SOVIET PHYSICS JETP VOLUME 37 (10), NUMBER 2 FEBRUARY, 1960

ON THEORIES WITH AN INDEFINITE METRIC

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Submitted to JETP editor February 13, 1959

J. Exptl. Theoret. Phys. (U.S.S.R.) **37**, 467-469 (August, 1959)

The conditions of unitarity and macro-causality for the Lee model with an indefinite metric are investigated.

We consider the Lee model with an indefinite metric,[1,2] regarding the coordinates of the "heavy" N and V particles as fixed. The unrenormalized Hamiltonian has the form

$$H = -m_V \sum_i \psi^+_{V_i}\psi_{V_i} + \sum_{\mathbf{k}} \omega_k a^+_{\mathbf{k}} a_{\mathbf{k}} - g_0 \sum_{\mathbf{k}} \frac{f_k}{\sqrt{2\omega_k}} \left(\psi^+_{V_i}\psi_{N_i} a_{\mathbf{k}} e^{i\mathbf{k}\mathbf{R}_i} + \psi_{V_i}\psi^+_{N_i} a^+_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{R}_i}\right).$$

Here $\psi^+_{V_i}$, ψ_{V_i}, $\psi^+_{N_i}$, and ψ_{N_i} are the creation and absorption operators for the V and N particles at the point $\mathbf{R}_i$; the mass of the N particle is $m_N = 0$;[2] g_0 is pure imaginary; for point interactions the cut-off factor f_k goes to unity: $f_k \to 1$.

The constants m_V and g_0 are chosen such that the denominator of the Green's function of the V particles,

$$g_0^2 h(\varepsilon) = \varepsilon + m_V + g_0^2 \sum_{\mathbf{k}} \frac{f_k^2}{2\omega_k} \frac{1}{\omega_k - \varepsilon}$$

has a multiple root at $E_0 < \mu$ (dotted curve in the figure). Let us consider the problem of the bound states of the θ particles in the field of the two particles N_a and N_b. The distance $|\mathbf{R}_a - \mathbf{R}_b|$ is arbitrary. The state vector has the form

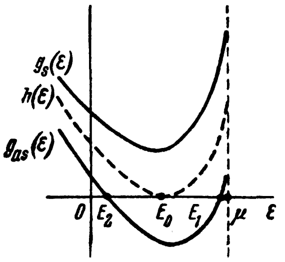
$$\Phi = c_a |V_a N_b\rangle + c_b |V_b N_a\rangle + \sum_{\mathbf{k}} \varphi_{\mathbf{k}} |N_a N_b \theta_{\mathbf{k}}\rangle.$$

Solving the Schrödinger equation $H\Phi = E\Phi$, we obtain a homogeneous system of equations for c_a and c_b for the bound states with $E < \mu$. The secular equation for this system yields

$$(h(\varepsilon) - J(\varepsilon))(h(\varepsilon) + J(\varepsilon)) \equiv g_{as}(\varepsilon) g_s(\varepsilon) = 0,$$

$$J(\varepsilon) = \sum_{\mathbf{q}} \frac{f_q^2}{2\omega_q} \frac{\exp\{i\mathbf{q}(\mathbf{R}_a - \mathbf{R}_b)\}}{\omega_q - \varepsilon}.$$

The dispersion curves $g_s(\epsilon)$ and $g_{as}(\epsilon)$ are given in the figure. The presence of a second center leads, of course, to the splitting of the "degenerate term" E_0. The function $g_s(\epsilon)$ has no real roots, where the function $g_{as}(\epsilon)$, has two real roots, E_1 and E_2; the state with energy E_2 has a negative norm. This consequence of the Lee model suggests that the presence of a multiple pole in the single nucleon Green's function gives rise to states with negative norm in the deuteron problem.



Bogolyubov, Medvedev, and Polivanov[3] proposed a method for the description of the more general case which allows for states with negative norm ("ghost states"). According to these authors, the state vector is given by a superposition of states with positive and negative norms. The contribution of the latter states is determined from the preceding ("preparatory") experiment as well as from the one that follows, and in such a way that the "ghost state" does not change in this later experiment. Then the presence of the state with negative norm cannot be detected in any experiment which measures the difference of physical quantities (for example, the particle flow) before and after the experiment, and the conservation of the total norm guarantees the unitarity of the observed S matrix. We shall show that this violates the condition of macro-causality. This condition is, for two remote centers, given by the relation $S(A+B) = S(B)S(A)$, where $S(A)$, $S(B)$, and $S(A+B)$ are the S matrices for the scatterers A, B, and A+B.

We assume that $h(\epsilon)$ has the distinct real roots E_p and $E_g < E_p$, and solve the problem of the scattering of two θ particles by the separated particles V_{ap} and N_b. According to the proposed scheme we have

$$\widetilde{S}^{\mathbf{k}_1\mathbf{k}_2 V_{ap} N_b}_{\cdots} = S^{\mathbf{k}_1\mathbf{k}_2 V_{ap} N_b}_{\cdots} + \sum_{\mathbf{q}_1\mathbf{q}_2} g^a_{\mathbf{q}_1\mathbf{q}_2}\delta(\omega_{q_1} + \omega_{q_2} + E_g - E)$$
$$\times S^{\mathbf{q}_1\mathbf{q}_2 V_{ag} N_b}_{\cdots} + \sum_{\mathbf{q}} g^b_{\mathbf{q}}\delta(\omega_q + E_p + E_g - E) S^{\mathbf{q} V_{ap} V_{bg}}.$$

The upper indices of S denote the incoming particles and the lower indices correspond to the possible outgoing channels; V_{ap}, V_{bp} and V_{ag}, V_{bg} are the physical and ghost states of the V particle at the points A and B; $S^{\cdots}_{\cdots}$ are standard S matrices obtained from the solution of the Schrödinger or Dyson equation; $\widetilde{S}^{\cdots}_{\cdots}$ is the observed S matrix. The functions $g^a_{\mathbf{q}_1\mathbf{q}_2}$, $g^b_{\mathbf{q}}$ describe the initial distribution of the θ particles in the "ghost" state. They are determined from the condition that there be no scattering of "ghost" states:

$$S^{\mathbf{k}_1\mathbf{k}_2 V_{ap} N_b}_{\mathbf{k}'\mathbf{k}'' V_{ag} N_b} + \sum_{\mathbf{q}_1\mathbf{q}_2} g_{\mathbf{q}_1\mathbf{q}_2}\delta(\omega_{q_1} + \omega_{q_2} + E_g - E)(S-1)^{\mathbf{q}_1\mathbf{q}_2 V_{ag} N_b}_{\mathbf{k}'\mathbf{k}'' V_{ag} N_b}$$
$$+ \sum_{\mathbf{q}} g^b_{\mathbf{q}}\delta(\omega_q + E_p + E_g - E) S^{\mathbf{q} V_{ap} V_{bg}}_{\mathbf{k}'\mathbf{k}'' V_{ag} N_b} = 0$$

and analogously for V_{bg}.

We restrict ourselves to the zeroth approximation with respect to $1/|\mathbf{R}_a - \mathbf{R}_b|$. The problem is readily solved by graphic methods. The answer is of the form

$$\widetilde{S}(A+B) = \widetilde{S}(B)\widetilde{S}(A) + R.$$

The meaning of R is the following. Suppose we have determined, in the absence of the center B, that admixture of ghost states which leads to no scattering of ghost states by the center A, i.e., we have found $\widetilde{S}(A)$. If now the particles N_b are added at the point B, the real scattering will be accompanied by the scattering of "ghost" states of the θ particles, so that the previous superposition has to be changed. The absence of "ghost" scattering in the complete problem can be achieved by changing the state vector, but this gives rise to additional transitions from ghost states to physical states at the point A, which are described by the matrix R. In the problem of the scattering of two θ particles, R does not decrease with the distance. This is precisely the reason why this problem was selected to illustrate the contradiction.

A unitary matrix $\widetilde{S}$ can also be constructed by calling the states which involve ghost states, "auxiliary" or "unobservable" states, which may therefore undergo arbitrary elastic processes as long as these conserve the norm in the ghost subspace. However, with this approach the theory has to forego uniqueness and a clear interpretation.

Let us now assume that the roots of $h(\epsilon)$ are complex (and conjugate to each other). Following Källén and Pauli,[1] we show by direct calculation that the S matrix is in this case automatically unitary. Let us consider, for example, the scattering of 2θ from N. From the equation $H\Phi = E\Phi$ with the usual boundary conditions we obtain

$$\Phi = \sum_{\mathbf{k}} \varphi_{\mathbf{k},\mathbf{k}_0\mathbf{q}_0}\left|V\theta_{\mathbf{k}}\right\rangle + \sum_{\mathbf{kq}} \psi_{\mathbf{kq},\mathbf{k}_0\mathbf{q}_0}\left|\theta_{\mathbf{k}}\theta_{\mathbf{q}}N\right\rangle,$$

$$\psi_{\mathbf{kq},\mathbf{k}_0\mathbf{q}_0} = \frac{1}{2}(\delta_{\mathbf{kk}_0}\delta_{\mathbf{qq}_0} + \delta_{\mathbf{kq}_0}\delta_{\mathbf{qk}_0})$$
$$+ \frac{g_0}{2}\frac{1}{\omega_k + \omega_q - E - i\gamma}\left(\frac{f_k}{\sqrt{2\omega_k}}\varphi_{\mathbf{q},\mathbf{k}_0\mathbf{q}_0} + \frac{f_q}{\sqrt{2\omega_q}}\varphi_{\mathbf{k},\mathbf{k}_0\mathbf{q}_0}\right),$$

$$h(E - \omega_k + i\gamma)\varphi_{\mathbf{k},\mathbf{k}_0\mathbf{q}_0} = -\varphi_{0\mathbf{k},\mathbf{k}_0\mathbf{q}_0}$$
$$-\sum_{\mathbf{q}}\frac{f_k f_q}{\sqrt{4\omega_k\omega_q}}\frac{\varphi_{\mathbf{q},\mathbf{k}_0\mathbf{q}_0}}{\omega_k + \omega_q - E - i\gamma} \equiv iU_{\mathbf{k},\mathbf{k}_0\mathbf{q}_0},$$

where

$$E = \omega_{k_0} + \omega_{q_0}, \quad \gamma \to +0,$$

$$\varphi_{0\mathbf{k},\mathbf{k}_0\mathbf{q}_0} = \frac{1}{g_0}\left(\frac{f_{q_0}}{\sqrt{2\omega_{q_0}}}\delta_{\mathbf{kk}_0} + \frac{f_{k_0}}{\sqrt{2\omega_{k_0}}}\delta_{\mathbf{kq}_0}\right).$$

Calculations similar to those of reference 1 lead to the equation

$$2\pi\delta(E - E')|g_0|^2\Big\{i\Big(\sum_{\substack{\mathbf{k}\\E-\omega_k<\mu}} + \sum_{\substack{\mathbf{k}\\E-\omega_k>\mu}}\Big)U^+_{\mathbf{k}_0'\mathbf{q}_0',\mathbf{k}}U_{\mathbf{k},\mathbf{k}_0\mathbf{q}_0}$$
$$\times\left(\frac{1}{h(E - \omega_k + i\gamma)} - \frac{1}{h(E - \omega_k - i\gamma)}\right)$$
$$- i\sum_{\mathbf{k}}(\varphi^+_{\mathbf{k}_0'\mathbf{q}_0',\mathbf{k}}\varphi_{0\mathbf{k},\mathbf{k}_0\mathbf{q}_0} - \varphi^+_{\mathbf{k}_0'\mathbf{q}_0',0\mathbf{k}}\varphi_{\mathbf{k},\mathbf{k}_0\mathbf{q}_0})$$
$$+ 2\pi\sum_{\mathbf{kq}}\frac{f_k f_q}{\sqrt{4\omega_k\omega_q}}\varphi^+_{\mathbf{k}_0'\mathbf{q}_0',\mathbf{k}}\varphi_{\mathbf{k},\mathbf{k}_0\mathbf{q}_0}\delta(\omega_k + \omega_q - E)\Big\} = 0.$$

The sum over $\mathbf{k}$ with $E - \omega_k < \mu$ in the first term, which describes transitions to bound states in the paper of Källén and Pauli, is here equal to zero, since $h(\epsilon)$ has no zeros on the real axis. With the notation

$$\langle\mathbf{kq}|R|\mathbf{k}_0\mathbf{q}_0\rangle = \pi i g_0\delta(\omega_k + \omega_q - E)$$
$$\times\left(\frac{f_k}{\sqrt{2\omega_k}}\varphi_{\mathbf{q},\mathbf{k}_0\mathbf{q}_0} + \frac{f_q}{\sqrt{2\omega_q}}\varphi_{\mathbf{k},\mathbf{k}_0\mathbf{q}_0}\right),$$

we obtain from the preceding equation

$$\langle\mathbf{k}_0'\mathbf{q}_0'|R + R^+ + R^+R|\mathbf{k}_0\mathbf{q}_0\rangle = 0.$$

The proof is easily generalized for the other sectors, as for example, for the problem of the scattering of two θ particles by two N particles, if, as before, the functions $g_s(\epsilon)$ and $g_{as}(\epsilon)$ do not intersect the real axis.

In the case of complex roots of $h(\epsilon)$ the stationary S matrix of the Lee model is therefore unitary[4] without any trivial violations of the causality principle. The scattering amplitude has a pole in the upper half of the energy plane, so that the dispersion relations have to be modified. In the sector $N + \theta$ we have two nonorthogonal state vectors with vanishing norm, $\Phi_{1,2}$, corresponding to the eigenvalues $\lambda_0 \pm i\lambda_1$ of the Hamiltonian. As solutions of the equation $i\partial\Phi/\partial t = H\Phi$ both of these vectors depend exponentially on the time: $\Phi_{1,2} \sim \exp[-i(\lambda_0 \pm i\lambda_1)t]$, but the norm of an arbitrary superposition is, of course, preserved.

The author thanks K. A. Ter-Martirosyan and L. A. Maksimov for their interest in this work and comments.

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Translated by R. Lipperheide
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