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GROUND STATES OF NONSPHERICAL ODD NUCLEI ACCORDING TO THE INDEPENDENT PARTICLE MODEL

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The level scheme of nucleons in a spheroidal well with vertical walls is computed by using the asymptotic expansions of spheroidal wave functions. The results obtained are in good agreement with the experimental data on the spins and parities of the ground and isomeric states of nonspherical odd nuclei.

THE energy level scheme for nucleons in a nonspherical axially symmetric nucleus proposed by Nilsson[1] agrees satisfactorily with experiment. Nilsson's scheme was obtained on the assumption that the effective field in which the nucleons move is described by the potential of an anisotropic oscillator. As is well known the approximation of a harmonic oscillator for the self-consistent nuclear field is good only for light nuclei, whereas for heavy ones the self-consistent potential is closer to that of a rectangular well (cf., for example, reference[2]). Therefore Nilsson did not simply use the harmonic oscillator potential in constructing his scheme, but added to the Hamiltonian a term proportional to the operator of the square of the orbital angular momentum of the nucleon, and thereby made his potential closer to that of a rectangular well.

Nevertheless, it is of interest to construct an energy level scheme for nucleons in a rectangular spheroidal well. This problem entails considerable mathematical difficulties. It was first solved by Moszkowski[3] for the lowest states of the nucleons by the perturbation theory method. Later Gottfried[4] constructed an energy level scheme for nucleons in a rectangular deformed well. In his paper Gottfried expanded the potential in powers of a parameter that describes the deviation from the spherical shape, and starting with the spherically symmetrical case as the zero-order approximation, he solved the secular equation which refers to all the nucleon states under consideration. However, as was pointed out by Kumar and Preston,[5] in doing this he incorrectly took into account terms proportional to the square of the deformation, which is important in the case of nuclei whose shape deviates appreciably from spherical. Moreover, in his calculations Gottfried did not take into account the existence of the continuous spectrum which, certainly in the case of the higher lying levels, may affect the result appreciably. Finally, the choice of the zero-order approximation for the functions was no improvement, and this, of course, also adversely affected the convergence of the method, particularly for large values of the parameter describing the deviation from the spherical shape. Thus, it has become necessary to carry out the calculation of the energy levels of nucleons in a rectangular spheroidal well by a method which would be applicable to large deformations and to high nucleon energy levels.

Such a method was proposed in the preceding papers[6,7] by the present author. The problem of finding the energy levels of nucleons of mass M in a rectangular potential well having the spatial shape of an ellipsoid of revolution reduces, as is well known, to the solution of the Schrödinger equation

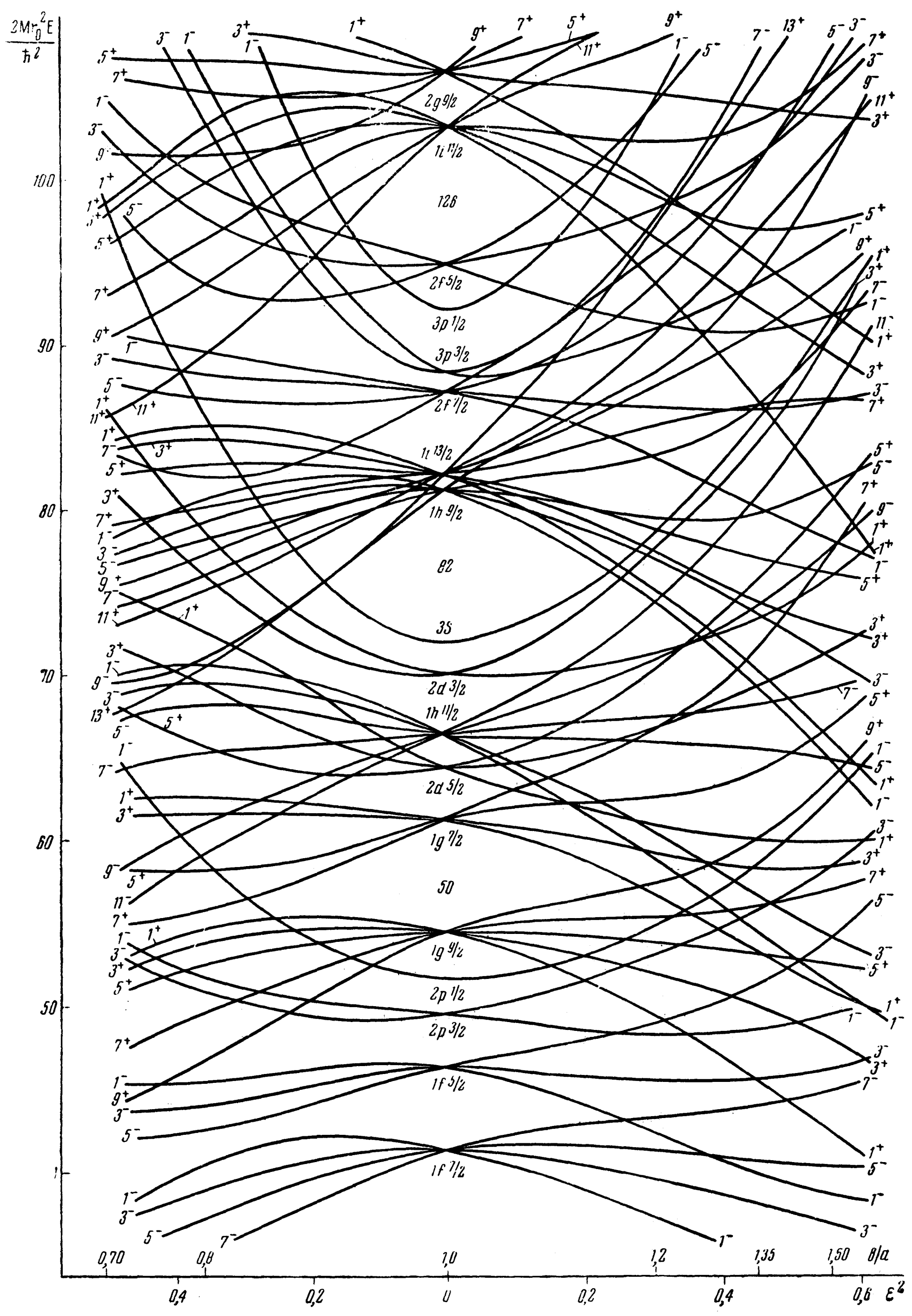
$$\left\{-\frac{\hbar^2}{2M}\Delta + V(\mathbf{r}) - \frac{\varkappa}{M^2c^2}\hat{s}\,[\nabla V(\mathbf{r})\cdot\hat{\mathbf{p}}]\right\}\psi = E\psi, \quad (1)$$

where c is the velocity of light, $\hat{\mathbf{s}}$ and $\hat{\mathbf{p}}$ are the nucleon spin and momentum operators, κ is a dimensionless constant. The potential V(r) has the form:

$$V(\mathbf{r}) = \begin{cases} 0 \text{ within the ellipsoid } (x^2+y^2)/a^2 + z^2/b^2 = 1 \\ V_0 \text{ outside the ellipsoid} \end{cases} \quad (2)$$

The semi-axes of the ellipsoid a and b are related by the condition that its volume is independent of the degree of deviation from the spherical shape:

$$a^2 b = r_0^3, \quad (3)$$



Nucleon energy level scheme in a rectangular spheroidal well. Along the horizontal axis are plotted the square of the eccentricity of the ellipsoid ε^2 and the corresponding ratio of the axes b/a, along the vertical axis is plotted the nucleon energy in units of $h^2/2Mr_0^2$. The numbers opposite the curves denote twice the value of the component of the angular momentum along the symmetry axis, while the sign signifies the parity of the state.

where r_0 is the radius of a sphere of equal volume.

The solution of Eq. (1) can be expressed in terms of spheroidal functions for which asymptotic expansions have been obtained in inverse powers of the parameter $\gamma = f(2ME)^{1/2}/\hbar$, where f is half the distance between the foci of the ellipsoid (2). The details of the calculation and numerical estimates of the range of applicability of this method are described in references 6 and 7.

Numerical calculations of the nucleon energy levels were carried out on the "Ural" electronic computer of the P. N. Lebedev Physics Institute for values of the constant V_0 equal to 35, 42 and 50 Mev. The spin-orbit coupling constant κ was fixed by the condition that in the case of a spherically symmetric well the best approximation should be obtained to the scheme of Klinkenberg[8] arrived at by means of an analysis of experimental data on the basis of the shell model. This condition is best of all satisfied by the value $\kappa = 30$. The constant r_0 was everywhere taken equal to 8.4×10^{-13} cm (which corresponds to $A \approx 200$ in the formula $r_0 = 1.4 \times 10^{-13} A^{1/3}$ cm, where A is the atomic weight). The calculations were carried out for values of the ratio of the semi-axes b/a taken

TABLE I

Nucleus	β	I_0^w experiment	I_0^w theory	State of the odd nucleon $\Omega^w(n, l, j)$: Z scheme (present work)	N scheme[1]	G scheme[4]	Notes
1	2	3	4	5	6	7	8
Eu^{153}	0.4	$5/2^+$	$5/2^+$	$5/2^+ (1g\,7/2)$	$5/2^+ (1g\,7/2)$	$5/2^+ (1g\,7/2)$	a
Gd^{155}	0.3	$3/2^-$	$3/2^\pm$	$3/2^+ (1i\,13/2)$ or $3/2^- (1h\,9/2)$	$3/2^+ (1i\,13/2)$ or $3/2^- (1h\,9/2)$	$3/2^+ (1i\,13/2)$ or $3/2^- (1h\,9/2)$	
Gd^{157}	0.3	$3/2^-$	$3/2^\pm$	$3/2^+ (1i\,13/2)$ or $3/2^- (1h\,9/2)$	$3/2^+ (1i\,13/2)$ or $3/2^- (1h\,9/2)$	$3/2^+ (1i\,13/2)$ or $3/2^- (1h\,9/2)$	
Tb^{159}	0.4	$3/2^+$	$3/2^+$	$3/2^+ (2d\,5/2)$	$3/2^+ (2d\,5/2)$	$3/2^+ (2d\,5/2)$	a
Dy^{161}	0.2	$5/2^+$	$5/2^+$	$5/2^+ (1i\,13/2)$	$5/2^+ (1i\,13/2)$	$5/2^+ (1i\,13/2)$	
Dy^{163}	0.2	$5/2^-$	$5/2^-$	$5/2^- (1h\,9/2)$	$5/2^- (1h\,9/2)$	$5/2^- (1h\,9/2)$	
Er^{167}	0.4	$7/2^+$	$7/2^+$	$7/2^+ (1i\,13/2)$	$7/2^+ (1i\,13/2)$	$7/2^+ (1i\,13/2)$	
Tm^{169}	0.3	$1/2^+$	$1/2^+$	$1/2^+ (2d\,3/2)$	$1/2^+ (2d\,3/2)$	$1/2^+ (2d\,3/2)$	b
Yb^{173}	0.3	$5/2^-$	$5/2^-$	$5/2^- (2f\,7/2)$	$5/2^- (1h\,9/2)$	$5/2^- (1h\,11/2)$	
Lu^{175}	0.4	$7/2^+$	$7/2^+$	$7/2^+ (1g\,7/2)$	$7/2^+ (1g\,7/2)$	$7/2^- (1h\,11/2)$	c
Hf^{177}	0.3	$7/2^-$	$7/2^-$	$7/2^- (1h\,9/2)$	$7/2^- (1h\,9/2)$	$7/2^- (2f\,7/2)$	
Hf^{179}	0.2	$9/2^+$	$9/2^+$	$9/2^+ (1i\,13/2)$	$9/2^+ (1i\,13/2)$	$9/2^+ (1i\,13/2)$	
Ta^{181}	0,3	$7/2^+$	$7/2^+$	$7/2^+ (1g\,7/2)$	$7/2^+ (1g\,7/2)$	$7/2^+ (1g\,7/2)$	a
W^{183}	0.2	$1/2^-$	$1/2^-$	$1/2^- (2f\,5/2)$	$1/2^- (2f\,5/2)$	?	d,e
$Re^{185,187}$	0.2	$5/2^+$	$5/2^+$	$5/2^+ (2d\,5/2)$	$5/2^+ (2d\,5/2)$	$5/2^+ (2d\,5/2)$	
Os^{189}	0.1	$3/2$	$3/2^-$	$3/2^- (3p\,3/2)$	$3/2^- (3p\,3/2)$	$1/2^- (3p\,3/2)$	f
$Ir^{191,193}$	0.2	$3/2^+$	$3/2^+$	$3/2^+ (2d\,3/2)$	$3/2^+ (2d\,3/2)$	$3/2^- (2f\,7/2)$	g
Ac^{227}	0.2	$3/2$	$3/2^\pm$	$3/2^+ (1i\,13/2)$ or $3/2^- (1h\,9/2)$	$3/2^+ (1i\,13/2)$ or $3/2^- (1h\,9/2)$	$3/2^+ (1i\,13/2)$ or $3/2^- (1h\,9/2)$	h
U^{233}	± 0.3	$5/2^+$	$5/2^+$	$5/2^- (2g\,9/2)$	$5/2^+ (1i\,11/2)$ or $5/2^+ (2g\,9/2)$	—	i
U^{235}	± 0.3	$7/2$	$7/2^+$	$7/2^+ (2g\,9/2)$ or $7/2^+ (2g\,7/2)$	$7/2^+ (2g\,9/2)$ or $7/2^- (1j\,15/2)$	—	j
Np^{237}	± 0.2	$5/2^+$	$5/2^+$	$5/2^+ (1i\,13/2)$	$5/2^+ (1i\,13/2)$	$5/2^+ (1i\,13/2)$	k
Pu^{239}	± 0.2	$1/2^+$	$1/2^+$	$1/2^+ (2g\,7/2)$ or $1/2^+ (2g\,9/2)$	$1/2^+ (3d\,5/2)$ or $1/2^+ (2g\,9/2)$	—	l,m
$Am^{241,243}$	± 0.3	$5/2^-$	$5/2^-$	$5/2^- (1h\,9/2)$	$5/2^- (2f\,7/2)$ or $5/2^- (1h\,9/2)$	$5/2^- (1h\,9/2)$	n

a. The odd nucleon state shown in the table which agrees with experiment is obtained in the N scheme if one supposes that the deformation is larger by a factor 1.5 or 2 than the experimentally observed one.

b. The coupling disruption factor in the Z scheme; $a_{theor} = -0.67$, $a_{exp} = -0.76$.

c. In the G scheme it is more natural to take $7/2^+(1g\,7/2)$, than $7/2^-(1h\,11/2)$ given in reference 4.

d. The coupling disruption factor in the Z scheme; $a_{theor} = -0.31$, $a_{exp} = -0.18$.

e. In the G scheme, no tolerable agreement with experiment can be obtained.

f. The result of the G scheme contradicts the experimental value of the spin of the ground state of Os^{189}.

g. In the G scheme it is more natural to take $3/2^+(2d\,3/2)$ than $3/2^-(2f\,7/2)$ given in reference 4.

h. As to the sign of the quadropole moment of Ac^{227}, see reference 17.

i. The identification of the state of the odd nucleon in the Z and N schemes does not depend on the sign of the deformation, however, in the N scheme the identification shown above is difficult to obtain for both states and for both signs of the deformation.

j. In the Z scheme the state $7/2^+(2g\,9/2)$ is obtained in the case $\beta > 0$, while $7/2^+(2g\,7/2)$ is obtained in the case $\beta < 0$. The states shown in the table for the N scheme refer to the case $\beta > 0$; in the case $\beta < 0$ the state of the odd nucleon is identified as $7/2^+(1i\,11/2)$.

k. In all three schemes the identification shown above for the state of the odd nucleon is more natural for $\beta > 0$, than for $\beta < 0$.

l. For the Z and N schemes the first of the states shown above refers to $\beta > 0$, the second refers to $\beta < 0$. In the Z scheme for $\beta > 0$ the coupling disruption factor is $a_{theor} = -0.61$, for $\beta < 0$ $a_{theor} = -0.64$, $a_{exp} = -0.58$.

m. In the case $\beta < 0$ the Z and N schemes admit the identification of the ground state as $1/2^-(3p\,1/2)$.

n. In all three schemes the result does not depend on the sign of the deformation.

TABLE II

Nucleus	β	I_0^w of the isomeric state (experiment)	State of the odd nucleon $\Omega^w(n, l, j)$			Notes
			Z scheme (present work)	N scheme[1]	G scheme[4]	
1	2	3	4	5	6	7
Er^{167}	0.4	$1/2^-$	$1/2^-(2f\,5/2)$	$1/2^-(2p\,3/2)$	$1/2^-(2f\,5/2)$	
Tm^{169}	0.3	$7/2^+$ and $7/2^-$	$7/2^+(1g\,7/2)$ and $7/2^-(1h\,11/2)$	$7/2^+(1g\,7/2)$ and $7/2^-(1h\,11/2)$	$7/2^+(1g\,7/2)$	b
Hf^{179}	0.2	$1/2^-$	$1/2^-(2f\,5/2)$	$1/2^-(2f\,5/2)$	$1/2^-(2f\,5/2)$	
W^{183}	0.2	unknown	$11/2^+(1i\,13/2)$	$11/2^+(1i\,13/2)$	?	a
Os^{189}	0.1	unknown	$13/2^+(1i\,13/2)$	$13/2^+(1i\,13/2)$	?	a
$Ir^{191,193}$	0.2	$11/2^-$	$11/2^-(1h\,11/2)$	$9/2^-(1h\,11/2)$	$11/2(1h\,11/2)$	
Ac^{227}	0.2	$1/2$	$1/2^+(3s\,1/2)$ or $1/2^-(2f\,7/2)$	$1/2^-(3p\,3/2)$	$1/2^-(2f\,5/2)$	
U^{235}	± 0.3	$1/2^+$	$1/2^+(2g\,7/2)$ or $1/2^+(2g\,9/2)$	$1/2^+(3d\,5/2)$ or $1/2^+(2g\,9/2)$	—	c
Np^{237}	± 0.2	$5/2^-$	$5/2^-(1h\,9/2)$	$5/2^-(1h\,9/2)$ or $5/2^-(2f\,7/2)$	$5/2^-(1h\,9/2)$	d

a. In the G scheme no isomeric state is obtained.

b. In the case of Tm^{169} two close long lived states are observed experimentally.

c. In the case of the Z and N schemes the first of the states shown refers to $\beta > 0$, while the second refers to $\beta < 0$. In the case $\beta < 0$ both schemes permit the isomeric state to be identified as $\frac{1}{2}(3p\,\frac{1}{2})$.

d. In all three schemes the identification shown is more natural in the case $\beta > 0$.

equal to 0.60; 0.70; 0.80 (oblate ellipsoid of revolution) and 1.20; 1.25; 1.35; 1.50; 1.70; 2.00 (prolate ellipsoid of revolution). For a spherically symmetric well the levels were calculated by the method described in reference 2. As shown by calculations, the behavior of the energy levels and their order do not depend strongly on the magnitude of the constant V_0, within the investigated limits of variation. A more significant effect on the order of levels was produced by variation of the spin-orbit coupling constant κ. In the diagram we show the nucleon energy level scheme in a rectangular spheroidal well calculated for the values $V_0 = 42$ Mev and $\kappa = 30$.

A comparison of the level scheme obtained here with Nilsson's[1] and Gottfried's[4] schemes shows that the behavior of the nucleon levels as a function of the deformation is qualitatively the same, but that there are certain differences of detail.

The results obtained were compared with experimental values of the spins and parities of the ground states of non-spherical odd nuclei. As usual it was assumed that the ground state of such nuclei is determined by the state of the odd nucleon, with the spin I_0 being equal to the component of Ω for this nucleon along the nuclear symmetry axis, with the exception of the case $\Omega = 1/2$ when for the determination of I_0 the so-called "coupling disruption factor" was computed (cf. reference 9, and also reference 1). The experimental values of the spins and the parities of the ground states of the nuclei were taken from the tables of Seaborg et al.[10] The values of the parameter β describing the deviation from the spherical shape were calculated from the experimentally determined values of the quadrupole moments.[10,11] The results of comparison with experiment are shown in Table I, which also gives a comparison of the same quantities with Nilsson's[1] and Gottfried's results.

It has been established experimentally that many of the nuclei considered in this paper have low lying isomeric states. These states can be identified according to our scheme. The results of such identification are shown in Table II.

From the results given in the tables it may be seen that the nucleon level scheme obtained in this paper is in good agreement with the experimentally found values of the spins and parities of the ground states and of the low lying isomeric states of nonspherical odd nuclei. In our case the agreement is better than in Gottfried's case.[4]

The present level scheme, generally speaking, gives as good an agreement with experiment as does Nilsson's scheme;[1] at the same time, it is free of the defect of Nilsson's scheme associated with the fact that to make the latter agree with data on the ground states of a number of nuclei it is necessary to assume for these nuclei considerably larger values of the parameter that describes the deviation from the spherical shape than are found experimentally.

A direct comparison of our nucleon level scheme with Nilsson's scheme shows that certain differences in level order existing in the spherical case, which are due to the different choice of potential, gradually disappear as the parameter β (describing the deviation from spherical shape) is increased. In the case of large deviations from the spherical shape, the order of levels in both schemes practically coincides. This fact indicates that the results of the shell model applied to strongly deformed nuclei are much less critical to the choice of the potential shape than in the case of spherical wells.

One should particularly note the odd isotopes of the actinides for which (with the exception of Ac^{227}) the sign of the quadrupole moment is either not known or is not known sufficiently reliably, and consequently the same uncertainty exists with respect to the sign of the parameter describing the deviation from the spherical shape. Neither our scheme nor Nilsson's scheme, contradicts the assumption of an oblate shape for some of these nuclei.

Let us investigate the conclusions to which we would be led by such an assumption in the case of U^{235}. First we note that it is in agreement with the measurements of B. Bleaney et al.,[12] who obtained for U^{235} a negative quadrupole moment. As seen from Table II, in this case (i.e. for $\beta < 0$) neither our scheme nor Nilsson's scheme contradicts the assumption that the isomeric state of U^{235} has negative parity (cf. note "c" of Table II). However, such an assumption would lead to changes in the properties of low lying levels of U^{235} and Pu^{239} from those adopted at present.[10,13]. Firstly, on the basis of data[13,14] on the multiplicity of the γ-transitions in U^{235}, we would have to change the parities of the low lying levels of this nucleus in the schemes of references 10 and 13. Secondly, on the basis of the data on the α-decay of Pu^{239}, one should expect that since the spins of the isomeric state of U^{235} and of the ground state of Pu^{239} coincide then, apparently, their parities also coincide. We should therefore ascribe $1/2^-$ to the ground state of Pu^{239}. According to both our scheme and Nilsson's scheme, such a possibility can be realized if the Pu^{239} nucleus is oblate (cf. note "m" to Table I). In this case, just as in the case of U^{235}, on taking into account the data[16] on the multiplicity of the γ-transitions in Pu^{239} one would also have to change the parities of the low lying levels of Pu^{239} in the schemes of references 10 and 13. However, the changes noted above would not contradict the available experimental data obtained from investigations of mutual transformations of the actinides (a detailed bibliography on this subject is contained in references 10 and 13). Thus, the problem of the shape of the actinides may be solved only by reliable measurement of the signs of their quadrupole moments.

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95