$$F_l(r) = (2\eta)^{1/6} v[(2\eta)^{-1/6}(\eta + \sqrt{\eta^2 + (l + 1/2)^2} - kr)],$$
$$G_l(r) = (2\eta)^{1/6} u[(2\eta)^{-1/6}(\eta + \sqrt{\eta^2 + (l + 1/2)^2} - kr)], \qquad (2)$$

where v and u are Airy[3] functions. These functions are related to Bessel functions of order $\frac{1}{3}$ in the following manner:

$$\left.\begin{matrix} u(t) \\ v(t) \end{matrix}\right\} = \sqrt{\frac{\pi}{3} t}\left\{I_{-1/3}\left(\frac{2}{3} t^{3/2}\right) \pm I_{1/3}\left(\frac{2}{3} t^{3/2}\right)\right\}, \quad t > 0,$$
$$\left.\begin{matrix} u(t) \\ v(t) \end{matrix}\right\} = \sqrt{\frac{\pi}{3} |t|}\left\{J_{-1/3}\left(\frac{2}{3} |t|^{3/2}\right) \mp J_{1/3}\left(\frac{2}{3} |t|^{3/2}\right)\right\}, \quad t < 0.$$

Since the Airy functions vary substantially when the magnitude of their argument changes by an amount on the order of unity, the root in the arguments of the functions (2) can be expanded in powers of $(l + \frac{1}{2})^2 \eta^{-2}$, retaining only the linear term. Furthermore, it is permissible to change in (1) from summation with respect to l to integration with respect to the variable t:

$$t = (l + 1/2)(2\eta)^{-4/3} + z_0, \qquad z_0 = (2\eta)^{-1/3}(2\eta - kr),$$
$$\sum_{l=0}^{\infty}(2l + 1)\ldots \rightarrow (2\eta)^{4/3}\int_{z_0}^{\infty} dt \ldots$$

For the evaluation of the integral thus obtained, we note that the quantities $u'(t)/u(t)$, $v'(t)/v(t)$, and $u(t)v(t)$ vary very slowly with t in comparison with $u^{-2}(t)$, [indeed, $u^{-2}(t)$ determines a substantial range of t] and they can be regarded as constants, with $t = z_0$. Now, changing to a new variable of integration, $x = v(t)/u(t)$, and noting that

$$v'(t)u(t) - v(t)u'(t) = 1, \qquad u^{-2}dt = dx,$$

we get

$$\sigma_c = \frac{8\pi\eta}{k^2}\left(\frac{k}{K}\right)\int_0^{x_0}\frac{dx}{(1 + x^2)}\left[\left(1 + \frac{\alpha x}{1 + x^2}\right)^2 + \left(\frac{\beta + \gamma x^2}{1 + x^2}\right)^2\right]^{-1},$$

where

$$\alpha = k/(2\eta)^{1/3} K u(z_0) v(z_0), \qquad \beta = u'(z_0)k/u(z_0)(2\eta)^{1/3}K;$$
$$\gamma = v'(z_0)/v(z_0)(2\eta)^{1/3}K.$$

Expanding the integrand in powers of $k/(2\eta)^{1/3}K$ and retaining only the first two terms, we have finally

$$\sigma_c = \frac{8\pi\eta}{k^2}\left(\frac{k}{K}\right)\left\{\tan^{-1}\frac{v(z_0)}{u(z_0)} - \frac{k}{(2\eta)^{1/3}K}\,\frac{v(z_0)}{u(z_0)[u^2(z_0) + v^2(z_0)]}\right\}.$$

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[2] Biedenhorn, Gluckstern, Hull, and Breit, Phys. Rev. **97**, 542 (1955).

[3] V. A. Fock, Таблицы функций Эйри (Tables of Airy Functions), M. 1946.

Translated by J. E. Viscardi
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THE ENERGY OF A COMPRESSED IMPERFECT FERMI GAS

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Submitted to JETP editor May 25, 1959

J. Exptl. Theoret. Phys. (U.S.S.R.) **37**, 585-587 (August, 1959)

WE consider a uniform degenerate Fermi gas whose particles interact according to a short-range law [two-body potential $V(r_{12})$, range of the forces a]. The mean distance between the particles is assumed small compared to a. There are no restrictions imposed upon the magnitude of the interaction, and we assume only that the Fourier transform of the potential $\nu(q)$ exists. The properties of such a simple model are of interest for the problem of nuclear matter where $\xi \sim 3$ or 4 (see below) and also for some astrophysical problems.

We evaluate in the following the energy ϵ per single fermion, which depends on the dimensionless "compression parameter"* $\xi = ap_0 \gg 1$ and "coupling constant" $\alpha = \nu(0)/a$. Here $p_0 = (3\pi^2\rho)^{1/3}$ is the Fermi momentum, and ρ the number density of the particles.

It is well known that the kinetic energy of the nonrelativistic gas is equal to $\epsilon_0 = 3p_0^2/10$. In the Hartree-Fock approximation the interaction energy corresponds to the first order of perturbation theory in α. Its non-exchange part is equal to

$$\varepsilon_1 = (\rho/2)\int V d\mathbf{r} \sim \alpha\xi p_0^2. \qquad (1)$$

The magnitude of the exchange term

$$\varepsilon_2 = -(8\pi^3\rho)^{-1}\int d\mathbf{p}_1 d\mathbf{p}_2 \nu(\mathbf{p}_1 - \mathbf{p}_2), \quad p_{1,2} < p_0,$$

depends on the behavior of V at small r. If V(0) is finite,

$$\varepsilon_2 = -V(0)/2 \sim \alpha p_0^2/\xi^2 \ll \varepsilon_1. \qquad (2)$$

To estimate the correlation energy (i.e., the higher terms of perturbation theory) we consider that the transition from the n-th to the n+1-st order in α occurs either through the addition of another interaction vertex between those particles which are already present in the n-th term, or through the increase by unity of the number of particles which are correlated. In the first case there occurs in the expression for the energy a superfluous integration over the momentum transfer $q \sim a^{-1}$, and in the second case over the momentum of the new particle $p \sim p_0$.

If $ap_0 \gg 1$, the ratio of the corresponding integrals is small and the main part is played by that chain where in each (n-th) order of perturbation theory the maximum number (equal to n) of particles is connected. The energy of such many-particle correlations was found by Gell-Mann and Brueckner[1] and can for $\xi \gg 1$ be put in the form†

$$\varepsilon_3 = (^5/_8\, \pi p_0^2) \int\limits_{-\infty}^{\infty} dv \int\limits_0^{p_0} dq q^3 (A + \ln(1 - A)),$$

$$A(q, v) = 8\pi p_0 \nu(q)(1 - v \tan^{-1}(v^{-1})) \sim \alpha\xi. \quad (3)$$

If α is so small that $\alpha\xi \ll 1$,

$$\varepsilon_3 = -8\pi^2(1 - \ln 2)\int\limits_0^{p_0} dq q^3 \nu^2(q). \quad (4)$$

For the most important kinds of interaction $\nu^2(q) \approx Cq^{-4}$ when $aq \gg 1$ [for a rectangular well, for instance, $C = V(0)^2 a^2/8\pi^4$]. The integral in (4) is thus equal to $C \ln \xi$.

A much more important case is $\alpha\xi \gg 1$ (but $\alpha/\xi \ll 1$). We introduce a momentum $q_0 \sim (\alpha\xi)^{1/2}/a$ by the condition $A(q_0, 1) \sim 1$. The main, logarithmic contribution to the integral over q in (3) gives a term of second order in A in the range from q_0 to p_0 (the range from 0 to q_0 gives a non-logarithmic contribution since $A \sim 1$). We therefore return to (4) but with the limits of integration from q_0 to p_0.

$$\varepsilon_3 = -4(1 - \ln 2)\pi^2 C \ln(\xi/\alpha) \sim \alpha^2 (p_0/\xi)^2 \ln \xi. \quad (5)$$

We now proceed to discuss the results.

a) Even for a strongly imperfect compressed gas the correlation energy is small and the Hartree-Fock approximation is permissible.‡

b) The equation of state of a compressed gas is of the form

$$P = \rho^2 \partial\varepsilon/\partial\rho = (\rho^2/2)\int V d\mathbf{r} + \tfrac{1}{5}(3\pi^2)^{2/3}\rho^{5/3} + \ldots \quad (6)$$

and is determined not by the kinetic energy but by the interaction. Making the gas relativistic makes this conclusion stronger. The more the gas is compressed, the further it will thus be from being perfect. This property is quite clearly pronounced: even when $\xi \simeq 1$, the first term in (6) will be an order of magnitude larger than the second one (for $\alpha \sim 1$). In a neutral electron-nuclear gas (Coulomb interaction) the first term in (6) falls out and the opposite situation occurs.

c) If $\int V d\mathbf{r} < 0$, i.e., if attractive forces predominate, $P < 0$ and the gas tends to an unbounded compression. It is thus necessary to require that $\int V d\mathbf{r} > 0$. This condition makes the well known considerations about the change in sign of the potential, which refers to the problem of the saturation of the nuclear forces, more precise.

To extend the results obtained to real systems one must bear in mind the inevitability of the occurrence in a strongly compressed gas of many-particle forces which depend moreover on the state of the interacting particles (momenta, spins, and so on).

One may, on the other hand, think that these results have a not too limited range of applicability. The smallness of the correlation energy is thus apparently a general property of compressed Fermi systems: because of the Pauli principle there is only a narrow range of momenta of the intermediate states over which one must integrate and it decreases with increasing compression. Moreover, conclusions based upon Eq. (6) are valid also for not too strong compressions, as was noted. In that region, however, the experimental data on the interaction of nucleons can be described by a two-particle potential with a not very sharp dependence on the state.**

The model under consideration does not enable us to describe a "hard core" corresponding to nuclear interactions. This problem will be considered separately.

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*We use units where $\hbar = M = 1$.

†In that paper a compressed electron gas was considered. The condition $\xi \gg 1$ (large range of the forces) brings that case nearer to the one investigated in the present paper. We note that in quantum field theory language the approximation under consideration corresponds to the lowest approximation in α for the polarization operator.

‡The opposite situation occurs in a well known sense for a rarefied gas: although the interaction is small in that case compared with the kinetic energy, all terms in an expansion in α (for $\alpha \gtrsim 1$) are equally important.

**It is easy to take this dependence into account in the formulas given above. For a system of particles of two kinds which are coupled by Serber forces, it is then necessary to use a coefficient 3/8 in (1) and (5) and −3/2 in (2).

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[1] M. Gell-Mann and K. A. Brueckner, Phys. Rev. 106, 364 (1957).

Translated by D. ter Haar
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