ANALYSIS OF EXPERIMENTAL DATA RELATING TO THE SURFACE IMPEDANCE OF SUPERCONDUCTORS

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The results of experimental measurement of the surface impedance of superconductors at various frequencies are compared with the new theory of superconductivity. Satisfactory agreement with the experiments has been found for all frequencies excluding the very lowest ones. At low frequencies the experimental values for the real part of the impedance near the critical temperature are several times larger than the theoretical values.

THE properties of superconductors in a high-frequency field have been discussed by the present authors[1] and by Mattis and Bardeen.[2] In this paper the theory will be compared with experimental results on the surface impedance of superconductors.

We first derive formulae for the surface impedance in various limiting conditions, which are appropriate for making comparison with experiment.* It is usual in experiments to measure the ratio of the impedance in the superconducting state $Z(\omega)$, to the real part of the impedance in the normal state, R_n. This ratio is given (in the Pippard limiting case) by the formula

$$Z(\omega)/R_n = -2i(\pi\omega/\Delta Q(\omega))^{1/3}. \tag{1}$$

The complex function $Q(\omega)$ is given by

$$Q(\omega) = \pi \int\limits_{max(1,\,\tilde{\omega}-1)}^{1+\tilde{\omega}} \tanh\frac{\varepsilon\Delta}{2T} \frac{\varepsilon(\varepsilon-\tilde{\omega})+1}{\sqrt{(\varepsilon^2-1)(1-(\varepsilon-\tilde{\omega})^2)}}\,d\varepsilon$$

$$-i\pi\theta(\tilde{\omega}-2)\int\limits_1^{\tilde{\omega}-1} \tanh\frac{\varepsilon\Delta}{2T} \frac{\varepsilon(\tilde{\omega}-\varepsilon)+1}{\sqrt{(\varepsilon^2-1)((\tilde{\omega}-\varepsilon)^2-1)}}\,d\varepsilon$$

$$-i\pi\int\limits_1^\infty \left\{\tanh\frac{(\varepsilon+\tilde{\omega})\Delta}{2T}\right.$$

$$\left.-\tanh\frac{\varepsilon\Delta}{2T}\right\}\frac{\varepsilon(\varepsilon+\tilde{\omega})+1}{\sqrt{(\varepsilon^2-1)((\varepsilon+\tilde{\omega})^2-1)}}; \tag{2}$$

$$\theta(x) = \begin{cases}1; & x>0\\ 0; & x<0\end{cases};\quad \tilde{\omega} = \omega/\Delta.$$

*We take this opportunity to point out that in Eq. (14) of reference 1 the real part of $Q(\omega)$ was incorrectly calculated. This error affects some limiting expressions for the impedance derived in reference 1. The present nomenclature is the same as in reference 1.

The magnitude of 2Δ gives the energy gap at the given temperature.

According to Eq. (1) the frequency dependence of the impedance is given by the expression

$$\frac{Z(\omega)}{R_n} = 2\left[\frac{\pi\omega}{\Delta|Q(\omega)|}\right]^{1/3}\left\{\sin\left(\frac{1}{3}\tan^{-1}\frac{\operatorname{Im}Q}{\operatorname{Re}Q}\right)\right.$$

$$\left.-i\cos\left(\frac{1}{3}\tan^{-1}\frac{\operatorname{Im}Q}{\operatorname{Re}Q}\right)\right\}. \tag{3}$$

For $T = 0$ the function Q has the values

$$Q(\omega,0) = \begin{cases} 2\pi E\left(\frac{\tilde{\omega}}{2}\right), & \tilde{\omega}<2\\ \pi\left[\tilde{\omega}E\left(\frac{2}{\tilde{\omega}}\right) - \frac{\tilde{\omega}^2-4}{\tilde{\omega}}K\left(\frac{2}{\tilde{\omega}}\right)\right] & \end{cases}$$

$$+i\pi\theta(\tilde{\omega}-2)\left[\tilde{\omega}E\left(\sqrt{1-\left(\frac{2}{\tilde{\omega}}\right)^2}\right)\right.$$

$$\left.-\frac{4}{\tilde{\omega}}K\left(\sqrt{1-\left(\frac{2}{\tilde{\omega}}\right)^2}\right)\right],\quad \tilde{\omega}>2. \tag{4}$$

Here E and K are complete elliptical integrals. Figure 1 shows the dependence of $Z/R_n = R/R_n + iX/R_n$ on $\tilde{\omega}$ at $T = 0$.

The temperature dependence for temperatures other than absolute zero will be analyzed for various frequencies:

a) $\omega \ll \Delta(0)$. At the lowest temperatures

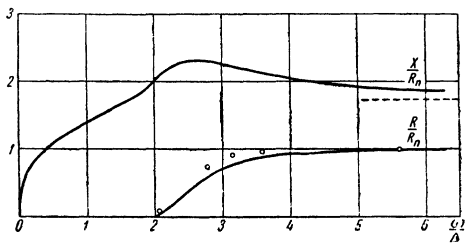

FIG. 1

there is a region where $T \ll \omega \ll \Delta$ in which Eqs. (3) and (4) apply. After the region where $T \sim \omega$ we arrive at $\omega \ll T \ll \Delta$, and the variation with T is given by

$$\frac{Z(\omega)}{R_n} = 2\left(\frac{\omega}{\pi\Delta}\right)^{1/3}\left\{\frac{4}{3}\sinh\frac{\omega}{2T}K_0\left(\frac{\omega}{2T}\right)e^{-\Delta/T}\right.$$
$$\left. - i\left[1 + \frac{1}{3}\left(\frac{\omega}{4\Delta}\right)^2 + \frac{2}{3}e^{-\omega/2T}I_0\left(\frac{\omega}{2T}\right)e^{-\Delta/T}\right]\right\}. \qquad (5)$$

On increasing T further we first have $\omega \ll \Delta \sim T$ and then the small region where $\omega \ll \Delta \ll T$. Here, until $\Delta/T \gg \omega/\Delta$, Q is little different from its value at $T = 0$, and we then obtain

$$\frac{Z(\omega)}{R_n} = 2\left[\frac{\omega}{\pi\Delta\tanh(\Delta/2T)}\right]^{1/3}\left\{\frac{2}{3\pi}\frac{\omega}{T\sinh(\Delta/T)}\ln\left(2\sqrt{\frac{2\Delta}{\omega}}\right)\right.$$
$$+\frac{1}{3\pi}\frac{\omega}{\Delta}\left(\coth\frac{\Delta}{2T}-1\right)$$
$$\left.-\frac{2}{3\pi}\frac{\omega}{T}P\left(\frac{\Delta}{T}\right)\coth\frac{\Delta}{2T} - i\left[1+\frac{1}{3}\left(\frac{\omega}{4\Delta}\right)^2\right]\right\}, \qquad (6)$$

where the function P (x) is the integral

$$P(x) = \int_1^\infty \frac{d\varepsilon}{\varepsilon^2-1}\frac{\cosh x\varepsilon - \cosh x}{(\cosh x\varepsilon + 1)(\cosh x + 1)}$$
$$= \begin{cases} e^{-x}\ln 2\gamma x; & \text{for } x \gg 1 \\ (7/2\pi^2)\zeta(3)\,x; & \text{for } x \ll 1. \end{cases} \qquad (6')$$

When this condition does not hold, one can obtain the following formula by making use of the fact that Δ is small:

$$Z(\omega)/R_n = -2i\left(-i + \pi\Delta^2/2T\omega\right)^{-1/3}. \qquad (7)$$

b) $\omega \sim \Delta(0)$. It is much more difficult to compare theory and experiment in this case, since in most of the temperature range $0 < T < T_c$, Δ, ω and T are of the same order of magnitude. The expression for $Q(\omega)$ can only be simplified at low temperatures, such that $T \ll \omega \sim \Delta$. There are then two possibilities: $\omega < 2\Delta(0)$ and $\omega > 2\Delta(0)$. In the first case $Q(\omega)$ is little different from the value of $\mathrm{Re}\,Q(\omega)$ at $T = 0$, equal to $2\pi E(\omega/2\Delta)$, and the impedance is given by

$$\frac{Z(\omega)}{R_n} = 2\left[\frac{\omega}{2\Delta E(\omega/2\Delta)}\right]^{1/3}\left\{\frac{e^{-\Delta/T}}{3E(\omega/2\Delta)}\sqrt{\pi T\left(\frac{1}{\omega}+\frac{1}{2\Delta}\right)}\right.$$
$$\left.-i\left[1+\frac{e^{-\Delta/T}}{3E(\omega/2\Delta)}\sqrt{\pi T\left(\frac{1}{\omega}-\frac{1}{2\Delta}\right)}\right]\right\}. \qquad (8)$$

For $\omega > 2\Delta(0)$ both parts of $Q(\omega)$ are of comparable magnitude, and it is simplest to use the general formula (3) with $Q(\omega)$ given by

$$Q(\omega, T) = Q(\omega, 0) + 2\pi e^{-(\omega-\Delta)/T}\sqrt{\pi T(1/2\Delta - 1/\omega)}$$
$$-2\pi i e^{-\Delta/T}\sqrt{\pi T}\left[\sqrt{1/\omega + 1/2\Delta}\right.$$
$$\left.-\sqrt{1/2\Delta - 1/\omega}\right]. \qquad (9)$$

With increasing temperature we pass the region where $\omega \sim \Delta \sim T$ and reach the region $\Delta \ll T \sim \omega$. Here we can make use of the fact that the limiting value of the impedance for $\omega \gg \Delta$ is independent of temperature and corresponds to the normal metal. Equation (9) can, therefore, be used for extrapolation purposes without appreciable error.

c) $\omega \gg \Delta(0)$. In this case only the relation between T and Δ varies, while ω is always large compared with them both. Bearing in mind that the real part of $Q(\omega)$ is small compared with the imaginary part, we obtain

$$\frac{Z(\omega)}{R_n} = 1 + \left(\frac{\Delta}{\omega}\right)^2\left[\frac{2}{3}\ln\frac{2\omega}{\Delta(0)} + \frac{1}{3} - \frac{\pi}{\sqrt{3}}\right]$$
$$- i\sqrt{3}\left[1 + \left(\frac{\Delta}{\omega}\right)^2\left(\frac{2}{3}\ln\frac{2\omega}{\Delta(0)} + \frac{1}{3} - \frac{\pi}{3\sqrt{3}}\right)\right]. \qquad (10)$$

The formulae we have derived make it possible to compare theory with the many experimental results in detail.

For this comparison we have used the data of several authors.[3-6]

There are no experimental data for very high frequencies $\omega \gg \Delta(0)$. The largest "effective frequency" $\omega/T_c = 3.04$ was used in the measurement of R for aluminum.[3] Since Δ decreases on approaching T_c, we may assume that the conditions for the applicability of Eq. (10) are approximately satisfied for $1 - T/T_c < 0.9$. Figure 2 shows the comparison between this equation and the experimental data, and it can be seen that near T_c there is agreement (the full curve is the theoretical curve for $\omega \gg \Delta$, T).

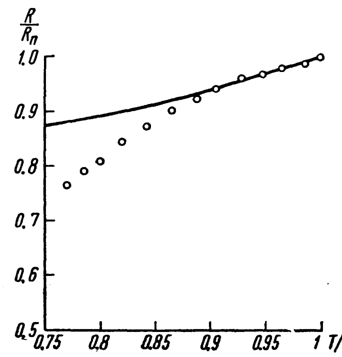

FIG. 2

In Khaĭkin's work[4] the surface impedance of single crystals of cadmium was measured at temperatures between 0.1° and $T_c = 0.56$° K at a frequency $\omega = 0.9\,T_c$ ($\Delta = 0.985$° K). In practice, the whole temperature range, except near T_c, is covered by Eq. (8) ($\omega \sim \Delta$). The theoretical curves for R/R_n and X/R_n are shown in Fig. 3, and the experimental data are plotted with the er-

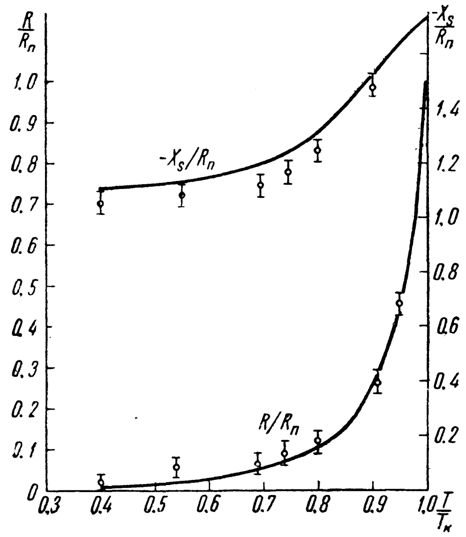

FIG. 3

ror limits indicated by the lines. The agreement between theory and experiment is quite good. It must be borne in mind that the effects of anisotropy, which may influence the results, are not considered in the theory.

The situation is worse when we analyze the low-frequency data of Pippard[5] and Prozorova.[6] There is considerable disagreement between the results for mercury ($\omega = 0.014\,T_c$) and tin ($\omega = 0.0155\,T_c$ in reference 5; $\omega = 0.11\,T_c$ in reference 6) and the theory, near the transition temperature.

At these low frequencies we are, near T_c, in the region where $\Delta/T \ll 1$, but the condition $\Delta/T \gg \omega/\Delta$ is not satisfied. For example, for mercury[5] the two conditions hold at $T - T_c \sim 10^{-3\circ}$ K. Equation (7) then simplifies to

$$\frac{R}{R_n} = 0.019\left(\frac{\omega/T_c}{1-T/T_c}\right)^{4/3}, \qquad \frac{X}{R_n} = 0{,}82\left(\frac{\omega/T_c}{1-T/T_c}\right)^{1/3},$$

$$\Delta \approx 3.06\ T_c(1 - T/T_c)^{1/2}. \tag{11}$$

These equations for T near T_c could have been obtained independently of the previous calculation, as can be seen from Eq. (7), which can really be considered an interpolation between a superconductor at $\omega = 0$ and the normal metal.

Equation (6) must be used for temperatures further away from T_c.

The following results come out of the calculation. R/R_n at $T/T_c \sim 0.6 - 0.7$ for Hg at 1200 Mcs,[5] calculated according to Eq. (6) is just half the experimental value. This discrepancy increases on approaching T_c, and where Eq. (11) is applicable, for $T - T_c \sim 0.01°$ K, the value of R/R_n obtained from Eq. (11) is one sixth of the measured value. The discrepancy for tin measured by Pippard is even greater. We may note that at these frequencies tin, unlike mercury, is not a clear example of a Pippard metal, but is, rather, an intermediate case.

A similar discrepancy is found between the calculated values and Prozorova's results for tin at higher frequencies, $\omega \simeq 0.11\,T_c$. At $T/T_c \sim 0.75$ the experimental value of R/R_n is only 30% higher than the calculated value, and this difference increases to a factor of 3 or 4 near T_c. Both curves, naturally, agree at $T = T_c$ with the normal metal. Comparison with the experimental data for X shows somewhat better agreement. The experimental[6] and theoretical values for $|X_n - X|/R_n$ do not differ by more than a factor of 2.

The reason for the discrepancy between the experimental data and the impedance calculated according to the new theory of superconductivity is at present not clear. We should point out that for mercury, near $T_c - T \sim 0.1 - 0.005°$, the data fit the relation $R/R_n \sim (1 - T/T_c)^{-4/3}$. This range is also described by Eq. (6) in the region $\Delta \sim T$, where the discrepancy with experiment is less. If the ratio R/R_n were calculated on the assumption that for this range of temperature and frequency the metal were of the London type, which is justifiable for tin, then the temperature dependence would essentially be $R/R_n \sim (1 - T/T_c)^{-2}$. As can be seen, this does not agree with Pippard's experimental results.

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