2,$$

so that this case is like that of the unique β-decay transitions, whose angular correlation functions do not depend on nuclear matrix elements and can be found exactly.

We note that the correlation functions do not depend on the matrix elements also in the case of a Hamiltonian (1) of the most general form.

If the neutrino emerges at the angle φ with the direction of emission of the electron, the probability function is

$$W(\varphi) = \sum_r a_r P_r(\cos\varphi), \quad 0 \leqslant r \leqslant 2;$$

$$a_0 = \tfrac{2}{5}[(|c_1|^2 + |c_2|^2) + (|c_3|^2 + |c_4|^2)|d|^2],$$

$$a_1 = \tfrac{2}{5}2\mathrm{Re}[(c_1c_3^* + c_2c_4^*)d] - \tfrac{2}{25}2\mathrm{Re}(c_1c_2^* + c_3c_4^*|d|^2),$$

$$a_2 = -\tfrac{2}{25}2\mathrm{Re}[(c_1c_4^* + c_2c_3^*)d]. \quad (5)$$

The probability of emission of a circularly polarized γ-quantum with polarization $\mu = \pm 1$ (right or left) at the angle θ with the direction of emission of the electron is given by

$$W(\theta) = \sum_R \beta_R \gamma_R P_R(\cos\theta), \quad 0 \leqslant R \leqslant 3;$$

$$\beta_0 = a_0, \quad \beta_1 = -\frac{\sqrt{6}}{15}2\mathrm{Re}\left[c_1c_2^* - \tfrac{3}{5}c_3c_4^*|d|^2\right],$$

$$\beta_2 = \frac{\sqrt{70}}{20}(|c_3|^2 + |c_4|^2), \quad \beta_3 = \frac{6}{25}\sqrt{\frac{2}{7}}2\mathrm{Re}(c_3c_4^*|d|^2). \quad (6)$$

We have found the quantity γ_R for an arbitrary γ-ray transition or cascade of γ-ray quanta in an earlier paper.[9] In the simplest case, in which β decay to an excited level j_2 is followed by a γ-ray transition with angular momentum L (of electric or magnetic type) to the level j_3,

$$\gamma_R = C_{L-\mu}^{L-\mu R0}\sqrt{(2j_2+1)(2L+1)}\,W(j_2j_3RL;\,Lj_2). \quad (7)$$

In Eqs. (5) and (6) the quantities c_1, c_2, etc. have the form

$$c_1 = 4\sqrt{(E+1)/E}\,(A'_{1/2} + B_{1/2}),$$

$$c_2 = 4\sqrt{(E-1)/E}\,(A'_{1/2} - B_{1/2}),$$

$$c_3 = 2\sqrt{(E+1)/E}\,(A'_{3/2} + B_{3/2}),$$

$$c_4 = 2\sqrt{(E-1)/E}\,(A'_{3/2} - B_{3/2}),$$

$$d = \frac{12\Gamma(\gamma_2 - i\alpha ZE/p)}{\Gamma(\gamma_1 - i\alpha ZE/p)}\frac{\Gamma(2\gamma_1+1)}{\Gamma(2\gamma_2+1)}\frac{p}{q}.$$

The β-ray spectrum of a second-forbidden Coulomb transition can be found from the usual formula

$$T = \frac{2\pi}{\hbar}|H|^2\rho(E), \quad |H|^2 = \frac{\pi^2}{2}a_0|M|^2,$$

$$|M|^2 = \frac{32}{9}G^2p^2q^2\frac{|\Gamma(\gamma_1 - i\alpha ZE/p)|^2}{|\Gamma(2\gamma_1+1)|^2}e^{\pi\alpha ZE/p}$$

$$\times|\sqrt{2}\langle rY_{2M}\rangle_2 + \lambda\langle \boldsymbol{\gamma}\gamma_5 rY_{2M}\rangle_2|^2.$$

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