ANGULAR AND POLARIZATION ANALYSES OF REACTIONS OF THE TYPE $a + b \to a' + b' + c'$

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A study is made of the invariant angular operators in terms of which the scattering matrix for reactions of the type $a + b \to a' + b' + c'$ can be expanded, and which are convenient for angular and polarization analyses of such reactions. The angular operators are obtained in explicit form for reactions in which the spin of the system does not exceed unity, and also for analogous reactions involving γ-ray quanta.

1. INTRODUCTION

IT is convenient to carry out angular and polarization analyses of nuclear reactions by means of the invariant angular operators in terms of which the scattering matrix or scattering amplitude can be expanded. We have previously[1] studied and constructed such operators for reactions of the type $a + b \to a' + b'$. Ciulli and Fischer[2] have constructed the first few angular operators for reactions of the type $a + b \to a' + b' + c'$ in which the spins of the initial and final system are equal to $\frac{1}{2}$. In the present paper the angular operators are constructed for reactions $a + b \to a' + b' + c'$ in which the spins of the initial and final systems do not exceed unity, and also for analogous reactions involving γ-ray quanta.

2. GENERAL REMARKS

The initial and final states of the system in the reaction $a + b \to a' + b' + c'$ are defined by the momenta $\mathbf{p}_a$, $\mathbf{p}_b$ and $\mathbf{p}'_a$, $\mathbf{p}'_b$, $\mathbf{p}'_c$ of the particles a, b and a′, b′, c′, and the spin coordinates α and α' of the spin of the system. The momenta of the particles are connected by the relations

$$\mathbf{p}_a + \mathbf{p}_b = \mathbf{p}'_a + \mathbf{p}'_b + \mathbf{p}'_c = \mathbf{P},$$

$$E_a + E_b = E'_a + E'_b + E'_c = E, \tag{1}$$

where $E_a = (p_a^2 + m_a^2)^{1/2}$ is the energy of particle a, and so on, and E and $\mathbf{P}$ are the energy and momentum of the system. In the center-of-mass system ($\mathbf{P} = 0$) we can introduce, instead of the momenta of the particles, the momenta

$$\mathbf{p} = \mathbf{p}_a - \mathbf{p}_b \tag{2}$$

and (cf. reference 3)

$$\mathbf{p}_1 = \mathbf{p}'_a - \mathbf{p}'_b, \qquad \mathbf{p}_2 = -(\mathbf{p}'_a + \mathbf{p}'_b) = \mathbf{p}'_c. \tag{2'}$$

To the momenta $\mathbf{p}$, $\mathbf{p}_1$, $\mathbf{p}_2$ there correspond the orbital angular momenta $\mathbf{l}$, $\mathbf{l}_1$, $\mathbf{l}_2$ ($\mathbf{l}$ is the orbital angular momentum of the system a + b, $\mathbf{l}_1$ that of the system a′ + b′, and $\mathbf{l}_2$ that of particle c′), and the total angular momenta $\mathbf{J}$, $\mathbf{J}'$ of the initial and final systems are connected with these orbital angular momenta by the relations $\mathbf{J} = \mathbf{l} + \mathbf{S}$, $\mathbf{J}' = \mathbf{l}_1 + \mathbf{l}_2 + \mathbf{S}'$, where $\mathbf{S}$, $\mathbf{S}'$ are the spins of the initial and final systems. Using the momenta (2), (2′) and the relations (1), we can see without difficulty that, as the independent variables of the initial system, we can choose the quantities

$$E,\ \mathbf{k},\ \alpha, \tag{3}$$

where $\mathbf{k} = \mathbf{p}/p$, and as the independent variables of the final system the quantities

$$E,\ p'_c,\ \mathbf{k}_1,\ \mathbf{k}_2,\ \alpha', \tag{3'}$$

where $\mathbf{k}_1 = \mathbf{p}_1/p_1$, $\mathbf{k}_2 = \mathbf{p}_2/p_2$.

In the center-of-mass system the scattering matrix S for the reaction $a + b \to a' + b' + c'$ depends on the variables (3) and (3′); it is a matrix (operator) in the spin variables α, α' and a function of the remaining variables. We shall be interested in the dependence of the scattering matrix on the variables $\mathbf{k}$, α and $\mathbf{k}_1$, $\mathbf{k}_2$, α'. Therefore we shall write the S matrix in the form $S(\mathbf{k}_1\mathbf{k}_2\alpha'; \mathbf{k}\alpha) \equiv [S(\mathbf{k}_1\mathbf{k}_2; \mathbf{k})]_{\alpha\alpha'}$, omitting the variables E, p'_c. If the matrix S is invariant with respect to rotations and reflections, it can be expanded in a series in the angular operators invariant with respect to rotations and reflections, $L_{J\nu'\nu}(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k})$:

$$S(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}) = \sum_{J\nu\nu'} S_{J\nu'\nu} L_{J\nu'\nu}(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}). \tag{4}$$

The coefficients $S_{J\nu'\nu}$ in this expansion are functions only of the total energy E and the absolute value p'_c of the momentum of particle c'.

The angular operators are scalar products of the J vectors $\psi_{JM\nu}(\mathbf{k})$ and $\psi_{JM\nu'}(\mathbf{k}_1\mathbf{k}_2)$ of the initial and final states:

$$L_{J\nu'\nu}(\mathbf{k}_1\mathbf{k}_2; \mathbf{k}) = \sum_M \psi_{JM\nu'}(\mathbf{k}_1\mathbf{k}_2)\psi^*_{JM\nu}(\mathbf{k}). \tag{5}$$

The numbers J, M, ν denote respectively the total angular momentum, its projection, and a set of eigenvalues of operators commuting with the total angular momentum, for example the spin of the system, the orbital angular momenta, etc. The general properties of the operators (5) are the same as in the case of the reactions a + b → a' + b' (cf. reference 1). In particular, these operators are Hermitian:

$$L^+_{J\nu'\nu}(\mathbf{k}_1\mathbf{k}_2; \mathbf{k}) = L_{J\nu\nu'}(\mathbf{k}; \mathbf{k}_1\mathbf{k}_2), \tag{6}$$

are orthogonal and normalized:

$$\int L^+_{J'\nu''\nu}(\mathbf{k}_1\mathbf{k}_2; \mathbf{k}) L_{J\nu'\nu_0}(\mathbf{k}_1\mathbf{k}_2; \mathbf{k}_0)\frac{d\mathbf{k}_1}{4\pi}\frac{d\mathbf{k}_2}{4\pi} = \delta_{JJ'}\delta_{\nu''\nu'}L_{J\nu\nu_0}(\mathbf{k}, \mathbf{k}_0), \tag{7}$$

$$\int L_{J'\nu\nu'}(\mathbf{k}'_1\mathbf{k}'_2; \mathbf{k}) L^+_{J\nu_0\nu''}(\mathbf{k}_1\mathbf{k}_2; \mathbf{k})\frac{d\mathbf{k}}{4\pi} = \delta_{JJ'}\delta_{\nu''\nu'}L_{J\nu\nu_0}(\mathbf{k}'_1\mathbf{k}'_2; \mathbf{k}_1\mathbf{k}_2) \tag{7'}$$

and have the property of completeness:

$$\sum_{J\nu}\int L^+_{J\nu'\nu}(\mathbf{k}_1\mathbf{k}_2; \mathbf{k}') L_{J\nu'\nu}(\mathbf{k}_1\mathbf{k}_2; \mathbf{k})\frac{d\mathbf{k}_1}{4\pi}\frac{d\mathbf{k}_2}{4\pi} = \delta(\mathbf{k}' - \mathbf{k}), \tag{8}$$

$$\sum_{J\nu'}\int L_{J\nu'\nu}(\mathbf{k}'_1\mathbf{k}'_2; \mathbf{k}) L^+_{J\nu'\nu}(\mathbf{k}_1\mathbf{k}_2; \mathbf{k})\frac{d\mathbf{k}}{4\pi} = \delta(\mathbf{k}'_1 - \mathbf{k}_1)\delta(\mathbf{k}'_2 - \mathbf{k}_2), \tag{8'}$$

where

$$\int \delta(\mathbf{k}' - \mathbf{k})\, dk/4\pi = 1 \text{ and so on}$$

Since

$$\mathrm{Sp}\, L_{J\nu\nu_0}(\mathbf{k}, \mathbf{k}) = (2J + 1)\delta_{\nu\nu_0},$$

$$\mathrm{Sp}\int L_{J\nu\nu_0}(\mathbf{k}_1\mathbf{k}_2; \mathbf{k}_1\mathbf{k}_2)\, dk_i/4\pi = (2J + 1)\delta_{\nu\nu_0}, \qquad i = 1.2,$$

the coefficients $S_{J\nu'\nu}$ of the expansion (4) can be calculated by the formulas

$$S_{J\nu'\nu} = \frac{1}{2J+1}\mathrm{Sp}\int \frac{d\mathbf{k}_1}{4\pi}\frac{d\mathbf{k}_2}{4\pi} L^+_{J\nu'\nu}(\mathbf{k}_1\mathbf{k}_2; \mathbf{k}) S(\mathbf{k}_1\mathbf{k}_2; \mathbf{k}); \tag{9}$$

$$S_{J\nu'\nu} = \frac{1}{2J+1}\mathrm{Sp}\int \frac{d\mathbf{k}}{4\pi}\frac{d\mathbf{k}_i}{4\pi} S(\mathbf{k}_1\mathbf{k}_2; \mathbf{k}) L^+_{J\nu'\nu}(\mathbf{k}_1\mathbf{k}_2; \mathbf{k}), \qquad i = 1.2. \tag{9'}$$

We shall describe the system a + b by the J vector

$$\psi_{JMlS}(\mathbf{k}) = \sum_\mu C^{lM-\mu;\, S\mu}_{JM} Y_{lM-\mu}(\mathbf{k}) Q_{S\mu}, \tag{10}$$

where $Y_{lm}(\mathbf{k})$ is the orbital function and $Q_{S\mu}$ the spin function of the system; $\mathbf{J} = \mathbf{l} + \mathbf{S}$. We shall describe the system a' + b' + c' either by the J vector

$$\psi_{JM\nu'}(\mathbf{k}_1\mathbf{k}_2) = \sum_{\mu'} C^{l'M-\mu';\, S'\mu'}_{JM} D^{l_1l_2}_{l'M-\mu'}(\mathbf{k}_1\mathbf{k}_2) Q_{S'\mu'},$$

$$D^{l_1l_2}_{l'm'} = \sum_{m_2} C^{l_1m_1;\, l_2m_2}_{l'm'} Y_{l_1m_1}(\mathbf{k}_1) Y_{l_2m_2}(\mathbf{k}_2), \tag{11a}$$

where S' is the spin of the system, l_1, l_2 are the orbital angular momenta, $\mathbf{l}' = \mathbf{l}_1 + \mathbf{l}_2$ is the total orbital angular momentum, and $\mathbf{J} = \mathbf{l}' + \mathbf{S}'$ is the total angular momentum, or else by the J vector

$$\psi_{JM\nu'}(\mathbf{k}_1\mathbf{k}_2) = \sum_m C^{l_2M-m;\, jm}_{JM} Y_{l_2M-m}(\mathbf{k}_2) D^{S'}_{jl_1m}(\mathbf{k}_1),$$

$$D^{S'}_{jl_1m}(\mathbf{k}_1) = \sum_{\mu'} C^{l_1m-\mu';\, S'\mu'}_{jm} Y_{l_1m-\mu'}(\mathbf{k}_1) Q_{S'\mu'}, \tag{11b}$$

where S' is the spin of the system, l_1, l_2 are the orbital angular momenta,* $\mathbf{j} = \mathbf{l}_1 + \mathbf{S}'$, $\mathbf{J} = \mathbf{j} + \mathbf{l}_2$.

By means of Eqs. (5), (10) and (11) one can construct the angular operators $L_{J\nu'\nu}(\mathbf{k}_1\mathbf{k}_2; \mathbf{k})$. For this purpose it is convenient to use the coordinate system with its x, y and z axes along the vectors $[\mathbf{k}\times\mathbf{k}_1]\times\mathbf{k}$, $[\mathbf{k}\times\mathbf{k}_1]$, $\mathbf{k}$ and the invariance of the angular operators.

3. THE EXPLICIT FORM OF THE ANGULAR OPERATORS

We shall present first the form of the angular operators (polynomials) for spinless particles.

Case S = S' = 0:

$$L_{l_1l_2l}(\mathbf{k}_1\mathbf{k}_2; \mathbf{k}) = \sqrt{(2l_1+1)(2l_2+1)(2l+1)}\Big[C^{l_10;\, l_20}_{l0} P_{l_1}(x_1)P_{l_2}(x_2) + 2\sum_{m>0}^{l_2} C^{l_1-m;\, l_2m}_{l0}\sqrt{\frac{(l_1-m)!\,(l_2-m)!}{(l_1+m)!\,(l_2+m)!}} \times a^{(q)}_m P^{(m)}_{l_1}(x_1)P^{(m)}_{l_2}(x_2)\Big], \tag{12}$$

where

$$a^{(q)}_m = \tfrac{1}{2}\{(x_1x_2 - x_{12} - i([\mathbf{k}_1\times\mathbf{k}_2]\mathbf{k}))^m + q(x_1x_2 - x_{12} + i([\mathbf{k}_1\times\mathbf{k}_2]\mathbf{k}))^m\},$$

$$x_1 = (\mathbf{k}\mathbf{k}_1), \quad x_2 = (\mathbf{k}\mathbf{k}_2), \quad x_{12} = (\mathbf{k}_1\mathbf{k}_2),$$

$$q = (-1)^{l_1+l_2-l}; \quad P^{(m)}_l = d^mP_l/dx^m; \tag{13}$$

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*We shall not consider here the coupling $\mathbf{j} = \mathbf{l}_2 + \mathbf{S}'$, $\mathbf{J} = \mathbf{j} + \mathbf{l}_1$, since it is equivalent to the coupling $\mathbf{j} = \mathbf{l}_1 + \mathbf{S}'$, $\mathbf{J} = \mathbf{j} + \mathbf{l}_2$ and the notations $\mathbf{p}_1 = -(\mathbf{p}'_a + \mathbf{p}'_b) \equiv \mathbf{p}'_c$, $\mathbf{p}_2 = \mathbf{p}'_a - \mathbf{p}'_b$ instead of Eq. (2').

$q = +1$ or -1, corresponding to conservation or change of the intrinsic parity of the system in the reaction. The angular polynomials (12) satisfy the conditions

$$L_{l_1 l_2 l}(\mathbf{k}_1\mathbf{k}_2;\, \mathbf{k}) = (-1)^{l_1 - l}\sqrt{\frac{2l+1}{2l_1+1}}\, L_{l_2 l l_1}(\mathbf{k}_2\mathbf{k};\, \mathbf{k}_1)$$
$$= (-1)^{l_2 - l}\sqrt{\frac{2l+1}{2l_2+1}}\, L_{l l_1 l_2}(\mathbf{k}\mathbf{k}_1;\, \mathbf{k}_2)$$
$$= (-1)^{l_1 + l_2 - l} L_{l_2 l_1 l}(\mathbf{k}_2\mathbf{k}_1;\, \mathbf{k}). \qquad (14)$$

A. Coupling $\mathbf{l}_1 + \mathbf{l}_2 = \mathbf{l}'$, $\mathbf{l}' + \mathbf{S}' = \mathbf{J}$

We now note that the angular operators $L_{J\nu'\nu}(\mathbf{k}', \mathbf{k})$ for the reaction $a + b \to a' + b'$ can be represented in the form of a certain invariant differential operator $O_{J\nu'\nu}(\mathbf{L}', \mathbf{T})$, which depends on the orbital angular momentum operator $\mathbf{L}' = -i[\mathbf{k}'\partial/\partial\mathbf{k}']$ and the vector matrix $\mathbf{T}$ in the spin space of the system, and acts on the angular polynomial for spinless particles:

$$L_{Jl'S'lS}(\mathbf{k}', \mathbf{k}) = O_{Jl'S'lS}(\mathbf{L}', \mathbf{T})\, L_{l'}(\mathbf{k}', \mathbf{k}). \qquad (15)$$

Starting from Eqs. (11a), (10), and (5), we can show without difficulty that the angular operators for reactions $a + b \to a' + b' + c'$ can be represented in the form

$$L_{Jl'l_1l_2S'lS}(\mathbf{k}_1\mathbf{k}_2;\, \mathbf{k}) = O_{Jl'S'lS}(\mathbf{L}', \mathbf{T})\, L_{l_1 l_2 l'}(\mathbf{k}_1\mathbf{k}_2;\, \mathbf{k}), \qquad (16)$$

where $O_{Jl'S'lS}$ is the same operator as in Eq. (15), $\mathbf{L}' = \mathbf{L}_1 + \mathbf{L}_2$, and $L_{l_1 l_2 l'}(\mathbf{k}_1\mathbf{k}_2;\, \mathbf{k})$ is the angular polynomial (12) for spinless particles.* With regard to Eq. (16) we note the following. Since parity is conserved in the reaction $a + b \to a' + b' + c'$, the quantity $l_1 + l_2 - l$ is either even or odd, depending on whether the intrinsic parity of the system is conserved or changes in the reaction. Therefore the angular operators $L_{J\nu'\nu}(\mathbf{k}_1\mathbf{k}_2;\, \mathbf{k})$ are either scalars (if the intrinsic parity is conserved and the quantity $q \equiv (-1)^{l_1 + l_2 - l}$ is equal to +1) or pseudoscalars (if the intrinsic parity changes and $q = -1$). As for the angular polynomials $L_{l_1 l_2 l'}$ and the operators $O_{Jl'S'lS}$ that appear in Eq. (16), they can be either scalars or pseudoscalars, independently of the conservation or change of the intrinsic parity in the reaction; namely, $L_{l_1 l_2 l'}$ is a scalar (pseudoscalar) if $q' \equiv (-1)^{l_1 + l_2 - l}$ is equal to $+1(-1)$, and $O_{Jl'S'lS}$ is a scalar (pseudoscalar) if $q_0 \equiv (-1)^{l' - l}$ is equal to $+1(-1)$. We note that $q = q_0 q'$.

We shall give the explicit forms of the operators $O_{Jl'S'lS}(\mathbf{L}', \mathbf{T})$ for the cases in which S and S′ do not exceed 1.

Case $S = S' = 1/2$:

$$O_{Jl'\frac{1}{2}l\frac{1}{2}}(\mathbf{L}', \boldsymbol{\sigma}) = \frac{1}{2(2l'+1)}\left[J + \frac{1}{2} + (-1)^{l' - J + 1/2}(\boldsymbol{\sigma}\mathbf{L}')\right][(q_0 + 1) + (q_0 - 1)(\boldsymbol{\sigma}\mathbf{k})]; \qquad (17)$$

Case $S = 0$, $S' = 1$:

$$O_{Jl'1l}(\mathbf{L}', \mathbf{T}) = \begin{cases} \dfrac{1}{2l'+1}\sqrt{\dfrac{2J+1}{J + 1/2 \mp 1/2}}\left[-i(\mathbf{T}[\mathbf{k}\times\mathbf{L}']) \pm \left(J + \dfrac{1}{2} \mp \dfrac{1}{2}\right)(\mathbf{T}\mathbf{k})\right], & J = l = l' \pm 1, \quad (18)\\[2ex] \dfrac{1}{\sqrt{J(J+1)}}(\mathbf{T}\mathbf{L}'), & J = l = l'; \quad (19)\end{cases}$$

Case $S = 1$, $S' = 0$:

$$O_{Jl'l1}(\mathbf{L}', \mathbf{T}) = \begin{cases} \dfrac{1}{2l'+1}\sqrt{\dfrac{2J+1}{J + 1/2 \mp 1/2}}\left[i(\mathbf{T}[\mathbf{k}\times\mathbf{L}']) \pm \left(J + \dfrac{1}{2} \mp \dfrac{1}{2}\right)(\mathbf{T}\mathbf{k})\right], & J = l \pm 1 = l', \quad (20)\\[2ex] \dfrac{1}{\sqrt{J(J+1)}}(\mathbf{T}\mathbf{L}'), & J = l = l'; \quad (21)\end{cases}$$

Case $S = S' = 1$:
a) $q_0 = +1$,

$$O_{Jl'1l1}(\mathbf{L}', \mathbf{S}) = \begin{cases} \dfrac{1}{(2l'+1)(J + 1/2 \mp 1/2)}\left\{\pm\left(J + \dfrac{1}{2} \mp \dfrac{1}{2}\right)[1 + (\mathbf{S}\mathbf{L}')] + (\mathbf{S}\mathbf{L}')(\mathbf{S}\mathbf{L}')\right\}, & J = l \pm 1 = l' \pm 1,\\[2ex] \dfrac{1}{(2l'+1)\sqrt{J(J+1)}}\left\{-2\left(J + \dfrac{1}{2} \pm \dfrac{1}{2}\right)^2 + (2J+1)\left[\left(J + \dfrac{1}{2} \pm \dfrac{1}{2}\right)(\mathbf{S}\mathbf{k})\right.\right. & \\ \left.\left. \pm i(\mathbf{S}[\mathbf{k}\times\mathbf{L}'])\right](\mathbf{S}\mathbf{k}) \pm \left(J + \dfrac{1}{2} \mp \dfrac{1}{2}\right)(\mathbf{S}\mathbf{L}') + (\mathbf{S}\mathbf{L}')(\mathbf{S}\mathbf{L}')\right\}, & J = l \pm 1 = l' \mp 1,\\[2ex] \dfrac{1}{J(J+1)}\{J(J+1) - (\mathbf{S}\mathbf{L}')(\mathbf{S}\mathbf{L}')\}, & J = l = l'; \quad (22)\end{cases}$$

*A similar representation for $S = S' = 1/2$ is used in reference 2.

b) $q_0 = -1$,

$$O_{Jl'1l1}(\mathbf{L}',\mathbf{S}) = \begin{cases} \frac{1}{(2l'+1)(J+{}^1/_2\mp{}^1/_2)}\sqrt{\frac{2J+1}{J+{}^1/_2\pm{}^1/_2}}\Big[\mp\Big(J+\frac{1}{2}\mp\frac{1}{2}\Big)(\mathbf{Sk})- \\ \qquad -i(\mathbf{S}[\mathbf{k}\times\mathbf{L}'])\Big](\mathbf{SL}'), \qquad J = l\pm 1 = l'. \\ \frac{1}{(2l'+1)(J+{}^1/_2\mp{}^1/_2)}\sqrt{\frac{2J+1}{J+{}^1/_2\pm{}^1/_2}}\Big[\mp\Big(J+\frac{1}{2}\mp\frac{1}{2}\Big)(\mathbf{Sk})\mp \\ \mp\Big(J+\frac{1}{2}\pm\frac{1}{2}\Big)(\mathbf{SL}')(\mathbf{Sk})+i(\mathbf{SL}')(\mathbf{S}[\mathbf{k}\times\mathbf{L}'])\Big], \quad J = l = l'\pm 1. \end{cases} \tag{23}$$

B. Coupling* $l_1 + S' = j$, $j + l_2 = J$

If, on the other hand, we use the formulas (11b), (10), and (5), then for integral S and S′ the angular operator for the reaction a + b → a′ + b′ + c′ can again be reduced to a certain invariant differential operator acting on an angular polynomial of the form $L_{jl_2J}(k_1k_2; k)$ [Eq. (12)]; we have

$$L_{Jjl_1S'l_2lS}(\mathbf{k}_1\mathbf{k}_2;\mathbf{k}) = O_{jl_1S',JlS}(\mathbf{L},\mathbf{L}_1,\mathbf{T})\,L_{jl_2J}(\mathbf{k}_1\mathbf{k}_2;\mathbf{k}). \tag{24}$$

In particular, we get for the operators $O_{jl_1S';JlS}$

Cases S = 0, S′ = 1 and S = 1, S′ = 0:

$$O_{jl_11} = (\mathbf{O}_{jl_1}(\mathbf{k}_1,\mathbf{L}_1)\mathbf{T}), \qquad O_{Jl1} = (\mathbf{O}_{Jl}(\mathbf{k},\mathbf{L})\mathbf{T}); \tag{25}$$

Case S = S′ = 1:

$$O_{jl_11;Jl1} = (\mathbf{O}_{Jl}(\mathbf{k},\mathbf{L})\,\mathbf{O}_{jl_1}(\mathbf{k}_1,\mathbf{L}_1)) - (\mathbf{O}_{Jl}(\mathbf{k},\mathbf{L})\mathbf{S})(\mathbf{O}_{jl_1}(\mathbf{k}_1,\mathbf{L}_1)\mathbf{S}), \tag{26}$$

where

$$\mathbf{O}_{JJ\pm1}(\mathbf{k},\mathbf{L}) = \frac{1}{\sqrt{(2J+1)(J+{}^1/_2\pm{}^1/_2)}}\Big[-i[\mathbf{k}\times\mathbf{L}]\mp\Big(J+\frac{1}{2}\pm\frac{1}{2}\Big)\mathbf{k}\Big],$$

$$\mathbf{O}_{JJ}(\mathbf{k},\mathbf{L}) = \frac{\mathbf{L}}{\sqrt{J(J+1)}}. \tag{27}$$

If, on the other hand, S and S′ are half-integral, the angular operator can be constructed directly, as in Eq. (12). In particular we have:

Case S = S′ = ½:

$$L_{Jjl_1{}^1/_2l_2l{}^1/_2}(\mathbf{k}_1\mathbf{k}_2;\mathbf{k}) = (-1)^{j+l_2-J}\sqrt{\Big(J+\frac{1}{2}\Big)(2l_2+1)}\sum_{m={}^1/_2}^{j}\left\{\frac{(j-m)!(l_2-m+{}^1/_2)!}{(j+m)!(l_2+m-{}^1/_2)!}\right\}^{1/2}$$

$$\times\Big\{\Big(l_1\pm m+\frac{1}{2}\Big)P_{l_1}^{(m-1/2)}(x_1)\mp i(\sigma[\mathbf{k}_1\times\mathbf{k}])P_{l_1}^{(m+1/2)}(x_1)\Big\}$$

$$\times\Big\{a_{m-1/2}^{(q)}-(\sigma\mathbf{k})a_{m-1/2}^{(-q)}\Big\}\Big\{C_{J\,{}^1/_2}^{l_2\,{}^1/_2-m;\,jm}P_{l_2}^{(m-1/2)}(x_2)$$

$$+(-1)^{J+1/2-l}\Big[\Big(l_2-m+\frac{1}{2}\Big)\Big(l_2+m+\frac{1}{2}\Big)\Big]^{-1/2}$$

$$\times C_{J-1/2}^{l_2-1/2-m;\,jm}\,i(\sigma[\mathbf{k}_2\times\mathbf{k}])P_{l_2}^{(m+1/2)}(x_2)\Big\}. \tag{28}$$

The signs ± correspond to $j = l_1 \pm \frac{1}{2}$.

We note that for the reactions π + N → 2π′ + N′ and $N_1 + N_2 \to N_1' + N_2' + \pi'$ the formulas more convenient in practice are Eqs. (28) and (24), and not Eq. (16), since in this case the quantum numbers l_1, j describing the final system correspond to the quantum numbers l', J describing the final system in the reactions π + N → π′ + N′ and $N_1 + N_2 \to N_1' + N_2'$. Therefore if the matrix elements $S_{J_0l_0'S_0'l_0S_0}$ are important for the reactions π + N → π′ + N′ and $N_1 + N_2 \to N_1' + N_2'$, it can be assumed that an important part in the expansions (4) for the reactions π + N → 2π′ + N′ and $N_1 + N_2 \to N_1' + N_2' + \pi'$ is played by the matrix elements $S_{Jjl_1S'l_2lS}$ for which $j = J_0$, $l_1 = l_0'$, $S' = S_0'$, $l = l_0$, $S = S_0$.

*Cf. footnote *, page 154.

4. REACTIONS INVOLVING γ-RAY QUANTA

The wave function $(\mathbf{e}\mathbf{D}_{lm}^{(\lambda)}(\mathbf{k}))$ of a photon with momentum **k**, angular momentum l, polarization **e**, and multipole character λ (λ = 1 or 0 for multipoles of electric or magnetic type, respectively) is connected with the function $Y_{lm}(\mathbf{k})$ of a particle of spin zero by the relation

$$(\mathbf{e}\mathbf{D}_{lm}^{(1)}(\mathbf{k})) = \frac{1}{\sqrt{l(l+1)}}\Big(\mathbf{e}\frac{\partial}{\partial\mathbf{k}}\Big)Y_{lm}(\mathbf{k}), \qquad \text{parity} = (-1)^l,$$

$$(\mathbf{e}\mathbf{D}_{lm}^{(0)}(\mathbf{k})) = \frac{1}{\sqrt{l(l+1)}}\Big(i[\mathbf{k}\times\mathbf{e}]\frac{\partial}{\partial\mathbf{k}}\Big)Y_{lm}(\mathbf{k}),$$

$$\text{parity} = (-1)^{l+1}. \tag{29}$$

Therefore the construction of the angular operators for a reaction involving a photon reduces to the application of the "polarization" operator $[l(l+1)]^{-1/2}(\mathbf{e}\partial/\partial\mathbf{k})$ or $[l(l+1)]^{-1/2}(i[\mathbf{k}\times\mathbf{e}]\partial/\partial\mathbf{k})$ to the angular operators of the analogous reaction, in which the photon is replaced by a scalar or pseudoscalar particle with spin zero (cf. reference 1). That is, if one has to construct the angular operators $L_{J\nu'\nu\lambda}(\mathbf{k}_1\mathbf{k}_2;\mathbf{k})$ for the reaction γ + b → a′ + b′ + c′, one must construct the angular operators $L_{J\nu'\nu}^{(q)}(\mathbf{k}_1\mathbf{k}_2;\mathbf{k})$ or $L_{J\nu'\nu}^{(-q)}(\mathbf{k}_1\mathbf{k}_2;\mathbf{k})$ for the reaction a + b → a′ + b′ + c′, where a is a scalar or pseudoscalar particle with spin zero, and then apply to them the operators $[l(l+1)]^{-1/2}(\mathbf{e}\partial/\partial\mathbf{k})^*$ or $[l(l+1)]^{-1/2}\cdot(i[\mathbf{k}\times\mathbf{e}]\partial/\partial\mathbf{k})^*$, respectively:

$$L_{J\nu'\nu\lambda}(\mathbf{k}_1\mathbf{k}_2;\ \mathbf{k}) = \begin{cases} [l(l+1)]^{-1/2}(\mathbf{e}\ \partial/\partial\mathbf{k})^* L^{(q)}_{J\nu'\nu}(\mathbf{k}_1\mathbf{k}_2;\ \mathbf{k}), & \lambda = 1, \\ [l(l+1)]^{-1/2}(i[\mathbf{k}\times\mathbf{e}]\,\partial/\partial\mathbf{k})^* L^{(-q)}_{J\nu'\nu}(\mathbf{k}_1\mathbf{k}_2;\ \mathbf{k}), & \lambda = 0. \end{cases} \quad (30)$$

Similarly, the angular operators $L_{J\nu'\lambda'\nu}(\mathbf{k}_1\mathbf{k}_2;\ \mathbf{k})$ for the reaction a + b → a′ + b′ + γ′ are connected with the operators $L^{(q)}_{J\nu'\nu}(\mathbf{k}_1\mathbf{k}_2;\ \mathbf{k})$ and $L^{(-q)}_{J\nu'\nu}(\mathbf{k}_1\mathbf{k}_2;\ \mathbf{k})$ of the reaction a + b → a′ + b′ + c′, where c′ is a scalar or pseudoscalar particle with spin zero, by the relations

$$L_{J\nu'\lambda'\nu}(\mathbf{k}_1\mathbf{k}_2;\ \mathbf{k}) = \begin{cases} [l_2(l_2+1)]^{-1/2}(\mathbf{e}'\partial/\partial\mathbf{k}_2)\, L^{(q)}_{J\nu'\nu}(\mathbf{k}_1\mathbf{k}_2;\ \mathbf{k}), & \lambda' = 1, \\ [l_2(l_2+1)]^{-1/2}(i[\mathbf{k}_2\times\mathbf{e}']\,\partial/\partial\mathbf{k}_2)\, L^{(-q)}_{J\nu'\nu}(\mathbf{k}_1\mathbf{k}_2;\ \mathbf{k}), & \lambda' = 0. \end{cases} \quad (31)$$

Here it is assumed that the angular momentum and momentum of the photon are l_2 and $\mathbf{p}_2$. Finally, the angular operators $L_{J\nu'\lambda'\nu\lambda}(\mathbf{k}_1\mathbf{k}_2;\ \mathbf{k})$ for the reaction γ + b → a′ + b′ + γ′ are obtained by applying the operators

$$[l_2(l_2+1)\,l(l+1)]^{-1/2}(\mathbf{e}'\partial/\partial\mathbf{k}_2)(\mathbf{e}\,\partial/\partial\mathbf{k})^*$$

and $[l_2(l_2+1)\,l(l+1)]^{-1/2}(i[\mathbf{k}_2\times\mathbf{e}']\,\partial/\partial\mathbf{k}_2)(i[\mathbf{k}\times\mathbf{e}]\,\partial/\partial\mathbf{k})^*$

to the angular operators for the reaction a + b → a′ + b′ + c′, in which the particles a and c′ have spin zero and the same parity and by applying the operators

$$[l_2(l_2+1)l(l+1)]^{-1/2}(\mathbf{e}'\partial/\partial\mathbf{k}_2)(i[\mathbf{k}\times\mathbf{e}]\partial/\partial\mathbf{k})^*$$

and $[l_2(l_2+1)\,l(l+1)]^{-1/2}\ (i[\mathbf{k}_2\times\mathbf{e}']\,\partial/\partial\mathbf{k}_2)(\mathbf{e}\partial/\partial\mathbf{k})^*$

to the angular operators for the reaction a + b → a′ + b′ + c′, in which particles a and c′ have spin zero and opposite parities.

5. CONCLUSION

The angular operators completely determine the angular distribution and polarization of the particles in the transition $J\nu \to J\nu'$. Since the differential cross section $d\sigma/d\Omega$ and the polarization of the scattered particles (i.e., the mean values $\langle\Omega\rangle'$ of certain spin operators Ω) are connected with the scattering amplitude T = S − 1 by the relations

$$d\sigma/d\Omega = \mathrm{Sp}\,(T\rho T^+), \qquad \langle\Omega\rangle' = \mathrm{Sp}\,(\Omega T\rho T^+)/\,\mathrm{Sp}\,(T\rho T^+),$$

where ρ is the density matrix of the incident particles, by measuring $d\sigma/d\Omega$ and $\langle\Omega\rangle'$ experimentally and using the expansion (4) for the scattering matrix, one can find the coefficients $S_{J\nu'\nu}(E,\ p'_c)$, i.e., carry out the phase-shift analysis.

On the other hand, if the scattering matrix obeys some known equation, the problem of finding this matrix is made easier if, by means of the expansion (4), one effects a separation of the angular and spin variables from the variables E, p'_c, and reduces the problem to that of finding the $S_{J\nu'\nu}(E,\ p'_c)$, which depend only on E and p'_c.

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