SUPERFLUIDITY AND THE MOMENTS OF INERTIA OF NUCLEI

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A method is developed for the treatment of superfluidity of nuclei. A formula which agrees satisfactorily with experiment is obtained for the moment of inertia of a nucleus. An expression is found for the change in the energy of "pairing" in the transition from an even-even to an even-odd nucleus, and also for the change in the moment of inertia associated with this transition.

1. INTRODUCTION

SYSTEMS consisting of interacting Fermi particles can be divided into two classes, depending on the type of excited states. When forces of repulsion prevail between the particles, the resultant excitations in the system are the same as in the case of free Fermi particles, but with an effective mass that depends on the forces of interaction between the particles. In the case of attractive forces, "correlated pairs" are formed and lead to an energy gap and to superfluidity.[1-3] One should think that the second type of single particle excitations takes place in nuclei.

In particular, this follows experimentally from the fact that the energy of the first single-particle excitation in even-even nuclei is several times larger than what one should expect for systems of the first type, and is larger than in the case of even-odd nuclei. The presence of a "pairing" energy, determined from mass defects, indicates this same phenomenon.

The existence of correlated pairs and superfluidity is evidenced most clearly in nuclear moments of inertia. The moments of inertia of nuclei are two or three times smaller than those computed from the formula for the moment of inertia of a solid, and this is the most direct evidence for the superfluidity of nuclear matter. Therefore, there is fundamental interest in the calculation of moments of inertia of nuclei on the basis of the modern theory of superfluidity of Fermi-systems. The formalism of this theory has been developed for homogeneous, unbounded systems.[4] However, in the case of a nucleus, the finite dimensions of the system are found to be very important.

A method is developed below, which permits us to study superfluidity in systems of finite dimensions. The moments of inertia are calculated by this method in a quasiclassical approximation and are in satisfactory agreement with the observed values of the moments of inertia.

The computed value of the moment of inertia in the transition from even-even to even-odd nucleus, and also the gyromagnetic ratio for rotating nuclei, are found to be in agreement with experiments.

These results thus confirm the assumption of the superfluidity of nuclear matter.

We note that the superfluidity of nuclear matter can lead to interesting macroscopic phenomena if stars with neutron cores exist. Such a star would be in a superfluid state with a transition temperature corresponding to 1 Mev.

2. METHOD OF ANALYZING A SYSTEM OF FINITE DIMENSIONS

1. For a study of superfluidity in a system of finite dimensions, it is convenient to make use of a method advanced by Gor'kov.[5]

In addition to the usual Green's function of a single particle

$$G(1,2) = -i(\Phi_N^0, T\Psi(1)\Psi^+(2)\Phi_N^0), \tag{1}$$

we introduce a function $F(1, 2)$, defined by the formula

$$F(1, 2) = (\Phi_N^0, \ T\Psi^+(1)\Psi^+(2)\Phi_{N-2}^0)e^{-2i\varepsilon_0 t}. \tag{2}$$

Here Φ_N^0 and Φ_{N-2}^0 are the eigenfunctions of the ground state of systems of N and $N-2$ interacting particles, T is the chronological operator, and $\Psi(\mathbf{r}, t)$, $\Psi^+(\mathbf{r}, t)$ are the operators of annihilation and creation of particles in the Heisenberg representation.

Gor'kov obtained a system of equations which relate G and F for weak δ-type interaction between the particles.

With the aid of methods of field theory, a similar set of equations can be obtained for an arbitrary interaction between the particles.

Another way of obtaining the system of equations for an arbitrary interaction is to apply the method of Gor'kov to a system of interacting quasiparticles.

The following system of equations is obtained for G and F:

$$[\varepsilon - H(\mathbf{r})]\,G(\mathbf{r}, \mathbf{r}', \varepsilon) = \delta(\mathbf{r} - \mathbf{r}') + i\Delta(\mathbf{r})\,F(\mathbf{r}, \mathbf{r}', \varepsilon),$$
$$[\varepsilon + H^*_{\bullet}(\mathbf{r}) - 2\varepsilon_0]\,F(\mathbf{r}, \mathbf{r}', \varepsilon) = -i\Delta^*(\mathbf{r})\,G(\mathbf{r}, \mathbf{r}', \varepsilon),$$
$$\Delta(\mathbf{r}) = \gamma(\mathbf{r}) \int_G F^*(\mathbf{r}, \mathbf{r}', \varepsilon)\,d\varepsilon/2\pi. \tag{3}$$

Here the functions $G(\mathbf{r}, \mathbf{r}', \epsilon)$ and $F(\mathbf{r}, \mathbf{r}', \epsilon)$ represent the Fourier transform of the functions $G(\mathbf{r}, \mathbf{r}', t-t')$ and $F(\mathbf{r}, \mathbf{r}', t-t')$, determined by Eqs. (1) and (2), with respect to the last variable. The value of ϵ_0 is given by the expression $\epsilon_0 = (E_N^0 - E_{N-2}^0)$, where E_N^0 is the ground-state energy of a system of N particles. The value of ϵ_0 determined in this way differs from the chemical potential $\epsilon_0' = E_{N+1}^0 - E_N^0$ by the "pairing" energy. The value of $\gamma(\mathbf{r})$ is determined by a function describing the interaction between the quasiparticles.

The boundary conditions for the functions G and F in $\mathbf{r}$, $\mathbf{r}'$ are determined by the boundary conditions for the eigenfunctions of the operator $H = p^2/2M_{\text{eff}} + U(\mathbf{r})$.

2. We assume that a solution of the set (3) satisfying the boundary conditions has been found. Knowing the value of $G(\mathbf{r}, \mathbf{r}', \epsilon)$, it is easy to find the single-particle density matrix

$$\rho(\mathbf{r}, \mathbf{r}') = (\Phi_N^0, \Psi^+(\mathbf{r})\,\Psi(\mathbf{r}')\,\Phi_N^0), \tag{4}$$

with the help of which we can determine the average value over the ground state of an arbitrary quantity A given in the form of a sum over the particles, $A = \sum_j A(\mathbf{r}_j, p_j)$.

The mean value of A is equal to

$$\langle A\rangle = (\Phi_N^0, \int \Psi^+ A(\mathbf{r}, p)\,\Psi\,d\mathbf{r}\,\Phi_N^0) = \mathrm{Sp}\,A\rho.$$

Comparing (4) with the definition (1) of the function G, we find

$$\rho(\mathbf{r}, \mathbf{r}') = -iG(\mathbf{r}, \mathbf{r}', \tau)\,|_{\tau\to -0} = \int_C G(\mathbf{r}, \mathbf{r}', \varepsilon)\,d\varepsilon/2\pi i, \tag{5}$$

where the contour C consists of the real axis and an infinite semicircle in the upper half plane.

3. The last equation of (3) contains the function $\gamma(\mathbf{r})$, which cannot be found without a detailed knowledge of the forces of interaction between the quasiparticles. The effective interaction between quasiparticles can be computed only in the form of a series in terms of some small parameter (for example, in the form of the series of perturbation theory, if the interaction between the particles is small). In the case of a nucleus, there is no small parameter to permit the use of approximate methods. Therefore the function $\gamma(\mathbf{r})$ in a nucleus cannot be calculated by analytic methods.

4. The system of equations (3) is materially simplified if the self-consistent potential entering into H does not depend upon the coordinates of the particle, i.e., in the case of a square potential well. Then the density of particles is constant over the volume and the function $\gamma(\mathbf{r})$, which depends upon the coordinates only through the medium of the density of the particles, is also a constant. It can be established that the function $F(\mathbf{r}, \mathbf{r}', \epsilon)$ is also independent of $\mathbf{r}$, and therefore the quantity $\Delta(\mathbf{r})$ is constant over the volume of the system.

Solutions of Eq. (3) for constant Δ are easily obtained if we expand the functions G and F in the eigenfunctions of the Hamiltonian H, which satisfy the boundary conditions for G and F:

$$G(\mathbf{r}, \mathbf{r}', \varepsilon) = \sum_{\lambda\lambda'} G_{\lambda\lambda'}(\varepsilon)\,\varphi_\lambda(\mathbf{r})\,\varphi^*_{\lambda'}(\mathbf{r}'),$$
$$F(\mathbf{r}, \mathbf{r}', \varepsilon) = \sum_{\lambda\lambda'} F_{\lambda\lambda'}(\varepsilon)\,\varphi_\lambda(\mathbf{r})\,\varphi^*_{\lambda'}(\mathbf{r}'), \qquad H\varphi_\lambda = \varepsilon_\lambda\varphi_\lambda. \tag{6}$$

Denoting

$$\varepsilon - \varepsilon_0 \to \varepsilon, \qquad H - \varepsilon_0 \to H, \qquad \varepsilon_\lambda - \varepsilon_0 \to \varepsilon_\lambda, \tag{7}$$

we obtain from (3)

$$(\varepsilon - H)\,G = \delta + i\Delta F, \qquad (\varepsilon + H^*)\,F = -i\Delta^* G,$$
$$\Delta = \gamma \int_C F^*\,d\varepsilon/2\pi. \tag{8}$$

Substitution of (6) into (8) gives,* for constant Δ,

$$G_{\lambda\lambda'} = \frac{\varepsilon + \varepsilon_\lambda}{\varepsilon^2 - \varepsilon_\lambda^2 - \Delta^2}\,\delta_{\lambda\lambda'}, \qquad F_{\lambda\lambda'} = -\frac{i\Delta}{\varepsilon^2 - \varepsilon_\lambda^2 - \Delta^2}\,\delta_{\lambda\lambda'}. \tag{9}$$

The poles of $G_{\lambda\lambda'}$ determine the energy levels of the single-particle excitations

$$\varepsilon = E_\lambda = \sqrt{\Delta^2 + \varepsilon_\lambda^2}. \tag{10}$$

5. To find the density matrix, it is necessary to know the singularities of G in the complex ϵ plane. In the denominator of $G_{\lambda\lambda'}$ we must substitute $\epsilon_\lambda - i\delta\epsilon_\lambda$, $\delta \to +0$ in place of ϵ_λ, which means the introduction of an infinitely small damping of the states that describe the particle and the hole. Then

$$E_\lambda = \sqrt{\Delta^2 + \varepsilon_\lambda^2} \to \sqrt{\Delta^2 + \varepsilon_\lambda^2 - 2i\delta\varepsilon_\lambda^2} = E_\lambda - i\delta_1, \qquad \delta_1 \to +0.$$

Thus, in the expression $\epsilon^2 - E_\lambda^2 = (\epsilon - E_\lambda)(\epsilon + E_\lambda)$,

*The constant phase of Δ in Eq. (8) can be chosen arbitrarily; therefore, for Δ = const we can set $\mathrm{Im}\Delta = 0$.

the first factor vanishes in the lower and the second in the upper half plane of ϵ.

According to (5), the density matrix is equal to

$$\rho_{\lambda\lambda'} = \int_C G_{\lambda\lambda'}\, d\varepsilon/2\pi i = (E_\lambda - \varepsilon_\lambda)\,\delta_{\lambda\lambda'}/2E_\lambda = \nu_\lambda \delta_{\lambda\lambda'}. \quad (11)$$

Similarly, we find

$$F_{\lambda\lambda'}(\tau = 0) = \int F_{\lambda\lambda'}(\varepsilon)\, d\varepsilon/2\pi = -\Delta/2E_\lambda.$$

Substituting this result in the last equation of (8), we find the link between γ and Δ:

$$1 = -\gamma \sum_\lambda \varphi_\lambda^*(\mathbf{r})\,\varphi_\lambda(\mathbf{r})/2E_\lambda. \quad (12)$$

6. We find the solution of the system (8) for a weak perturbation $H \rightarrow H + V$. The equations for terms of first order in V will be

$$(\varepsilon - H)\,G' - VG = i\Delta' F + i\Delta F',$$

$$(\varepsilon + H)\,F' + V^* F = -i\Delta'^* G - i\Delta G',$$

$$\Delta'^*(\mathbf{r}) = \gamma \int F'(\mathbf{r}, \mathbf{r}, \varepsilon)\, d\varepsilon/2\pi. \quad (13)$$

We find

$$G' = GVG + FV^*F + iG\Delta' F + iF\Delta'^* G,$$
$$F' = -DV^*F + FVG + iF\Delta' F - iD\Delta'^* G, \quad (14)$$

where

$$D = \frac{\varepsilon - H}{\varepsilon^2 - H^2 - \Delta^2}, \quad G = \frac{\varepsilon + H}{\varepsilon^2 - H^2 - \Delta^2},$$

$$F = -\frac{i\Delta}{\varepsilon^2 - H^2 - \Delta^2}.$$

Integrating the first equation of (14) with respect to ϵ, according to (5), we find the first-order correction to the density matrix. After simple algebraic manipulations, we obtain

$$\rho'_{\lambda\lambda'} = \int_C G'_{\lambda\lambda'} \frac{d\varepsilon}{2\pi i}$$
$$= \frac{(\varepsilon_\lambda \varepsilon_{\lambda'} - E_\lambda E_{\lambda'})\,V_{\lambda\lambda'} - \Delta^2 V^*_{\lambda\lambda'} + \Delta\,(\varepsilon_\lambda \Delta'_{\lambda\lambda'} + \varepsilon_{\lambda'} \Delta'^*_{\lambda\lambda'})}{2E_\lambda E_{\lambda'}\,(E_\lambda + E_{\lambda'})}. \quad (15)$$

In similar fashion we compute the quantity

$$\int F'_{\lambda\lambda'} \frac{d\varepsilon}{2\pi} = \frac{\Delta\,(\varepsilon_\lambda V^*_{\lambda\lambda'} + \varepsilon_{\lambda'} V_{\lambda\lambda'}) + \Delta^2 \Delta'_{\lambda\lambda'} - (E_\lambda E_{\lambda'} + \varepsilon_\lambda \varepsilon_{\lambda'})\,\Delta'^*_{\lambda\lambda'}}{2E_\lambda E_{\lambda'}\,(E_\lambda + E_{\lambda'})}. \quad (16)$$

The last equation of (13) can be written

$$\Delta'^*(\mathbf{r}) = \gamma \sum_{\lambda\lambda'} \varphi_\lambda(\mathbf{r})\,\varphi^*_{\lambda'}(\mathbf{r}) \int F'_{\lambda\lambda'}(\varepsilon)\, d\varepsilon/2\pi.$$

Here we have neglected changes in γ under the action of the perturbation V. The left side of this equation can be represented in the form

$$\Delta'^*(\mathbf{r}) = -\gamma \sum_{\lambda\lambda'} \Delta'^*_{\lambda\lambda'}\,\varphi_\lambda(\mathbf{r})\,\varphi^*_{\lambda'}(\mathbf{r})/2E_\lambda$$
$$= -\gamma \sum_{\lambda\lambda'} \Delta'^*_{\lambda\lambda'}\,\varphi_\lambda(\mathbf{r})\,\varphi^*_{\lambda'}(\mathbf{r})/2E_{\lambda'}.$$

with the help of (12). We obtain an integral equation for $\Delta'(\mathbf{r})$:

$$\sum_{\lambda\lambda'} \left[\int F'_{\lambda\lambda'}\, d\varepsilon/2\pi + \Delta'^*_{\lambda\lambda'}/4E_\lambda + \Delta'^*_{\lambda\lambda'}/4E_{\lambda'}\right] \varphi_\lambda \varphi^*_{\lambda'} = 0.$$

Making use of (16), we find

$$\sum_{\lambda\lambda'} \frac{2\Delta\,(\varepsilon_\lambda V^*_{\lambda\lambda'} + \varepsilon_{\lambda'} V_{\lambda\lambda'}) + 2\,\Delta^2 \Delta'_{\lambda\lambda'} + [2\Delta^2 + (\varepsilon_\lambda - \varepsilon_{\lambda'})^2]\,\Delta'^*_{\lambda\lambda'}}{E_\lambda E_{\lambda'}\,(E_\lambda + E_{\lambda'})}$$
$$\times \varphi_\lambda \varphi^*_{\lambda'} = 0. \quad (17)$$

3. METHOD OF CALCULATING THE MOMENT OF INERTIA

1. For the calculation of the moment of inertia, it is necessary to find the change in the Hamiltonian resulting from a transformation to a rotating set of coordinates.

Let $H = \mathscr{E}(\mathbf{p}) + U$, where $\mathbf{p}$ the momentum operator. To find the change in H upon transformation to the rotating system of coordinates, one must replace $\mathbf{p}$ by $\mathbf{p} - M\mathbf{r} \times \boldsymbol{\Omega}$, where M is the mass of the nucleon and Ω is the angular velocity of the system of coordinates. We obtain

$$V = -\frac{\partial \varepsilon}{\partial \mathbf{p}} M\,[\mathbf{r} \times \boldsymbol{\Omega}] = -\frac{M}{M_{eff}} \mathfrak{M}\Omega = -\frac{M}{M_{eff}} \mathfrak{M}_x\,\Omega, \quad (18)$$

where $M_{eff} = p/(\partial\epsilon/\partial p)$ is the effective mass of the quasiparticles; $\mathfrak{M} = \mathbf{r} \times \mathbf{p}$, and the x axis is chosen along $\boldsymbol{\Omega}$. Noting that $V^* = -V$, and denoting $\Delta'(\mathbf{r}) = if(\mathbf{r})\,\Omega M/M_{eff}$, we obtain from (15)

$$\frac{1}{\Omega}\rho'_{\lambda\lambda'} = \frac{(E_\lambda E_{\lambda'} - \varepsilon_\lambda \varepsilon_{\lambda'} - \Delta^2)\,\mathfrak{M}^x_{\lambda\lambda'} + i\Delta(\varepsilon_\lambda - \varepsilon_{\lambda'})\,f_{\lambda\lambda'}}{2E_\lambda E_{\lambda'}\,(E_\lambda + E_{\lambda'})} \frac{M}{M_{eff}}, \quad (19)$$

where

$$f_{\lambda\lambda'} = \int \varphi^*_\lambda(\mathbf{r})\, f(\mathbf{r})\, \varphi_{\lambda'}(\mathbf{r})\, d\mathbf{r}.$$

Multiplying (17) by i, we find

$$\sum_{\lambda\lambda'} \frac{2\Delta \mathfrak{M}^x_{\lambda\lambda'} + (\varepsilon_\lambda - \varepsilon_{\lambda'})\,f_{\lambda\lambda'}}{E_\lambda E_{\lambda'}\,(E_\lambda + E_{\lambda'})} \varphi_\lambda(\mathbf{r})\,\varphi^*_{\lambda'}(\mathbf{r}) = 0. \quad (20)$$

We find the mean value of the momentum

$$\langle \mathfrak{M}^x \rangle = \mathrm{Sp}\,\rho\,\mathfrak{M}^x = \sum_{\lambda\lambda'} \mathfrak{M}^x_{\lambda\lambda'}\,\rho_{\lambda'\lambda}.$$

The moment of inertia is obtained from the expression

$$J = \frac{\langle \mathfrak{M}^x \rangle}{\Omega} = \sum_{\lambda\lambda'} \frac{(E_\lambda E_{\lambda'} - \varepsilon_\lambda \varepsilon_{\lambda'} - \Delta^2) \, |\mathfrak{M}^x_{\lambda\lambda'}|^2 - \Delta f_{\lambda\lambda'} \dot{\mathfrak{M}}^x_{\lambda\lambda'}}{2E_\lambda E_{\lambda'} (E_\lambda + E_{\lambda'})} \frac{M}{M_{eff}}, \tag{21}$$

where f must be determined from the integral equation (20).

2. We note that the integral equation (20) for f can be obtained in another way. With the help of ρ we set up the mean value of the current density $\langle \mathbf{j}(\mathbf{r}) \rangle = \mathrm{Sp}\,(\rho' \mathbf{j} - \rho_0 \mathbf{r} \times \mathbf{\Omega})$; then Eq. (20) is obtained from the condition $\operatorname{div} \langle \mathbf{j}(\mathbf{r}) \rangle = 0$ without use of the third equation of (8).

This circumstance permits us to solve the first two equations of (8) with $\Delta = \mathrm{const}$ for an arbitrary initial Hamiltonian without danger of running into a contradiction, in spite of the fact that the last equation of (8) leads to $\Delta = \mathrm{const}$ only for a square potential well.

Let us consider the expression

$$J_1 = \sum_{\lambda\lambda'} \frac{E_\lambda E_{\lambda'} - \varepsilon_\lambda \varepsilon_{\lambda'} - \Delta^2}{2E_\lambda E_{\lambda'} (E_\lambda + E_{\lambda'})} \, |\mathfrak{M}^x_{\lambda\lambda'}|^2 \, \frac{M}{M_{eff}}. \tag{22}$$

It is not difficult to see that the quantity

$$\mathcal{L}(\varepsilon_\lambda, \varepsilon_{\lambda'}) = (E_\lambda E_{\lambda'} - \varepsilon_\lambda \varepsilon_{\lambda'} - \Delta^2) / 2E_\lambda E_{\lambda'} (E_\lambda + E_{\lambda'}) \tag{23}$$

as a function of ϵ_λ, for a fixed difference $d = \epsilon_\lambda - \epsilon_{\lambda'}$, has a sharp maximum of width $\sim \Delta$ at the point $\epsilon_\lambda + \epsilon_{\lambda'} = 0$. We denote

$$\int \mathcal{L}(\varepsilon_\lambda, \varepsilon_\lambda + d) \, d\varepsilon_\lambda = U(d/2\Delta).$$

It is easy to establish the fact that

$$U(x) = 1 - \ln (x + \sqrt{1+x^2}) / x \sqrt{1+x^2}. \tag{24}$$

When a sufficient number of levels are contained in the width Δ, we may replace the function $\mathcal{L}$ in (22) by

$$\mathcal{L}(\varepsilon_\lambda, \varepsilon_{\lambda'}) \to U((\varepsilon_\lambda - \varepsilon_{\lambda'})/2\Delta) \, \delta(\varepsilon_\lambda). \tag{25}$$

Making use of (24) and (25), we get in place of (22)

$$J_1 = \sum_{\lambda\lambda'} \left\{ 1 - g\left(\frac{\varepsilon_\lambda - \varepsilon_{\lambda'}}{2\Delta} \right) \right\} |\mathfrak{M}^x_{\lambda\lambda'}|^2 \, \delta(\varepsilon_\lambda) \, \frac{M}{M_{eff}}, \tag{26}$$

where

$$g(x) = (\sinh^{-1} x) / x (1 + x^2)^{1/2}. \tag{27}$$

4. We now show that the expression

$$J_0 = \sum_{\lambda\lambda'} |\mathfrak{M}^x_{\lambda\lambda'}|^2 \, \delta(\varepsilon_\lambda) \, M / M_{eff} \tag{28}$$

differs only by a factor from the value of the moment of inertia for a solid. We carry out a summation over λ' in (28), and denote

$$\rho(\varepsilon_0, \mathbf{r}) = \sum_\lambda \varphi^*_\lambda(\mathbf{r}) \, \varphi_\lambda(\mathbf{r}) \, \delta(\varepsilon_\lambda), \tag{29}$$

where $\rho(\epsilon_0, \mathbf{r})$ is the density of particles with energy ϵ_0. We obtain

$$J_0 = \int d\mathbf{r} \, \rho(\varepsilon_0, \mathbf{r}) \, \langle (\mathfrak{M}^x)^2 \rangle \, M / M_{eff}.$$

Here $(\mathfrak{M}^x)^2$ is replaced by the value of $\langle (\mathfrak{M}^x)^2 \rangle$ averaged over the angles of the momentum $\mathbf{p}$. Since all the directions of $\mathbf{p}$ are equally probable, then by averaging the operator $(\mathfrak{M}^x)^2 = y^2 p_z^2 + z^2 p_y^2 - y p_z z p_y - z p_y y p_z$ over the angles of $\mathbf{p}$, we obtain

$$\langle (\mathfrak{M}^x)^2 \rangle = {}^1/_3 \, p_0^2(\mathbf{r}) \, (y^2 + z^2).$$

We denote

$$n'(\mathbf{r}) = \rho(\varepsilon_0, \mathbf{r}) \, p_0^2 / 3M_{eff},$$

Then

$$J_0 = M \int n'(\mathbf{r}) (y^2 + z^2) \, d\mathbf{r}.$$

In the quasiclassical approximation, the density of particles is

$$n(\mathbf{r}) = C p_0^3(\mathbf{r}), \qquad p_0(\mathbf{r}) = \sqrt{2M_{eff}(\varepsilon_0 - U(\mathbf{r}))},$$

$$\rho(\varepsilon_0, \mathbf{r}) = \partial n(\mathbf{r}) / \partial \varepsilon_0 = 3M_{eff} \, n(\mathbf{r}) / p_0^2(\mathbf{r}), \tag{30}$$

i.e., $n'(\mathbf{r}) = n(\mathbf{r})$, and consequently J_0 coincides with the value of the moment of inertia for a solid:

$$J_{\mathrm{sol}} = \int M n(\mathbf{r}) (y^2 + z^2) \, d\mathbf{r}.$$

However, because of the existence of shells, the density of levels on the Fermi surface $\rho_0 = \int \rho(\epsilon_0, \mathbf{r}) \, d\mathbf{r}$ can differ from the classical value $\rho_0^{\mathrm{cl}} = \int (\partial n / \partial \epsilon) \, d\mathbf{r}$; therefore,

$$J_0 = (\rho_0 / \rho_0^{\mathrm{cl}}) \, J_{\mathrm{sol}} \tag{31}$$

We note that the effective mass does not enter into J_0, as expected.

For a sufficiently deformed nucleus ($\beta \gtrsim A^{-1/3}$) we have $\rho_0 / \rho_0^{\mathrm{cl}} \approx 1$ and $J_0 = J_{\mathrm{sol}}$. For small deformations ($\beta \ll A^{-1/3}$), when the shell structure substantially affects the density of levels, $\rho_0 / \rho_{\mathrm{cl}} > 1$.

5. Let us now consider the second component in the moment of inertia

$$J_2 = -\Delta \sum_{\lambda\lambda'} \frac{f_{\lambda\lambda'} \dot{\mathfrak{M}}^x_{\lambda\lambda'}}{2E_\lambda E_{\lambda'} (E_\lambda + E_{\lambda'})} \frac{M}{M_{eff}}.$$

Multiplying Eq. (20) by $f(\mathbf{r})$ and integrating over $\mathbf{r}$ we obtain

$$-\Delta \sum_{\lambda\lambda'} \frac{\dot{\mathfrak{M}}^x_{\lambda\lambda'} f_{\lambda'\lambda}}{2E_\lambda E_{\lambda'} (E_\lambda + E_{\lambda'})} = -\Delta \sum_{\lambda\lambda'} \frac{f_{\lambda\lambda'} \dot{\mathfrak{M}}^x_{\lambda'\lambda}}{2E_\lambda E_{\lambda'} (E_\lambda + E_{\lambda'})}$$

$$= \sum_{\lambda\lambda'} \frac{(\varepsilon_\lambda - \varepsilon_{\lambda'})^2 \, |f_{\lambda\lambda'}|^2}{4E_\lambda E_{\lambda'} (E_\lambda + E_{\lambda'})}.$$

Consequently,

$$J_2 = \sum_{\lambda\lambda'} \frac{(\varepsilon_\lambda - \varepsilon_{\lambda'})^2 |f_{\lambda\lambda'}|^2}{4E_\lambda E_{\lambda'}(E_\lambda + E_{\lambda'})} \frac{M}{M_{eff}}.$$

Making use of the sharp maximum of $1/E_\lambda E_{\lambda'}(E_\lambda + E_{\lambda'})$ with respect to ϵ_λ, similarly to what was done in the calculation of J_1, we obtain

$$J_2 = -\frac{1}{2\Delta}\sum_{\lambda\lambda'} f_{\lambda\lambda'} \dot{\mathfrak{M}}^x_{\lambda'\lambda} g\delta(\varepsilon_\lambda) M/M_{eff}$$

$$= \sum_{\lambda\lambda'} |f_{\lambda\lambda'}|^2 g\left(\frac{\varepsilon_\lambda - \varepsilon_{\lambda'}}{2\Delta}\right) \frac{(\varepsilon_\lambda - \varepsilon_{\lambda'})^2}{4\Delta^2} \delta(\varepsilon_\lambda) \frac{M}{M_{eff}}, \qquad (32)$$

where $g(x)$ is given by (27).

6. Using the sharp maximum of $1/E_\lambda E_{\lambda'}(E_\lambda + E_{\lambda'})$ we obtain, in place of (20),

$$\sum_{\lambda\lambda'} \left[\dot{\mathfrak{M}}^x_{\lambda\lambda'}/2\Delta + f_{\lambda\lambda'}(\varepsilon_\lambda - \varepsilon_{\lambda'})^2/4\Delta^2\right]$$

$$\times g((\varepsilon_\lambda - \varepsilon_{\lambda'})/2\Delta)\varphi_\lambda(\mathbf{r})\varphi^*_{\lambda'}(\mathbf{r})\delta(\varepsilon_\lambda) = 0. \qquad (33)$$

Thus the moment of inertia is calculated according to the formula

$$J = J_0 - \frac{M}{M_{eff}}\sum_{\lambda\lambda'} g\left(\frac{\varepsilon_\lambda - \varepsilon_{\lambda'}}{2\Delta}\right) \left|\dot{\mathfrak{M}}^x_{\lambda\lambda'}\right|^2 \delta(\varepsilon_\lambda)$$

$$+ \frac{M}{M_{eff}}\sum_{\lambda\lambda'} g\left(\frac{\varepsilon_\lambda - \varepsilon_{\lambda'}}{2\Delta}\right)\left(\frac{\varepsilon_\lambda - \varepsilon_{\lambda'}}{2\Delta}\right)^2 |f_{\lambda\lambda'}|^2 \delta(\varepsilon_\lambda), \qquad (34)$$

in which one must insert $f(\mathbf{r})$, found from the integral equation (33).

We introduce the quantity

$$\varkappa = \varepsilon_0\beta/\Delta p_0 R_0, \qquad (35)$$

where $\beta = 2(a-b)/(a+b)$ is a parameter of the deformation of the nucleus, a and b are the semi-axes of the spheroid, $R_0 = (a+b)/2$, and $\hbar = 1$.

It is not difficult to show that for $\kappa \gg 1$

$$J \to J_0. \qquad (36)$$

In the other limiting case, when $\kappa/\beta \ll 1$, assuming that the potential U has the form $U(z^2/a^2 + (x^2+y^2)/b^2)$, we can obtain

$$J = J_0\left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2 + C_1 J_0 \varkappa^2, \qquad (37)$$

where $C_1 \sim 1$.

Thus, for $\kappa/\beta \ll 1$, the moment of inertia of the system approaches the moment of inertia of an ideal liquid. This result is quite natural. The quantity κ/β is of the order of a_0/R, where $a_0 = p_0/M\Delta$ is the correlation radius of the paired particles. When the correlation radius is much smaller than the radius of the nucleus, the equations of the hydrodynamics of an ideal liquid should be applicable.

For $\beta \to 0$ and fixed Δ, $J \sim \beta^2$.

4. COMPUTATION OF THE MOMENT OF INERTIA FOR AN OSCILLATORY POTENTIAL

Let us write the potential $U(\mathbf{r})$ in the form

$$U(\mathbf{r}) = \tfrac{1}{2} M_{eff}[\omega_z^2 z^2 + \omega_y^2(x^2 + y^2)]. \qquad (38)$$

We have the operator relation

$$\dot{\mathfrak{M}}^x = z\partial U/\partial y - y\partial U/\partial z = (\omega_y^2 - \omega_z^2) yz M_{eff}.$$

Therefore,

$$\dot{\mathfrak{M}}^x_{\lambda\lambda'} = (\omega_y^2 - \omega_z^2) y_{mm'} z_{nn'} M_{eff}.$$

Thus the matrix element $\dot{\mathfrak{M}}^x_{\lambda\lambda'}$ is different from 0 only for $m' = m \pm 1$, $n' = n \pm 1$. With quasiclassical accuracy ($n \gg 1$, $m \gg 1$), all four possible values of $\dot{\mathfrak{M}}^x_{\lambda\lambda'}$ are identical.

Moreover, noting that $\epsilon_\lambda - \epsilon_{\lambda'} = \pm(\omega_z \pm \omega_y)$, we find from (26)

$$J_1 = J_0 - \sum_{\lambda\lambda'} g\left(\frac{\varepsilon_\lambda - \varepsilon_{\lambda'}}{2\Delta}\right)\left|\dot{\mathfrak{M}}^x_{\lambda\lambda'}\right|^2 \frac{1}{(\varepsilon_\lambda - \varepsilon_{\lambda'})^2}\delta(\varepsilon_\lambda)\frac{M}{M_{eff}}$$

$$= J_0 - \frac{1}{2}\left(\frac{g_1}{\nu_1^2} + \frac{g_2}{\nu_2^2}\right)\sum_{\lambda\lambda'} |\dot{\mathfrak{M}}^x_{\lambda\lambda'}|^2 \frac{1}{4\Delta^2}\frac{M}{M_{eff}}, \qquad (39)$$

where

$$\nu_1 = (\omega_z - \omega_y)/2\Delta, \qquad \nu_2 = (\omega_z + \omega_y)/2\Delta,$$

$$g_1 = g(\nu_1), \qquad g_2 = g(\nu_2).$$

Similarly, we obtain

$$J_0 = \sum_{\lambda\lambda'} |\dot{\mathfrak{M}}^x_{\lambda\lambda'}|^2 \delta(\varepsilon_\lambda)\frac{M}{M_{eff}}$$

$$= \frac{1}{2}\frac{\nu_1^2 + \nu_2^2}{\nu_1^2\nu_2^2}\frac{1}{4\Delta^2}\sum_{\lambda\lambda'} |\dot{\mathfrak{M}}^x_{\lambda\lambda'}|^2 \delta(\varepsilon_\lambda)\frac{M}{M_{eff}}.$$

Substituting this result in (39), we obtain

$$J_1 = J_0\left(1 - \frac{g_1\nu_2^2 + g_2\nu_1^2}{\nu_1^2 + \nu_2^2}\right). \qquad (40)$$

2. We shall show that Eq. (33) is satisfied for $f = \alpha yz$. As above, we make use of the independence of the matrix elements $\dot{\mathfrak{M}}^x_{\lambda\lambda'}$, $f_{\lambda\lambda'}$, and $\varphi_\lambda\varphi^*_{\lambda'}$ of the values of λ' for a given value of λ.

From Eq. (33) we obtain

$$\Delta(g_1 + g_2)\sum_{\lambda\lambda'} \dot{\mathfrak{M}}^x_{\lambda\lambda'}\varphi_\lambda\varphi^*_{\lambda'}\delta(\varepsilon_\lambda)$$

$$= -2\Delta^2(g_1\nu_1^2 + g_2\nu_2^2)\sum_{\lambda\lambda'} f_{\lambda\lambda'}\varphi_\lambda\varphi^*_{\lambda'}\delta(\varepsilon_\lambda). \qquad (40a)$$

Summing over λ', we obtain

$$f(\mathbf{r}) = -\frac{g_1 + g_2}{2\Delta(g_1\nu_1^2 + g_2\nu_2^2)}\dot{\mathfrak{M}}^x = -\frac{(g_1 + g_2)(\omega_y^2 - \omega_z^2)yz}{2\Delta(g_1\nu_1^2 + g_2\nu_2^2)}M_{eff}. \qquad (41)$$

Substituting (41) in (32), we find

$$J_2 = J_0(g_1 + g_2)\nu_1^2\nu_2^2/(\nu_1^2 + \nu_2^2)(\nu_1^2 g_1 + \nu_2^2 g_2). \qquad (42)$$

It is easy to see that the expression (40) goes over into J_0 when ν_1, $\nu_2 \gg 1$, while J_2, determined from (42), approaches 0 so that the asymptotic formula (36) is satisfied.

For ν_1, $\nu_2 \ll 1$, the relation (37) is satisfied, i.e., the moment of inertia approaches the moment of inertia of an ideal liquid.

3. Substitution of numerical values of ν_1 and ν_2 for the nuclei gives $\nu_2 \sim 10$, $\nu_1 \sim 1$, which permits us to write down an expression for J in the form

$$J \approx J_0 \left\{1 - g_1 + \frac{g_1^2 \nu_1^2}{\nu_1^2 g_1 + \ln 2\nu_2}\right\} = J_0 \Phi_1(\nu_1). \qquad (43)$$

The values of the function $\Phi_1(x)$ are given in Table I (for $\nu_2 = 10$).

5. COMPUTATION OF THE MOMENT OF INERTIA FOR A RECTANGULAR POTENTIAL WELL

1. From Eqs. (26) and (28), we have

$$J_0 - J_1 = \sum_{\lambda\lambda'} g\left(\frac{\varepsilon_\lambda - \varepsilon_{\lambda'}}{2\Delta}\right) |\mathfrak{M}^x_{\lambda\lambda'}|^2 \delta(\varepsilon_\lambda) \frac{M}{M_{eff}}. \qquad (44)$$

We shall show that only such values of λ' are to be kept in this sum for which $m' = m \pm 1$ and $\nu' = \nu$, where m is the eigenvalue of the projection of the angular momentum on the axis of symmetry and ν is the set of remaining quantum numbers.

The relation $m' = m \pm 1$ is the exact selection rule for the matrix elements of the operator $\mathfrak{M}^x$. We shall show that the components with $\nu' \neq \nu$ contribute little to the sum (44).

The index ν is determined by three quantum numbers; for example, in the case of a nucleus that is close to spherical, by the radial quantum number n, the orbital number L, and the value of the total momentum $j = l \pm \frac{1}{2}$. If one of these quantum numbers is changed, then the order of the energy change is $|\epsilon_\lambda - \epsilon_{\lambda'}| \sim \epsilon_0/p_0R \sim \epsilon_0/A^{1/3} \gg \Delta$, and the corresponding value of the function g is $\sim (\Delta p_0 R/\epsilon_0) \ll 1$. Therefore such components make a negligibly small contribution to the sum (44).

Small differences $|\epsilon_\lambda - \epsilon_{\lambda'}| \sim \epsilon_0 A^{-2/3}$ can be obtained upon an increase of one quantum number by several units simultaneous with a decrease of another. In this case, $g \sim 1$, but the matrix element $M^X_{\lambda\lambda'}$ will be small because of the difference in the number of nodes of the functions φ_λ and $\varphi_{\lambda'}$ and, moreover, will contain the factor β, inasmuch as $M^X_{\lambda\lambda'} = 0$ for a spherical well at $\nu' \neq \nu$. Therefore, only components with $\nu = \nu'$ and $m' = m \pm 1$ should remain in the sum (44):

$$J_0 - J_1 \approx \sum_{\nu, m} g\left(\frac{\varepsilon_{\nu, m+1} - \varepsilon_{\nu, m}}{2\Delta}\right) [|\mathfrak{M}^x_{\nu, m; \nu, m+1}|^2 + |\mathfrak{M}^x_{\nu, m; \nu, m-1}|^2] \delta(\varepsilon_{\nu m}) \frac{M}{M_{eff}}. \qquad (45)$$

2. Further calculations are carried out for a spherical well with effective diameter R_0. The quantity J/J_0 is a symmetric function of the semiaxes a and b, as can be seen, in particular, in the example of the calculations for an oscillatory potential. Therefore, choosing in the final result a radius equal to any mean symmetric in a and b, we determine J/J_0 with a small error of the order of β^2.

For the functions of the spherical well we have

$$|\mathfrak{M}^x_{\nu, m; \nu, m+1}|^2 + |\mathfrak{M}^x_{\nu, m; \nu, m-1}|^2 = {}^1/_2 (j^2 - m^2) + {}^1/_4 j \approx {}^1/_2 (l^2 - m^2), \qquad (46)$$

since the large values of j are important in the sum (45) and one can replace $j = l \pm \frac{1}{2}$ by l. The dependence of the energy of the particle on m is given by the equality

$$\varepsilon_{\nu m} = \varepsilon_\nu + \beta (m^2 / l^2 - {}^1/_3)(\varepsilon_0 + \varepsilon_\nu), \qquad (47)$$

where ϵ_ν is the energy of a particle in a spherical well, reckoned from ϵ_0.

Equation (47) can easily be obtained with the help of perturbation theory. Carrying out a coordinate transformation that converts the ellipsoid into a sphere, we obtain (in first order in β) the perturbation operator

$$Q = {}^1/_3 \beta (p^2 - 3p_z^2)/M_{eff}.$$

Computing the energy by the formula

$$\varepsilon_{\nu m} = \varepsilon_\nu + \int \varphi_\lambda^* Q \varphi_\lambda d\mathbf{r},$$

where φ_λ are the eigenfunctions in a spherical well, we arrive at Eq. (47).

Substituting (46) and (47) in (45), we get

$$J_0 - J_1 = \sum_{\nu, m} g\left(\frac{\beta m}{\Delta l^2}\varepsilon_0\right) \frac{l^2 - m^2}{2} \delta(\varepsilon_\nu) \frac{M}{M_{eff}}$$
$$= \sum_{lm} g\left(\frac{\beta m}{\Delta l^2}\varepsilon_0\right) \frac{l^2 - m^2}{2} \rho_{lm}(\varepsilon_0) \frac{M}{M_{eff}}, \qquad (48)$$

where $\rho_{lm}(\epsilon_0)$ is the energy density of the level with specified values of l and m. Computing the volume of the phase space corresponding to the given values of l and m, we readily obtain

$$\rho_{lm}(\varepsilon_0) = {}^3/_2\, l_0^{-3} \rho_0 \sqrt{l_0^2 - l^2}, \qquad (49)$$

in a way similar to what was done in the Thomas-Fermi theory; here $l_0 = p_0R_0$. Making use of (30) and (31), we obtain

TABLE I. Values of the functions Φ_1 and Φ_2, determined from Eqs. (43) and (51)

x	$\Phi_1(x)$	$\Phi_2(x)$	x	$\Phi_1(x)$	$\Phi_2(x)$	x	$\Phi_1(x)$	$\Phi_2(x)$	x	$\Phi_1(x)$	$\Phi_2(x)$
0	0	0	0.7	0.32	0.13	1.3	0.64	0.24	2.0	0.80	0.37
0.1	0.01	0.005	0.8	0.38	0.15	1.4	0.67	0.26	2.2	0.81	0.40
0.2	0.03	0.02	0.9	0.43	0.17	1.5	0.71	0.28	2.4	0.83	0.43
0.3	0.07	0.03	1.0	0.49	0.19	1.6	0.74	0.30	2.6	0.85	0.45
0.4	0.13	0.06	1.1	0.53	0.21	1.7	0.75	0.32	2.7	0.86	0.48
0.5	0.19	0.08	1.2	0.60	0.23	1.8	0.77	0.34	3.0	0.87	0.50
0.6	0.26	0.10				1.9	0.79	0.36			

$$J_0 = \frac{2}{5} NMR_0^2 = \frac{2}{15} = \frac{M}{M_{eff}} \rho_0 R_0^2. \quad (50)$$

Substituting (49), (50) in (48), and introducing the variables $\eta = m/l$, $\xi = l/l_0$, we obtain

$$J_1 = J_0 \Big\{ 1 - \frac{45}{4} \int_0^1 d\xi \xi^3 \sqrt{1-\xi^2}$$

$$\times \int_0^1 d\eta (1-\eta^2)\, g\left(\varkappa \frac{\eta}{\xi}\right) \Big\} = J_0 \Phi_2(\varkappa), \quad (51)$$

where κ is determined by Eq. (35).

The values of the function $\Phi_2(\kappa)$ are given in Table I.

3. In the case of a rectangular potential well, the solution of Eq. (33) is found in analytic form only for the limiting cases $\kappa \ll 1$ and $\kappa \gg 1$.

The value of J_2 is shown to be much smaller than that of J_1; therefore, such a solution of Eq. (33) is not of practical interest.

We shall write down without derivation an expression for J_2 in the cases $\kappa \ll 1$ and $\kappa \gg 1$;

$$\varkappa \to 0: J_2 = J_0 \varkappa^2 / 6 \ln (5\varepsilon_0 / \Delta p_0 R_0),$$

$$\varkappa \to \infty: J_2 = J_0 5 \ln^3 (a_1 \varkappa) / 21 \varkappa^2 \ln (a_2 \varepsilon_0 / \Delta p_0 R_0), \quad (52)$$

where a_1 and a_2 are numbers of the order of unity.

The equations (52) allow us to conclude that $J_2/J_0 \lesssim 5\%$ and consequently $J_1 \approx J$.

6. NEUTRON AND PROTON MOMENTS OF INERTIA. GYROMAGNETIC RATIO

1. The calculations carried out above were performed for particles of a single kind. For comparison with experiment it is necessary to consider both types of nuclear particles.

When $Z > 20$, the Fermi surfaces for neutrons and protons move apart and the coupling of neutrons with protons is made impossible. Thus for $Z > 20$, there are two liquids — neutron and proton — which cannot exchange angular momentum since the excitations in each of them have gaps.

Therefore, the moment of inertia of the nucleus is equal to the sum of the moments of inertia of the neutrons and protons. Denoting by $\Phi(\kappa)$ the ratio $J/J_0 = \Phi(\kappa)$ computed above, we obtain the following expression for the moment of inertia

$$\frac{J}{J_0} = \frac{N}{A} \Phi(\varkappa_n) + \frac{Z}{A} \Phi(\varkappa_p), \quad (53)$$

where κ_n and κ_p are the values of κ for neutrons and protons.

2. The quantities Δ_n and Δ_p entering into κ_n and κ_p cannot be computed theoretically and must be taken from experiment.

As will be shown below, Δ_n is smaller than Δ_p; therefore, Δ_n is determined from the first single-particle levels. The value of Δ_p can be found from the experimental gyromagnetic ratio for rotating nuclei. The gyromagnetic ratio is shown to be much smaller than Z/A.*

The gyromagnetic ratio for rotating nuclei is evidently equal to

$$g_r = \frac{\int [\mathbf{r} \times \mathbf{j}_p]\, d\mathbf{r}}{\int [\mathbf{r} \times (\mathbf{j}_n + \mathbf{j}_p)]\, d\mathbf{r}} = \frac{J_p}{J_n + J_p} = \frac{Z\Phi(\varkappa_p)}{N\Phi(\varkappa_n) + Z\Phi(\varkappa_p)}. \quad (54)$$

The measured[6] value of g_r for Sm^{152}, Sm^{154}, Nd^{150} is $g_r \approx 0.21 \pm 0.04$ when $J_n/J_p = N\Phi(\kappa_n)/Z\Phi(\kappa_p) \approx 3.5 \pm 1$.

The function $\Phi_1(\kappa)$ is represented in the interval $\kappa = 1-2$ by $\Phi_1 = C\kappa$ with good accuracy, whence we obtain

$$J_n/J_p = (N/Z)^{1/3} \Delta_p/\Delta_n \approx 3.5 \pm 1,$$

which gives $\Delta_p/\Delta_n = 1.5$ to 2.5 [$\Phi_2(\kappa)$ gives a somewhat larger value]; such a value of Δ_p/Δ_n agrees within the limits of error with the value of Δ_p/Δ_n found from mass defects in this region of the table ($\Delta_p/\Delta_n = 1.5 \pm 0.3$).

7. MOMENTS OF INERTIA OF ODD NUCLEI. MOMENTS OF INERTIA IN EXCITED STATES.

1. Integrating Eq. (12) over the volume and denoting $\gamma/V = \gamma_1$, we obtain

*The attention of the author was directed to this fact by D. F. Zaretskiĭ.

$$1 = -\gamma_1 \sum_\lambda (1/2E_\lambda). \qquad (12')$$

The value of γ_1 is determined by the density of nuclear matter and by the nuclear forces, and should not change appreciably from nucleus to nucleus.

We shall show that the Δ determined from (12′) experiences significant fluctuations. We rewrite (12′) in the form

$$1 = -\gamma_1 \int_{-\varepsilon_0}^{\varepsilon_1} \rho(\varepsilon) d\varepsilon / 2\sqrt{\Delta^2 + \varepsilon^2}, \qquad (12'')$$

where $\epsilon_1 \sim \epsilon_0$, $\rho(\epsilon)$ is the energy level density.

If in the transition from one nucleus to another a change $\delta\rho$ takes place in the density of levels close to the Fermi surface, then it follows from (12″) that the corresponding change in Δ is given by the relation

$$\int_{-\varepsilon_0}^{\varepsilon_1} \delta\rho\, d\varepsilon / 2\sqrt{\Delta^2 + \varepsilon^2} - \int_{\varepsilon_0}^{\varepsilon_1} \rho(\varepsilon)\, \Delta\delta\Delta\, d\varepsilon / 2(\Delta^2 + \varepsilon^2)^{3/2} = 0.$$

Since

$$\int \Delta^2 d\varepsilon / 2 (\Delta^2 + \varepsilon^2)^{3/2} = 1,$$

then

$$\frac{\delta\Delta}{\Delta} = \frac{1}{2\rho_0} \int \delta\rho \frac{d\varepsilon}{\sqrt{\Delta^2 + \varepsilon^2}}. \qquad (55)$$

It is natural to expect the fluctuations in ρ to be such that $\int \delta\rho\, d\epsilon \sim 1$; therefore, $\delta\Delta/\Delta \sim 1/3\rho_0\Delta \sim 10\%$ if we take $\rho_0 = 3N/2\epsilon_0 \approx 4$ (in the region of the rare earths) and $\Delta = 0.8$ Mev. This estimate agrees well with the fluctuating values of Δ found from mass defects or from the primary levels.

The fluctuations in the quantity Δ, together with the fluctuations of the sum (34) which determines J/J_0, lead to appreciable fluctuations in the moment of inertia, $\delta J/J \sim 20\%$.

2. Along with the fluctuations that lead to a random change in the moment of inertia, there is a systematic effect of decreasing Δ, and consequently of increasing moment of inertia, in the transition to an odd element.

We shall consider the excited state of the system. It can be shown that the equation for Δ changes in the following fashion:[5]

$$1 = -\gamma_1 \sum \frac{1}{2E_\lambda} (1 - 2n_\lambda), \qquad (56)$$

where n_λ is the number of quasiparticles.

The transition to an odd nucleus must be considered as the appearance of a single quasiparticle with an energy closest to ϵ_0, i.e., $n_\lambda^- = 0$, $n_\lambda^+ = \delta_{\lambda\lambda_1}$, $E_{\lambda_1} \approx \Delta$. From (56) we get, by analogy to (55),

$$(\Delta' - \Delta'') / \Delta = 1/2\rho_0\Delta'', \qquad (57)$$

where Δ' and Δ'' are the corresponding values for even and odd nuclei, $\Delta = (\Delta' + \Delta'')/2$.

The change in the moment of inertia for a change in N is given by the expression

$$\frac{J'' - J'}{J'} = \frac{J_n'' - J_n'}{J'} = \frac{J_n'}{J'} \varkappa_n \frac{d \ln \Phi(\varkappa_n)}{d\varkappa_n} \frac{\Delta_n' - \Delta_n''}{\Delta_n}$$

$$= 0.9 \frac{\Delta_n}{\Delta_n''} \frac{J_n'}{J'} \frac{1}{2\rho_0\Delta_n''}, \qquad (58)$$

The latter equation is obtained with the help of (57) and the approximate formula $\dfrac{\kappa_n d \ln \Phi_{1,2}(\kappa_n)}{d\kappa_n} \approx 0.9$.

Similarly, we obtain the change of moment of inertia in transition to odd Z:

$$\frac{J'' - J'}{J'} = \frac{J_p'' - J_p'}{J'} = \frac{J_p'}{J'} 0.9 \frac{\Delta_p}{\rho_0^p \Delta_p''^2}. \qquad (59)$$

As will be shown below, Eqs. (58) and (59) agree well with with the experimental data.

3. An appreciable increase in the moment of inertia should occur in single-particle excitations of the nucleus. Thus in excitation with formation of a single hole and a single quasiparticle we get from (56) a relative decrease in Δ that is twice as large as in the transition from even-even to even-odd nuclei, and consequently a twofold relative change in the moment of inertia.

8. DETERMINATION OF Δ FROM THE MASSES OF THE NUCLEI

From the definition of Green's function, it is not difficult to obtain

$$G_\lambda = \sum_s \frac{|(\Phi_{N+1}^s, a_\lambda^+ \Phi_N^0)|^2}{\varepsilon - E_s(N+1) + E_0(N) + i\delta} + \sum_s \frac{|(\Phi_{N-1}^s, a_\lambda \Phi_N^s)|^2}{\varepsilon + E_s(N-1) - E_0(N) - i\delta}, \qquad (60)$$

a_λ, a_λ^+ are the operators of annihilation and creation of particles in the state λ.

Comparing (60) with (9) we obtain

$$E_0(N+1) - E_0(N) = \Delta + \varepsilon_0, \qquad (62)$$

$$E_0(N) - E_0(N-1) = -\Delta + \varepsilon_0. \qquad (61)$$

Equations (61) and (62) permit us to find Δ from the binding energy of the nuclei. A much more exact expression can be obtained if we eliminate from E (N) the dependence on N that is not connected with pairing. For this purpose we set up an expression in which the components with first and second derivatives in $N' = N$ are eliminated. The relation

TABLE II. Values of Δ_n and Δ_p computed from mass defects for certain elements according to Eqs. (57) and (63)

Element	Δ_n, Mev	Δ_p, Mev	Element	Δ_n, Mev	Δ_p, Mev	Element	Δ_n, Mev	Δ_p, Mev	Element	Δ_n, Mev	Δ_p, Mev
Sm^{150}	1.32	1.64	Gd^{157}	0.72	—	Hf^{178}	0.62	—	Th^{231}	0.56	0.81
Gd^{154}	1.15	—	Dy^{162}	0.82	—	Hf^{179}	0.42	—	Th^{232}	0.55	0.78
Gd^{156}	0.92	—	Dy^{163}	0.62	—	Th^{230}	0.76	1.05	U^{238}	0.64	0.85

TABLE III. Values of the moments of intertia, the parameters of deformation β, the quantities κ_n and κ_p determined from (64)*

Element	β^7	κ_p	κ_n	$(J/J_0)_{rect}$	$(J/J_0)_{osc}$	$(J/J_0)_{exp^7}$
Nd^{150}	0.26	0.54	0.94	0.15	0.38	0.35
Sm^{152}	0.24	0.65	1.02	0.17	0.43	0.38
Gd^{154}	0.26	0.52	0.88	0.13	0.35	0.36
Gd^{156}	0.33	0.87	1.37	0.22	0.57	0.48
Gd^{157}	0.29	0.93	1.60	0.22	0.64	0.60
Dy^{162}	0.30	0.84	1.43	0.23	0.57	0.50
Hf^{179}	0.20	0.99	1.75	0.27	0.66	0.52
Os^{186}	0.18	0.44	0.69	0.09	0.26	0.28
Th^{230}	0.22	0.63	0.95	0.15	0.40	0.43
Th^{232}	0.22	0.84	1.42	0.24	0.60	0.44
U^{238}	0.24	0.83	1.29	0.22	0.54	0.43

*To calculate κ_p when Δ_p is not known from mass defects, we assume $\Delta_p/\Delta_m = 1.5$. $(J/J_0)_{osc}$ is the computed value of the moment of inertia for an oscillatory potential. $(J/J_0)_{rect}$ is the same for a rectangular potential well.

$$\tfrac{1}{4}[3E(N+1)-3E(N)$$
$$+E(N-1)-E(N+2)]=\Delta. \tag{63}$$

satisfies this condition. The values of $\Delta'_{n,p}$ and $\Delta''_{n,p}$ found from (57) and (63) are given in Table II.

9. COMPARISON WITH EXPERIMENT

1. To compare Eqs. (43) and (51) with the experimental moments of inertia, we express κ and ν_1 in terms of observed quantities. As was pointed out above, one should use for R_0 in (35) a mean radius, for example, $R_0 = (a+b)/2$. We obtain $R_0 = R(1+\beta/3)$ where $R^3 = ab^2$. Taking $R = 1.2\times10^{-13}A^{1/3}$ cm, we find

$$\varepsilon_0^n = 52(M/M_{eff})(N/A)^{2/3},\quad p_0^n R = 1.9\cdot N^{1/3},$$

$$\varkappa_n = \frac{\beta}{1+\beta/3}\frac{27}{\Delta_n A^{1/3}}\left(\frac{N}{A}\right)^{1/3}\frac{M}{M_{eff}}. \tag{64}$$

here Δ_n is in Mev. We have similar expressions for protons.

We select $\omega_0 = (\omega_y+\omega_z)/2$ from the requirement of coincidence of the mean square $\langle r^2\rangle$ for rectangular and oscillatory potentials. We find $\langle y^2\rangle = \epsilon_0/4\omega_0^2 = R_0^2/5$. It is easy to obtain expressions similar to (64):

$$\varepsilon_0^n = 76(M/M_{eff})(N/A)^{2/3},\quad p_0^n R = 2.3N^{1/3},$$
$$\nu_1^n = \beta\omega_0/2\Delta_n = 0.95\varkappa_n \approx \varkappa_n. \tag{65}$$

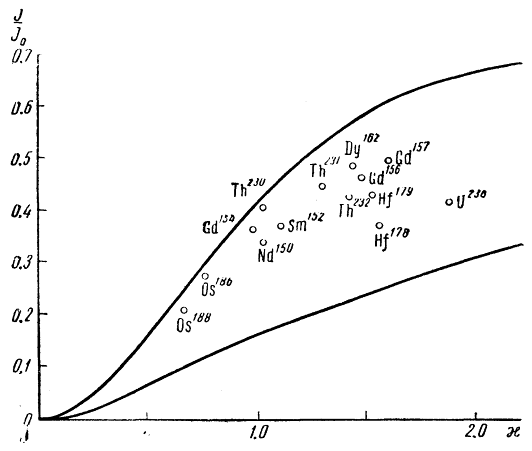
There are reasons for thinking* that the effective mass M_{eff} differs slightly from the mass of the nucleon M.

In Table III and in the drawing are shown the values of κ for $M_{eff}/M = 1$. As is seen from the drawing, all the experimental values of J/J_0 lie between the two theoretical curves.

2. To compare Eqs. (58) and (59) with experimental data, we compute the mean value of the observed moment of inertia for a group of nuclei with neighboring values of N and Z. The value averaged over five elements in rare-earth group is equal to: for Z, N even — $(J/J_0) = 0.42$; for Z even N odd — $(J/J_0) = 0.53$, $(J''-J')/J' = 0.26$. From Eq. (58) we obtain $(J''-J')/J' = 0.32$. For the transition to even N and odd Z we obtain from (58) and (59)

$$(J''-J')_{\text{even}\,Z}/(J''-J')_{\text{even}\,N} \approx J_n\Delta_p/J_p\Delta_n \approx 4$$

*The effective mass also enters into the expression for the orbital magnetic moment of the nucleon. For a significant difference between M_{eff} and M, the magnetic moments of the nuclei would not fit between the Schmidt curves.



Experimental values of J/J_0 as functions of κ_n. Theoretical curves of J/J_0 for an oscillating potential (upper curve) and a rectangular potential well for $J_n/J_p = 2.5$.

in agreement with the fact that the transition to an odd proton changes the moment of inertia much less than the transition to an odd neutron.

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[1] L. Cooper, Phys. Rev. **104**, 1189 (1956).

[2] J. Bardeen, L. Cooper, and J. Schrieffer, Phys. Rev. **108**, 1175 (1957).

[3] N. N. Bogolyubov, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 58, 73 (1958); Soviet Phys. **7**, 41, 51 (1958).

[4] N. N. Bogolyubov, V. V. Tolmachev, and D. V. Shirkov, Новый метод в теории сверхпроводимости (A New Method in the Theory of Superconductivity) U.S.S.R. Acad. of Sci. Press, Moscow, 1958.

[5] L. P. Gor'kov, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 735 (1958); Soviet Phys. **7**, 305 (1958).

[6] G. G. Goldring and R. P. Scharenberg, Phys. Rev. **110**, 701 (1958).

[7] K. Alder, A. Bohr, T. Huss, B. Mottelson, and A. Winther, Revs. Modern Phys. **28**, 432 (1956).

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