DYNAMICS OF A CONDUCTING GASEOUS SPHERE IN A QUASI-STATIONARY ELECTROMAGNETIC FIELD

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Submitted to JETP editor, February 10, 1959

J. Exptl. Theoret. Phys. (U.S.S.R.) **37**, 224-228 (July, 1959)

An investigation is made of the small oscillations of a plasma sphere of infinite conductivity.

VEKSLER[1] and Knox[2] have called attention to the possibility of using ac electromagnetic fields for the stabilization of hot plasmas.

Below we consider a uniform gaseous sphere with sharp boundaries located in a quasi-stationary, spatially homogeneous electromagnetic field

$$H_k = H_0 \exp(i\Omega_k^H t), \quad E_k = E_0 \exp(i\Omega_k^E t), \tag{1}$$

(k = x, y, z) where the frequencies Ω_k^H and Ω_k^E are all assumed to be different from each other. If the skin effect is large the ac field does not penetrate the plasma, the electrical conductivity of which is assumed to be infinite. The time-average of the pressure at the surface of a weakly deformed, ideally conducting sphere in an ac field (1) has been determined in reference 3:

$$\bar{p}(\vartheta, \varphi) = \frac{9H_0^2}{32\pi}\Big[\Big(1 - 2\frac{E_0^2}{H_0^2}\Big) + \sum_{l=2}^{\infty}\sum_{m=-l}^{l}\frac{(2l-1)(l-1)}{2l+1} \times \Big(1 - \frac{4}{3}\frac{l(l+1)}{l-1}\frac{E_0^2}{H_0^2}\Big)\alpha_{lm}Y_l^m(\vartheta, \varphi)\Big]. \tag{2}$$

Here the α_{lm} are the coefficients in the expansion, in spherical functions, for the radial deviation from a sphere of radius r_0 of points at the surface of the plasma:

$$\delta r(\vartheta, \varphi)/r_0 = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\alpha_{lm}Y_l^m(\vartheta, \varphi).$$

For equilibrium the internal gas kinetic pressure at the boundary of the plasma must be equal to the electromagnetic forces:

$$(9H_0^2/32\pi)(1 - 2E_0^2/H_0^2) = p > 0.$$

Since there is no electromagnetic field inside the bunch, in analyzing small deviations we start with the usual hydrodynamic equation (neglecting viscosity). Then, a monochromatic component of the displacement potential

$$\xi(r, \vartheta, \varphi)e^{i\omega t} = \nabla\Phi(r, \vartheta, \varphi)e^{i\omega t}$$

of an individual particle obeys the following wave equation in the linear perturbation approximation:

$$\omega^2\Phi + c^2\Delta\Phi = 0,$$

where the adiabatic velocity of sound is $c^2 = \gamma p/\rho$. The particular solution

$$\Phi_{lm}(r, \vartheta, \varphi) = \text{const}\cdot r^{-1/2}J_{l+1/2}(\omega r/c)Y_l^m(\vartheta, \varphi), \tag{3}$$

in which the frequency ω is still arbitrary, must satisfy the boundary condition

$$-\gamma p\{\operatorname{div}\xi\}_{r=r_0} = \frac{9H_0^2}{32\pi}\frac{(2l-1)(l-1)}{2l+1}f_l\Big\{\frac{\xi_r}{r_0}\Big\}_{r=r_0} \quad (l \geqslant 2),$$

$$f_l(E_0/H_0) = 1 - \frac{4}{3}\frac{l(l+1)}{l-1}\frac{E_0^2}{H_0^2}, \tag{4}$$

which states that the Lagrangian variations of the hydrodynamic and electromagnetic pressures at the surface of the perturbed sphere must be equal. Converting from the displacement amplitude to the potential in Eq. (4) we obtain the dispersion relation

$$\omega^2 = F_l(\omega) \equiv \frac{9H_0^2}{32\pi}\frac{c^2}{\gamma p r_0}\frac{(2l-1)(l-1)}{2l+1}f_l\frac{\sqrt{r_0}}{J_{l+1/2}(\omega r_0/c)} \times \Big\{\frac{d}{dr}\frac{J_{l+1/2}(\omega r/c)}{\sqrt{r}}\Big\}_{r=r_0}, \quad l \geqslant 2. \tag{5}$$

For radial spherically symmetric oscillations of the sphere ($l = 0$) and reciprocating displacements of the sphere as a whole ($l = 1$) the internal pressure remains unchanged. Consequently we have the boundary condition

$$J_{l+1/2}(\omega r_0/c) = 0, \quad l = 0,1. \tag{6}$$

The roots of the transcendental equations (5) and (6) determine the spectrum of eigenvalues ω_l for a given perturbation (3). The absence of solutions with $\omega^2 < 0$ for positive f_l indicates that the bunch is stable against perturbations characterized by wavelengths $\lambda_s = 2\pi r_0/l$ along the surface of the sphere for which $f_l(E_0/H_0) > 0$. Instability is possible for a perturbation characterized by $f_l < 0$. In this case the solution with $\omega^2 < 0$ signifies random deviations which increase exponentially in the course of time. In the particular case[2] $E_0 = 0$ and $f_l = 1$ a spherical plasma has internal stability against arbitrary perturbations. The limiting case $c \to \infty$ in Eq. (5) gives the simple formula

$$\omega_l^2 = \frac{9H_0^2}{32\pi}\frac{1}{\rho r_0^2}\frac{(2l-1)(l-1)l}{2l+1}f_l,$$

which characterizes the dynamics of a highly conducting incompressible fluid.* The equilibrium criteria formulated do not apply when λ_S becomes comparable with the skin depth (for large values of l) since a perturbation of this kind is not compatible with the original assumption of infinite conductivity for the deformed sphere.

We have discussed here the behavior of a plasma sphere when the pressure of the external electromagnetic field is constant (time average). In order to calculate the effect of the alternating component of the high-frequency field we consider rapid motion of a plasma produced, for example, by the z component of the magnetic field (E_0 is assumed to be zero). As before we assume that $\alpha_{lm} \ll 1$. The magnetic pressure

$$p_{H_z}(\vartheta, \varphi, t) = \frac{3H_z^2}{16\pi}\left[1 - Y_2^0 + \sum_{l=2}^{\infty}\sum_{m=-l}^{l}\Lambda_{lm}\alpha_{lm}Y_l^m\right],$$

$$\Lambda_{lm} = \frac{12l^3(l^2-m^2) + 2l^2(l^2-3m^2) - 18l(l^2-m^2) + 10l^2 - 9m^2}{(2l+3)(2l+1)l(2l-1)} \quad (7)$$

is computed in the same way as for (2). In Eq. (7) we separate out the harmonic time component

$$\tilde{p}_{H_z}(\vartheta, \varphi, t) = \frac{p}{3}\left[1 - Y_2^0 + \sum_{l=2}^{\infty}\sum_{m=-l}^{l}\Lambda_{lm}\alpha_{lm}Y_l^m(\vartheta, \varphi)\right]\cos 2\Omega_z t,$$

and, in the acoustic approximation, obtain the boundary condition for the displacement potential $\xi(r, \vartheta, \varphi, t) = \nabla\Psi$, which is governed by the wave equation $c^2\Delta\Psi = \partial^2\Psi/\partial t^2$:

$$\frac{3\gamma}{c^2}\left\{\frac{\partial^2\Psi}{\partial t^2}\right\}_{r=r_0} = \begin{cases} -\cos 2\Omega_z t \quad \text{for } l = 0; \\ Y_2^0\cos 2\Omega_z t - \left(\frac{52}{35}\cos 2\Omega_z t + \frac{9}{5}\right)\frac{1}{r_0}\left\{\frac{\partial\Psi}{\partial r}\right\}_{r=r_0} \\ \quad \text{for } l = 2,\ m = 0; \\ -\left(\Lambda_{lm}\cos 2\Omega_z t + \frac{3(2l-1)(l-1)}{2l+1}\right)\frac{1}{r_0}\left\{\frac{\partial\Psi}{\partial r}\right\}_{r=r_0} \\ \quad \text{for } l = 2,\ m \neq 0;\ l \geqslant 3. \end{cases} \quad (8)$$

*The lowest limiting frequency found (ω_2) is approximately the same as that determined by Eq. (5) in the general case of a compressible fluid.

Whence we find the solution for $l = 0$:

$$\frac{\xi_r(r_0, t)}{r_0} = \frac{1}{3\gamma\nu_0}\frac{\sqrt{\nu_0}}{J_{1/2}(\nu_0)}\left\{\frac{d}{d\nu}\frac{J_{1/2}(\nu)}{\sqrt{\nu}}\right\}_{\nu=\nu_0} Y_0\cos 2\Omega_z t,$$

where the following notation has been introduced:

$$2\Omega_z r/c = \nu, \quad 2\Omega_z r_0/c = \nu_0.$$

An investigation of the other equations ($l \geq 3$; $l = 2$, $m \neq 0$) by the method of successive approximations shows that the oscillations are stable for frequencies characterized by $\Omega_z > \omega_l$. Unstable solutions are possible close to the values $\Omega_z = \omega_l/n$, where $n = 1, 2, 3, \ldots$ (parametric resonance). In the region of stable solutions for spheroidal perturbation ($l = 2$, $m = 0$) the motion of the sphere boundary is given approximately by the expression

$$\frac{\xi_r(r_0, t)}{r_0} = \frac{5}{9}\frac{F_2(2\Omega_z)}{F_2(2\Omega_z) - (2\Omega_z)^2}Y_2^0(\vartheta, \varphi)\cos 2\Omega_z t,$$

$$F_2 \neq n^2(2\Omega_z)^2.$$

By proper choice of the frequency Ω_z it should be possible to meet the requirements for small induced surface oscillations with respect to simple Y_0 and Y_2^0-deformation. For example, with $x_0 \gg 1$ the oscillations of the volume and the shape of the plasma are insignificant $[r_0^{-1}|\xi_r(r_0, t)|_{max} \ll 1]$ if the inequality $|\nu_0 \cot \nu_0 - 1| \ll \nu_0^2$ is satisfied.

If we assume that the plasma is incompressible the boundary conditions in (8) reduce to Mathieu equations (with a right-hand member in the case of a spheroidal deformation)

$$\frac{d^2\alpha_{20}}{d\tau^2} + (a_2 + 16\,q_{20}\cos 2\tau)\alpha_{20} = \frac{2p\cos 2\tau}{3\rho\Omega_z^2 r_0^2},$$

$$\frac{d^2\alpha_{lm}}{d\tau^2} + (a_l + 16q_{lm}\cos 2\tau)\alpha_{lm} = 0, \quad l = 2, \quad m \neq 0; \quad l \geqslant 3,$$

where

$$\tau = \Omega_z t, \quad a_l = \omega_l^2/\Omega_z^2, \qquad q_{lm} = l\Lambda_{lm}p/48\rho\Omega_z^2 r_0^2.$$

The motion of the surface of an incompressible liquid is stable with respect to the first three harmonics ($l = 2, 3, 4$) for example with $\Omega_z^2 \approx p/3\rho r_0^2$. In general, with increasing values of m and l the instability zones become smaller.

The stability criteria for high frequencies in the presence of an electric field may be obtained in a similar manner if, in place of Eq. (7), we use

$$p_{E_z}(\vartheta, \varphi, t) = -\frac{3E_z^2}{4\pi}\left[\frac{1}{2} + Y_2^0 + \sum_{l=2}^{\infty}\sum_{m=-l}^{l}\Xi_{lm}\alpha_{lm}Y_l^m\right], \quad (9)$$

where

$$\Xi_{lm} = \frac{4l^3(l^2-m^2) + 8l^2(l^2-m^2) - l(l^2-5m^2) - 6(l^2-m^2) - l}{(2l-1)(2l+1)(2l+3)}.$$

We can now consider the qualitative features of the physical results. First we consider the mechanism for stabilization of a plasma by a magnetic field. The application of one field (for example H_Z) causes, in addition to the anisotropic pressure $p(\vartheta) = 3H_0^2(1 - Y_2^0)/32\pi$, an instability for perturbations characterized by $m = l$ [cf. Eq. (7)]. In these deformations the magnetic force lines which bend around the sphere along the meridians, are, (without twisting) spread apart at protruding regions of the surface and concentrated in regions of indentation; as a result a magnetic pressure differential is created, which tends to increase the deformation. With $m \neq l$ another effect predominates; this is the increase in the magnetic pressure as a consequence of twisting of the force lines, which provides stability for all simple perturbations. A rapid change in the direction of the field, realized above by superposition of three fields, breaks up the correlation of the motion produced by the instability with $m = l$, and the field configuration; rotation of the field leads to a time average dynamic stability with respect to a weak perturbation.* The application of electric fields only weakens the stability of the spherical shape. This is because the electric force lines formed at the induced surface charges are concentrated in surface regions of high curvature and any deformation leads to an additional negative electric pressure which tends to increase the deformation [cf. Eq. (9)].

In an inhomogeneous magnetic field a plasma of infinite conductivity is diamagnetic; in an inhomogeneous electric field it behaves like a pure dielectric. Thus, the average force acting on the sphere is given by

$$\mathbf{F} = r_0^3 \,\mathrm{grad}\,(2\overline{E^2} - \overline{H^2})/4.$$

By satisfying the stability requirements of the position of the sphere as a whole as well as those for volume and shape it is possible to make an isolated plasma stable in a given region of the external field.

The author is indebted to M. S. Rabinovich for valuable discussions of the present work.

*The containment of a plasma by a rotating magnetic field has also been considered by Butler et al.[4]

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[3] V. V. Yankov, J. Exptl. Theoret. Phys. (U.S.S.R.) **36**, 560 (1959), Soviet Phys. JETP **9**, 388 (1959).

[4] Butler, Hatch, and Ulrich, Second United Nations Intern. Conf. on the Peaceful Uses of Atomic Energy, p. 350, U.S.A., 1958.

Translated by H. Lashinsky
35