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ION OSCILLATIONS IN A PLASMA

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The effect of a high-frequency electromagnetic field on ion oscillations in a plasma is considered. It is shown that the frequencies of the quasi-acoustic longitudinal plasma oscillations are functions of the field amplitude. Possible instability mechanisms are discussed.

1. The effect of a high-frequency electromagnetic field on plasma oscillations has been considered by us in the hydrodynamic approximation in an earlier paper.[1] In the present paper we develop a kinetic approach to this problem. It is assumed that the plasma density is low so that collisions can be neglected.

The problem may be formulated as follows. A monochromatic plane electromagnetic wave of arbitrary amplitude is propagated through a uniform plasma. If the density of the plasma is perturbed by any external effect the electromagnetic field is partially reflected from points at which the plasma density is high and is increased at points of reduced density. This process can be of interest if the characteristic dimensions of the perturbation are of the order of the electromagnetic wavelength. The increased intensity of the electromagnetic field leads to an interaction with the plasma oscillations in which the phase velocity of the plasma waves are functions of the amplitude of the electromagnetic field. Under certain conditions this mechanism can result in an instability, that is to say, in perturbations which increase in time.

2. We consider the one-dimensional problem, assuming that all quantities depend on the single spatial coordinate z. The motion of the ions and electrons is described by the kinetic equations (collisions are neglected):

$$\frac{\partial f_\alpha}{\partial t} + v'_z \frac{\partial f_\alpha}{\partial z'} + \frac{e_\alpha}{m_\alpha}\left(E_x - \frac{v'_z}{c} H_y\right)\frac{\partial f_\alpha}{\partial v'_x} + \left(E_z + \frac{v'_x}{c} H_y\right)\frac{\partial f_\alpha}{\partial v'_z} = 0. \quad (1)$$

Here $\alpha = i$ or e, for ions or electrons respectively.

The electromagnetic field satisfies Maxwell's equations

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{4\pi}{c^2}\frac{\partial}{\partial t}\sum_\alpha e_\alpha \int v'_x f_\alpha d\mathbf{v}' + \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2}, \quad (2)$$

$$\frac{\partial E_z}{\partial z} = 4\pi \sum_\alpha e_\alpha \int f_\alpha d\mathbf{v}', \quad (3)$$

$$\frac{\partial E_x}{\partial z} = -\frac{1}{c}\frac{\partial H_y}{\partial t}. \quad (4)$$

It is well known[2,3] that in an inhomogeneous high-frequency electromagnetic field the particles execute rapid oscillations in the direction of the electric field and drift slowly along the field gradient. We separate the field into high-frequency components and low-frequency components. It will be assumed that the E_x is created by external sources and is characterized by a frequency Ω which is higher than the Langmuir frequency $\omega_0 = (4\pi N_0 e^2/m_e)^{1/2}$. In this field the particles execute rapid oscillations along the x axis. On the other hand, E_z results from the separation of charges in the slow drift motion of the particles in the z direction. The frequency of this field is of the same order as that of the plasma oscillations which, hereinafter, will be assumed to be much smaller than Ω. Introducing the following notation in the equations of motion

$$v'_j = v_j + \tilde{v}_j, \qquad j = x, y, z \quad (5)$$

(where v_j corresponds to the slow motion and $\tilde{v}_j$ to the fast oscillations) and assuming that in one oscillation period the particle is displaced by a distance much smaller than the characteristic distance in which the external field changes, we have

$$m_\alpha \frac{d\tilde{v}_x}{dt} = e_\alpha E_x(z, t), \qquad m_\alpha \frac{dv_z}{dt} = \frac{e_\alpha}{c}\overline{\tilde{v}_x H_y} + e_\alpha E_z \quad (6)$$

(the bar denotes averaging over the fast oscillation period).

Introducing the substitution of variables (5) in the kinetic equations (1), after averaging we obtain

$$\frac{\partial \bar{f}_\alpha}{\partial t} + v_z \frac{\partial \bar{f}_\alpha}{\partial z} + \frac{e_\alpha}{m_\alpha} E_z \frac{\partial \bar{f}_\alpha}{\partial v_z} + \frac{e_\alpha}{m_\alpha c}\overline{\tilde{v}_x H_y \frac{\partial f_\alpha}{\partial v_z}} = 0. \quad (7)$$

From Eqs. (2) and (3) we have

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{4\pi}{c^2}\frac{\partial}{\partial t}\sum_\alpha e_\alpha \int \widetilde{v}_x \bar{f}_\alpha \, d\mathbf{v} + \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2},$$

$$\frac{dE_z}{\partial z} = 4\pi \sum_\alpha e_\alpha \int \bar{f}_\alpha \, d\mathbf{v} \tag{8}$$

(hereinafter we will omit the bar over the averaged distribution function).

We seek a solution of the system (4), (7), and (8) by successive approximations, assuming that

$$f_\alpha = f_\alpha^{(0)} + f_\alpha^{(1)} + \ldots, \qquad E = E^{(0)} + E^{(1)} + \ldots,$$
$$E_z = E_z^{(1)} + \ldots . \tag{9}$$

The field $E_z^{(0)}$ will be assumed to be zero because the plasma is quasi-neutral. The quantity $f_\alpha^{(0)}$ is the Maxwell distribution function:

$$f_\alpha^{(0)} = (m_\alpha / 8\pi T_\alpha)^{3/2} N_0 \exp(-m_\alpha v^2 / 2T_\alpha). \tag{10}$$

In the zeroth approximation Eq. (8) assumes the form

$$d^2 E^{(0)} / dz^2 + \varkappa^2 E^{(0)} = 0$$

($\kappa^2 = \Omega^2 \epsilon / c^2$, $\epsilon = 1 - \omega_0^2/\Omega^2$ and the motion of the ions in the high-frequency field is neglected). Whence

$$E^{(0)} = E_0 e^{i\varkappa z + i\Omega t}. \tag{11}$$

As a first approximation, in Eq. (8) we write

$$E^{(1)} = E(z, t)\, e^{i\Omega t}.$$

It is assumed that the time dependence of the amplitude is slow since it is determined by the change in E_x due to the plasma oscillations. In solving Eq. (8), in general we can neglect the dependence of $E^{(1)}$ on time, taking t as a parameter.

In the first approximation we have from Eq. (8)

$$\frac{d^2 E^{(1)}}{dz^2} + \varkappa^2 E^{(1)} = \frac{4\pi}{c^2} E^{(0)} \sum_\alpha \frac{e_\alpha^2}{m_\alpha} \int f_\alpha^{(1)} d\mathbf{v}. \tag{12}$$

Whence*

$$E^{(1)} = \frac{4\pi}{c^2 \varkappa} \int^z E^{(0)} \Big(\sum_\alpha \frac{e_\alpha^2}{m_\alpha} \int f_\alpha^{(1)} d\mathbf{v} \Big) \sin \varkappa (z - \zeta)\, d\zeta. \tag{13}$$

Using Eqs. (11) and (13), we can find H_y from Eq. (4). Substituting in the kinetic equations (7) and averaging, we have

$$\frac{\partial \varphi_\alpha^{(1)}}{\partial t} + v_z \frac{\partial \varphi_\alpha^{(1)}}{\partial z} + \frac{e_\alpha}{m_\alpha} E_z \frac{\partial \varphi_\alpha^{(0)}}{\partial v_z} - \frac{2\pi e_\alpha^2 E_0^2}{m_\alpha^2 c^2 \Omega^2} \frac{\partial \varphi_\alpha^{(0)}}{\partial v_z}$$
$$\times \int^z \Big(\sum_\alpha \frac{e_\alpha^2}{m_\alpha} \int_{-\infty}^{\infty} \varphi_\alpha^{(1)} dv_z \Big) \cos 2\varkappa (z - \zeta)\, d\zeta = 0. \tag{14}$$

In order to obtain the complete system it is necessary to add the first-approximation equation for

$$\frac{\partial E_z}{\partial z} = 4\pi \sum_\alpha e_\alpha \int_{-\infty}^{\infty} \varphi_\alpha^{(1)} dv_z, \qquad \varphi_\alpha \equiv \int f_\alpha dv_x dv_y. \tag{15}$$

We seek a solution of the system (14) and (15) which is proportional to exp (ist + ikz). The quantity s is assumed to be complex with a positive real part, corresponding to the Laplace transform.[4] The condition which must be satisfied to solve the system of homogeneous algebraic equations which we have obtained is given by the dispersion equation

$$1 + (\theta k^{*-2} - p\theta m_e^2 / m_i^2)\, \psi_i$$
$$+ (k^{*-2} - p)\, \psi_e - p\theta k^{*-2} \psi_e \psi_i = 0. \tag{16}$$

where we have introduced the following notation:

$$k^* = kD, \quad \varkappa^* = \varkappa D, \quad D = (T_e / 4\pi N_0 e^2)^{1/2}, \quad \theta = T_e / T_i,$$
$$\xi = v_e / v_i, \quad v_e = (T_e / m_e)^{1/2}, \quad v_i = (T_i / m_i)^{1/2},$$
$$p = \omega_0^2 E_0^2 / 8\pi m_e c^2 N_0 \Omega^2 (k^{*2} - 4\varkappa^{*2}), \quad \psi_e = \frac{1}{\sqrt{2\pi}} \int \frac{t e^{-t^2/2} dt}{t + Z},$$
$$\psi_i = \frac{1}{\sqrt{2\pi}} \int \frac{t e^{-t^2/2} dt}{t + \xi Z}, \quad Z = \frac{s^*}{k^*}, \quad s^* = \frac{s}{\omega_0}.$$

Following Landau,[4] the integration in the expressions for ψ_e and ψ_i is carried out along a contour which is parallel to the real axis and which passes the singular points $t = -Z$ and $t = -\xi Z$ from below. We then obtain (cf. reference 5)

$$\psi_e = F(Z) + i\sqrt{\pi/2}\, Z e^{-Z^2/2},$$
$$\psi_i = F(\xi Z) + i\sqrt{\pi/2}\, Z\xi e^{-Z^2\xi^2/2}, \tag{17}$$

where

$$F(Z) = 1 - Z e^{-Z^2/2} \int_0^Z e^{-\tau^2/2} d\tau.$$

3. We now investigate the dispersion equation. Consider the case in which $k^{*2} \ll 1$ and $k^{*2}p \ll 1$. In practice this means that we are considering perturbations characterized by wavelengths considerably longer than the Debye radius but not as large as a half-wavelength of the electromagnetic wave, because when $k \to 2\kappa$, p approaches ∞. Equation (16) assumes the form

$$\theta\psi_i + \psi_e = p\theta\psi_e\psi_i. \tag{18}$$

Suppose further that $Z = \alpha + i\beta$ ($\alpha = \omega/kv_e$ is the dimensionless frequency and $\beta = \gamma/kv_e$ is the decay factor). We seek a solution for which $\alpha \gg \beta$. Expanding in powers of β/α, we have from Eq. (18)

$$\theta F(\xi\alpha) + F(\alpha) = p\theta F(\xi\alpha) F(\alpha), \tag{19}$$

$$\beta = \sqrt{\frac{\pi}{2}}\, \alpha \frac{p\theta [F(\xi\alpha) e^{-\alpha^2/2} + \xi F(\alpha) e^{-\xi^2\alpha^2/2}] - \theta\xi e^{-\xi^2\alpha^2/2} - e^{-\alpha^2/2}}{F'(\alpha) + \theta F'(\xi\alpha) - p\theta [F(\alpha) F(\xi\alpha)]'} \tag{19a}$$

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*The field $E^{(1)}$ is bounded at infinity.

(the primes denote differentiation with respect to α).

Equation (19) can be used to determine α. Once α is known we can use Eq. (19a) to determine β. We make the assumption (which is justified by the results) that $\alpha \ll 1$ and $\xi\alpha \gg 1$; thus, using the expansion of the function $F(x)$ for small and large values of the argument (cf. reference 5), we have when $\theta \gg 1$

$$\alpha^2 = (m_e / m_i)(1 - p) \tag{20}$$

or, in the usual units,

$$\left(\frac{\omega}{ka}\right)^2 = 1 - \frac{1}{\gamma(k^2/\varkappa^2 - 4)}, \quad a^2 = \frac{T_e}{m_i}, \quad \gamma = \frac{\varepsilon}{1-\varepsilon}\frac{T_e}{\Phi}, \tag{21}$$

where Φ is the effective potential, which is

$$\Phi = e^2E_0^2 / m_e\Omega^2 \approx 4.8 \cdot 10^{-8} E_v \lambda^2, \qquad \lambda = c/\Omega.$$

The ion mass is taken as 3.3×10^{-24} g and E_V is the amplitude of the electric field in volts per centimeters.

Computing the damping factor we have

$$\beta = \sqrt{\frac{\pi}{8}} \frac{m_e}{m_i} \frac{1+p}{1-p}. \tag{22}$$

Equation (21) coincides with the corresponding hydrodynamic formula which has been obtained earlier.[1] In contrast with the hydrodynamic case, the oscillations characterized by the dispersion relation in Eq. (20) exist, as is shown by an investigation of Eq. (19), if θ is larger than some number of order unity (an estimate shows that the relation $\theta \gtrsim 3.5$ must be satisfied). When $\theta \sim 1$, $1 - p > 0$, and $p \sim 1$ these oscillations are highly damped. If $p = 0$, we have $\omega = (T_e/m_i)^{1/2}k$, i.e., these oscillations become quasi-acoustic ion oscillations such as those which have been considered by a number of authors.[6-8]* While these oscillations take place the electrons move, without collisions, in the potential well which is formed because the ions are shifted from their equilibrium positions. The electrons and the electromagnetic field produce an effective pressure and the effective density is determined by the ions. The ions move slowly; in one period they traverse a path which is $1/\xi$ of the wavelength, i.e., the ions do not leave the region of the perturbation. When these oscillations are produced long-range electric forces arise in the plasma. As is well known, in a gas consisting of neutral particles the production of acoustic oscillations without collisions is impossible. The oscillations characterized by the dispersion relation in Eq. (21) are analogous to magneto-acoustic oscillations but the role of the magnetic pressure is played by the average pressure of the high-frequency electromagnetic field. When $p > 1$, we have $\mathrm{Re}\, s < 0$, i.e., there is an instability which, as in the hydrodynamic case, arises by virtue of a resonance if the wavelength of the perturbation is equal to half the wavelength of the electromagnetic wave. Since the Fourier spatial spectrum of any localized perturbation contains the harmonic with $k = 2\varkappa$, it may be assumed that a uniform plasma in the field of a high-frequency traveling electromagnetic wave becomes unstable when $p > 1$ $(\theta \gg 1)$ regardless of the amplitude of the electromagnetic field.

*This problem has also been considered in the author's Diploma paper, Moscow Engineering Physics Institute, 1953.

In the present paper we have considered the linearized equations. Hence we cannot determine the final results of the appearance of an instability within the framework of the present analysis. However, it may be assumed that if the pressure of the electromagnetic field $E_0^2/8\pi$ is much smaller than the plasma pressure the instability will only lead to small oscillations.

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[1] T. F. Volkov, Сб. Физика плазмы и проблема управляемых термоядерных реакций (Plasma Physics and the Problem of a Controlled Thermonuclear Reaction), 4, Acad. Sci. U.S.S.R., Moscow, 1958, p. 98.

[2] R. Z. Sagdeev, ibid., vol. III, 346 (1958).

[3] A. V. Gaponov and M. A. Miller, J. Exptl. Theoret. Phys. (U.S.S.R.) 34, 242 (1958), Soviet Phys. JETP 7, 168 (1958).

[4] L. D. Landau, J. Exptl. Theoret. Phys. (U.S.S.R.) 16, 574 (1936).

[5] V. N. Faddeeva and N. M. Terent'ev, Таблицы значений интеграла вероятностей от комплексного аргумента (Tables of Values of the Probability Integral for Complex Arguments), Gostekhizdat, Moscow, 1954.

[6] J. B. Bernstein, Phys. Rev., 109, 10 (1958).

[7] S. I. Braginskiĭ and A. P. Kazantsev, Физика плазмы и проблема управляемых термоядерных реакций (Plasma Physics and the Problem of a Controlled Thermonuclear Reaction) 4, Acad. Sci. U.S.S.R., Moscow, 1958, p. 24.

[8] K. N. Stepanov, J. Exptl. Theoret. Phys. (U.S.S.R.) 34, 1292 (1958), Soviet Phys. JETP 7, 892 (1958).

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