SOVIET PHYSICS JETP VOLUME 37 (10), NUMBER 3 MARCH, 1960

CAPTURE MECHANISM IN BETATRONS

I. M. SAMOĬLOV

Submitted to JETP editor March 5, 1959

J. Exptl. Theoret. Phys. (U.S.S.R.) **37**, 705-712 (September, 1959)

The motion of electrons in a betatron is considered, with account of the Coulomb repulsion of the particles in the beam injected into the chamber during one revolution of an electron. It is shown that electron capture is due to the change in the mode of the radial oscillations, caused by the repulsion of the particles in the beam, and to the loss of electrons as a result of collisions with the injector or chamber walls.

AFTER the first attempts to explain electron capture in the betatron on the basis of a single electron analysis,[1,2] a number of authors proposed various capture mechanisms in which collective particle interactions were taken into account. Many workers[3-12] indicate that the most important effect is the Coulomb interaction. Usually in analyzing the injection process one considers the average effect of the space charge in the chamber. This procedure leads to capture mechanisms which, however, are characterized by low efficiencies. In this kind of analysis the effects due to the Coulomb repulsion of electrons in the beam in the first few revolutions, when the motion of the particles is relatively well ordered, is neglected.

We consider an arbitrary time of injection, at which there is some electron space charge present in the chamber. We isolate a group of injected particles which form a complete loop. This group will be affected by the space charge in the chamber, Q, and by the interactions between the particles themselves. If the interaction in the loop is taken into account by including the charge of the loop in Q, taking averages over the volume of the chamber as is usually done, the results do not give a true picture of the motions. Each of the particles in the loop is subjected to a force which depends on the position of the particle with respect to the space charge whereas in actuality this force is determined by the position relative to the central part of the loop, i.e., the well known transverse repulsion of electrons is a ribbon-shaped loop. (Obviously the beam has this shape, at least for some fraction of a revolution after the electrons leave the injector). Since the motion of the particles in the loop can remain ordered for several revolutions, it is of interest to investigate the effect of this ordered motion on the motion of electrons in the injection process. The Coulomb repulsion causes expansion of the beam, but because of the magnetic focusing field the expansion cannot go on indefinitely. As a result there is an increase in the amplitude of the oscillations and a shift of the focal points (if the latter continue to exist) in the azimuthal direction. The transverse repulsion of particles in a loop has been considered in a recently published paper by Matveev.[12] This work treats a particular case which is not usually realized in betatrons, i.e., the case in which the electrons leave the injector in a parallel beam. In the present paper we consider the more general case in which the electron beam is characterized by an angular divergence.

1. EFFECT OF ELECTRON REPULSION IN THE BEAM

It is convenient to treat the effect of repulsion by considering the motion of a group of particles isolated in such a way as to form a single loop. The presence of the space charge in the chamber (except for the charge of the loop itself) results in a value, n_e, which differs from the original magnetic field decay index n $(H \sim r^{-n})$. If we neglect the change in n_e during one or two revolutions the charge in the chamber does not affect the interaction of particles in the loop in any way other than a change in the effective value of n.

Consider the problem of Coulomb repulsion in that part of the beam which forms a single loop and is injected into a chamber which is free from space charge. An estimate based on the usual equations of motion indicates that the Coulomb repulsion force (F) in the loop (here we consider the radial component) does not change substantially over the greater part of the electron path. The only noticeable exceptions are the focal points, where F is several times larger than in the other parts of the trajectory. For this reason we first estimate the effect of repulsion in the focal regions.

The available evidence indicates the existence of only a single focus as the beam leaves the injector; the further behavior of the particles is determined by interaction effects in the loop. In analyzing the repulsion of particles in the vicinity of the first focus, which occupies an angle γ of the azimuthal coordinate φ, we make the following assumptions. 1) The force F is different from zero for $\varphi < \gamma$ and in the other regions the particles execute free betatron oscillations. 2) The electron distribution over flight angle α (total angular divergence 2α) is uniform; for an infinite beam height this means $F = f\alpha$ where $f = \text{const}$ and α is taken from the vertical symmetry plane of the beam. 3) The axial electrons ($\alpha = 0$) leave the injector along a tangent to the radius of the circle r_0 on which the injector is located. 4) The magnetic field is constant from the time at which the electrons leave the injector. 5) The motion is one-dimensional, i.e., we consider only the radial motion of particles in the loop.

In the region of the first focus the equations of motion in a magnetic field described by $H \sim r^{-n}$ can be conveniently written (in the first approximation) in the form*

$$\eta'' + k^2\eta = -\delta_H + q\alpha, \tag{1.1}$$

where $\eta = (r - r_0)/r_0 = \rho/r_0$; the derivative is taken along the azimuth; $k^2 = 1 - n$; $\delta_H = \delta H/H_0 \ll 1$ is a parameter which characterizes the instantaneous orbit, where H_0 is the field at the radius r_0 and $eH_0r_0/mvc = 1$ (the usual notation is used here), $v \ll c$ and $\delta H = H(r_0) - H_0$. The Coulomb repulsion is given by the $q\alpha$ term

$$q = fr_0/mv^2 = \varkappa 4\pi r_0 e j_\alpha/mv^3 h, \tag{1.2}$$

where j_α is the current in a unit angle of the beam; κ is a coefficient which takes account of the finite height of the beam h. The solution of Eq. (1.1) can be written in the form

$$\eta_1 = q\alpha/k^2 + A_1 \sin k(\varphi + \psi_1) - (\delta_H/k^2)(1 - \cos k\varphi), \tag{1.3}$$

where

$$A_1 = (\alpha/k)\sqrt{1 + (q/k)^2}, \quad \tan k\psi_1 = -q/k.$$

Equation (1.3) can be divided into two parts; one of these, $\eta_\delta = -(\delta_H/k^2)(1 - \cos k\varphi)$, describes the motion of the beam as a whole while the other, which depends on $\alpha(\eta_\alpha)$, determines the motion of the particles in the beam with respect to the axial trajectory η_δ, so that

$$\eta = \eta_\delta + \eta_\alpha. \tag{1.3a}$$

*The radius at the injector is taken as the reference.

Since ψ_1 is independent of α and $A_1 \sim \alpha$, in the next section of the trajectory η_2 has the form given by Eq. (1.3a) (for $q = 0$) where, as in the first part, ψ_2 is independent of α and $A_2 \sim \alpha$. It is easy to show that $\eta_{\alpha 2}$ is symmetric with respect to the azimuth φ_m at which $\eta_{\alpha 2}$ is a maximum and $\eta'_{\alpha 2} = 0$. Hence for the azimuth $\beta = 2\varphi_m - \gamma$ we have $\eta_{\alpha 2}(\beta) = \eta_{\alpha 1}(\gamma)$ and $\eta'_{\alpha 2}(\beta) = -\eta'_{\alpha 1}(\gamma)$. Since the force F is "turned off" at $\varphi = \gamma$, i.e., over the width of the beam $\eta_{\alpha 1}(\gamma)$, it must be "turned on" again in the next part, $\varphi > \beta$. The equation of motion will be similar to Eq. (1.3), where it can be shown that $A_3 = A_1$ and the phase ψ_3, as before, is independent of α. From the symmetry of the trajectory with respect to φ_m it follows (this can easily be shown by direct calculation) that when $\varphi = 2\varphi_m$, $\eta_{\alpha 3} = 0$ for any α and $\eta_{\alpha 3} = -\eta'_{\alpha 1}(0) = -\alpha$, i.e., the electron beam is focused again and the original angular divergence is maintained. It is apparent that in the next portion of the path the behavior of the beam with respect to the axial trajectory η_δ will repeat that given above since the only difference from the original initial conditions is the change in the sign of F at the focus, i.e., where the electron trajectories intersect. Thus η_α is a periodic function with period $\Phi = 4\varphi_m$ which exceeds the period of the free betatron oscillations $(2\pi/k)$ by an amount θ, where

$$\tan(k\theta/4) = (1 - \cos k\gamma)/(k/q + \sin k\gamma). \tag{1.4}$$

An expression for η_α which is valid for any φ and convenient for calculation can be obtained if we represent η_α in a form of a series using the expressions $\eta_{\alpha(1-3)}$, each of which applies for the appropriate sections:

$$\eta_\alpha = \sum_{i=0}^{\infty} [8q\alpha(1 - \cos\lambda_i\gamma)/\lambda_i\Phi(k^2 - \lambda_i^2)] \sin\lambda_i\varphi, \tag{1.5}$$

where $\lambda_i\Phi = (2i + 1)2\pi$.

Limiting ourselves to the first term in the expansion $\lambda = 2\pi/\Phi$ and small values of γ, using $\rho_\alpha = r_0\eta_\alpha$, we have from Eqs. (1.4) and (1.5)

$$\rho_\alpha \simeq (1 + q\gamma)(\alpha r_0/k)\sin\lambda\varphi. \tag{1.6}$$

We shall not discuss these results but estimate the effect of the Coulomb repulsion over the entire electron trajectory. It is apparent that this case is the particular case of the problem considered above for which $\gamma \to (\frac{1}{4})\Phi$. In the expression for $F = f\alpha$ we choose some average value of f along the trajectory. When $\gamma \to (\frac{1}{4})\Phi$, Eqs. (1.4) and (1.5) yield

$$\tan(k\theta/4) = q/k, \tag{1.7}$$

$$\rho_\alpha = B\alpha \sin \lambda\varphi, \quad (1.8)$$

where

$$B = 4qr_0/\pi (k^2 - \lambda^2) \approx (qr_0/k^2)(1 + 4/k\theta). \quad (1.9)$$

In making a numerical estimate we note that at injection $q = q_f = 1.5$ in the focal regions and the minimum value of q along the trajectory is approximately $0.25\, q_f$. In this case, with $\gamma \approx 0.5$ and $n = 0.5$, it follows from Eqs. (1.4) and (1.6) that there is an increase in the amplitude by 50 — 75% while the increase in the period is about 3 — 5%; for the entire trajectory, from Eqs. (1.7) and (1.9) (with $q = 0.25\, q_f$), the increase in amplitude is 35 — 60% and in period 20 — 30%. Thus, aside from the significant increase in q_f over the minimum mean value of q along the trajectory, taking account of the Coulomb repulsion in the focal regions only is not sufficient since, as will be shown below, the change in the period of oscillation is very important in the capture mechanism.

We note that taking formal averages of the charge of the turn over the volume of the chamber also yields a change in the amplitude and frequencies of the electron oscillations. We shall not consider this effect quantitatively, but indicate the important qualitative difference of the picture described above from the "average" analysis. In the latter case each of the particles in the turn (consequently in the beam as a whole) executes oscillations at the same frequency, corresponding to some effective value of n_e. Actually, as has been shown above, the resulting trajectory has everywhere the form given by Eq. (1.3a) or

$$\rho = B\alpha \sin \lambda\varphi - a(1 - \cos k\varphi), \quad (1.10)$$

where $a = \delta_H r_0/k^2$, i.e., each of the particles in the turn participates simultaneously in two oscillatory motions (with different frequencies) and, as is apparent from Eq. (1.10), the motion of the beam as a whole at the frequency of the usual betatron oscillations is maintained.

The relation between the parameter a and the time of injection τ can be easily obtained for $H = \xi H_m \sin \omega t$ (ξ is a coefficient which takes account of the behavior of the magnetic field close to zero)

$$a \approx (\xi e H_m \omega r_0^2 / mvck^2)\tau, \quad (1.11)$$

where $\tau = t - t_0$ and t_0 is the time at which $H = H_0$.

In conclusion we may note that as a consequence of the nonuniform distribution of electrons over flight angle, the difference in the coefficient κ for particles with different α, and the vertical oscillations (which have not been considered in the problem), the electron beam will obviously not be focused ideally. However it will be apparent that the above analysis is qualitatively correct and that the idealized picture of the motion can be used in the calculations.

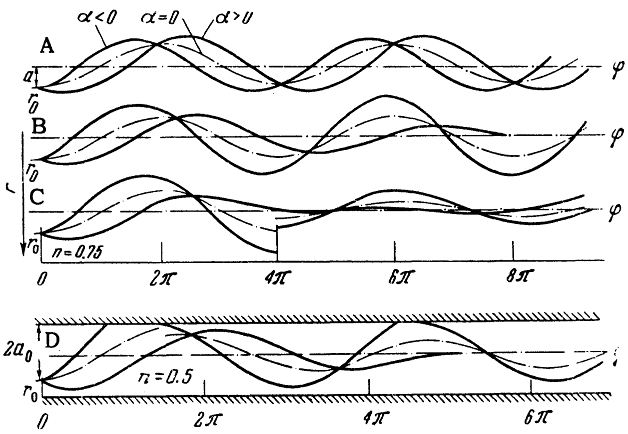

FIG. 1

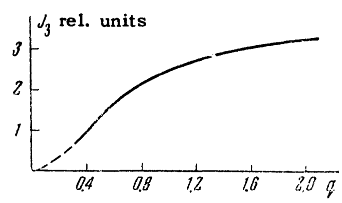

FIG. 2

2. ELECTRON CAPTURE MECHANISM

The Coulomb repulsion of electrons in a turn changes the usual particle trajectories (Fig. 1A) in such a way that for half of the electrons in the beam the motion (1.10) assumes the form of rapidly damped oscillations, Fig. 1B. However, if the conditions are not changed, as follows from the periodicity of Eq. (1.10) the amplitude of the oscillations returns to the initial value. An irreversible change in amplitude, which leads to particle capture, results from loss of part of the electrons in the turn as a result of collisions with the injector or with the chamber walls.

If there is a collision with the injector at some value $\varphi = \varphi_1$, in the remaining part of the turn the new axial electrons (which are emitted from the injector at some angle $\alpha = \alpha_c$) are no longer subject to the effect of the repulsive forces and execute free betatron oscillations with a new amplitude A_c which can be determined (for $\alpha = \alpha_c$ and $\varphi = \varphi_1$) from the expression

$$A = \{a^2 + (B\alpha)^2 [\sin^2 \lambda\varphi + (\lambda/k)^2 \cos^2 \lambda\varphi] - 2Ba\alpha [(\lambda/k) \sin k\varphi \cos \lambda\varphi - \cos k\varphi \sin \lambda\varphi]\}^{1/2}. \quad (2.1)$$

At the same time the remaining electrons in the beam now move with respect to the new axial particles with the old increase in period (Fig. 1C) since the quantity q is not changed. As has been noted above, the motion of the axial particles describes the motion of the beam as a whole; hence, after loss of part of the electrons of the turn, as before, the remaining particles participate in two oscillatory motions with different frequencies and amplitudes A_c ($a \rightarrow A_c$) and $B\alpha_i$ where α_i is now taken from α_c. Since the electrons move about the instantaneous orbit, remaining at a distance $a - \Delta$ from the edge of the injector anode (Δ is the thickness of the injector element), the injector will not be struck by those particles for which

$$A_c + B\alpha_i \leqslant a - \Delta. \tag{2.2}$$

It is apparent from Eq. (2.2) that the capture angle increases with increasing a; hence electrons will be captured most effectively at the injection time for which $a \approx a_0$, half the distance between the inner wall of the chamber and the center of the cathode. In estimating the capture angle it is necessary to take account of the collision of the beam with the chamber walls because when $a \approx a_0$ a significant fraction of the electrons are lost in the first turn, Fig. 1D.

The general picture of the collision of the beam with the wall is rather complicated because the loss of electrons is "stretched" over the azimuth. In estimating the capture angle in this case, we assume for simplicity that the electrons which strike the wall are lost "abruptly" (as in collisions with the injector) for some value of the azimuthal angle φ_0. In this case, in contrast with collisions in the turn, particles are lost both for $\alpha < 0$ and $\alpha > 0$. In determining α_c and A_c we assume that the electrons do not strike the wall if the amplitude determined from Eq. (2.1) for $\varphi = \varphi_0$ is smaller than $2a_0 - a$. Assuming that $\alpha_c = (\alpha_1 + \alpha_2)/2$ where $\alpha_{1,2}$ is the root of the expression $A \leq 2a_0 - a$, we obtain from Eq. (2.1)

$$A_c = ca, \tag{2.3}$$

where

$$c = [\sin k\varphi_0 \sin \lambda\varphi_0 + (\lambda / k) \cos k\varphi_0 \cos \lambda\varphi_0] / [\sin^2 \lambda\varphi_0 + (\lambda / k)^2 \cos^2 \lambda\varphi_0]^{1/2}. \tag{2.3a}$$

Determining the capture angle as in Eq. (2.2), we have for the two regions

$$a \leqslant a_0 + \Delta / 2, \qquad \alpha_{cap} = (a - \Delta - A_c) / B; \tag{2.4a}$$

$$a \geqslant a_0 + \Delta / 2, \qquad \alpha_{cap} = (2a_0 - a - A_c) / B. \tag{2.4b}$$

We note that after the first passage of the beam the part which remains may experience new collisions with the walls of the chamber or with the injector; as a result, the capture angle can be larger than indicated by Eq. (2.4). However it is extremely difficult to analyze the multiple losses of electrons; for this reason we take the capture angle to be the minimum value as determined by the first collision only.

Equation (2.4) allows us to estimate the critical values of the injector currents (J_{cr}) at which the collective capture mechanism starts to become effective. As the azimuth φ_0 we take the azimuth defined by $k\varphi_0 = \pi$, where the axial electrons of the beam pass closest to the inner wall of the chamber. In estimating J_{cr} we must take account of the fact that when $a < a_0$, for values of α which are not too large and values of q which are not too small, the beam passes through the azimuth $k\varphi_0 = \pi$ essentially without loss of particles and the determining azimuths become $k\varphi_0 = 2\pi$ to 3π or larger. It is apparent from Eq. (2.4) that the criterion for the operation of collective capture is the condition $(1 - c) \geq \Delta/a$ whence, expressing eq. (2.3a) in a series in powers of $\beta = k\theta/2\pi \ll 1$ and limiting ourselves to the first terms of the expansion $c \approx 1 - (\beta k\varphi_0)^2/2$, we obtain

$$q_{cr} \leqslant \pi \sqrt{\Delta / 2a} / \varphi_0. \tag{2.5}$$

With $\Delta = 0.2$ cm, $a = 1$ cm, $k^2 = 0.5$, and $k\varphi_0 = 2\pi$ to 3π we have $q_{cr} = 0.1 - 0.07$, which for $h \approx 1$ cm and $\kappa \simeq 0.4$ gives $j_{\alpha cr} = 2.5$ ma/deg for an injection energy U = 10 kv. Thus at the 30 Mev synchrotron of the Institute of Physics of the Academy of Sciences[11] ($k^2 = 0.25$) the experimental value $J_{cr} \approx 3$ ma. If we assume $2\alpha \approx 6°$, calculation shows that $J_{cr} = 5$ to 8 ma for U = 10 kv and 2 to 3.5 ma for U = 5 kv.

Equation (2.4) can be used to determine the dependence of the capture current J_{cap} on the injection current J. Since $J_{cap} \sim \alpha_3 j_\alpha$, taking account of Eq. (2.4) (where $\Delta \ll a$) and Eqs. (1.2) and (1.9), we have

$$J_{cap} \sim ak^2(1 - c) / (1 + 4/k\theta). \tag{2.6}$$

It is apparent from Eq. (2.6) that the further the instantaneous orbit is from the injector, the larger the capture current. The quantities c and $k\theta$ are determined in the final analysis by the dimensionless parameter q/k, i.e., taking account of Eq. (1.2), Eq. (2.6) gives the qualitative dependence of J_{cap} on J shown in Fig. 2, (computed for $k\varphi_0 = \pi$). For $k^2 = 0.5$, $a = a_0 = 2$ cm, $\Delta = 0.2$ cm, $r_0 = 20$ cm, $k\varphi_0 = \pi$ and q = 0.5 the total capture angle is approximately 1.5°. Thus the capture mechanism due to Coulomb repulsion of the particles and the loss of part of the electrons from the turn serves to explain the observed high efficiency of injection and the threshold nature of capture.

Above we have considered the behavior of particles in a beam which is injected into the chamber during the course of one turn τ_0. In considering injection times greater than τ_0 it is necessary to take account of the interaction between turns. As we have already noted, the motion of electrons in an isolated turn with account taken of the space charge of the other turns, can be considered in terms of an effective n_e. When this effect is taken into account the dependence of J_{cap} on J will differ from that given in Fig. 2 since the function $(1-c)/(1+4/k\theta)$ will reach its limiting value for values of q which are smaller than those in Fig. 2. For large values of q, Eq. (2.6) becomes the usual curve, i.e., if we take account of the growth of space charge with q due to the previous turns the nature of the curve (2.6) is in agreement with that observed in betatron curves which shows the dependence of gamma yield on injection current.

CONCLUSION

In this paper we have considered the behavior of particles in a beam injected into a chamber to form one turn, treating the averaged motion of the particles in the space charge field of the other turns; qualitatively this analysis would seem to indicate that capture is not changed in multi-turn injection. Starting from this result one can draw several conclusions which are of practical interest. 1) The capture angle is limited by collisions of the beam with the walls of the chamber which result in a loss of a considerable fraction of the electrons, as is observed experimentally.[9] 2) It is feasible to inject a sharply focussed beam since smaller emission currents are required and this is an important factor, especially at high injection energies. 3) In view of the relatively small effect of the thickness of the injector edge Δ (especially in large machines) capture should take place effectively when electrons are injected from outside the chamber through the use of deflection plates. This method of injection appears promising from the point of view of increasing the gamma yield since it is relatively easy to use an injector at several hundred kilovolts. 4) It is possible to achieve injection in a magnetic field which is constant in time; this technique has been used successfully for the preliminary adjustment of betatrons.[9] It is also possible to achieve injection in a uniform magnetic field. 5) Electron capture can also be realized when the injector is located inside the equilibrium orbit, close to the inner wall of the chamber. It is also possible to locate the injector above (or below) the plane of the equilibrium orbit in such a way that the cathode (long-dimension) is along a radius and close to the top (bottom) of the chamber. The use of an injector of this kind may be useful for the extraction of electron beams from a betatron.

In conclusion we may note that even if a rigorous analysis of the injection process (many turns) shows that the presence of space charge due to the other turns in any way reduces the efficiency of particle capture, the single-turn injection scheme given above is still of interest. In this scheme the electrons are injected into a chamber which is space-charge free; because of the high injection efficiency (one turn) it is possible to achieve capture currents which are sufficient for producing high gamma yields. We may note that for injection from outside the chamber (through the use of deflection plates) a one-turn injection system would not represent any special technical difficulties.

The author is indebted to A. A. Sokolov for fruitful discussions.

[1] D. W. Kerst and R. Serber, Phys. Rev. **60**, 53 (1941).

[2] J. A. Rajchman and W. H. Cherry, J. Franklin Inst. **243**, 261 (1947).

[3] D. W. Kerst, Phys. Rev. **74**, 503 (1948).

[4] R. Wideroe, J. Appl. Phys. **22**, 362 (1951).

[5] S. E. Barden, Proc. Phys. Soc. **64B**, 85 (1951).

[6] Kerst, Adams, Koch, and Robinson, Rev. Sci. Instr. **21**, 462 (1950).

[7] V. D. Rusanov, Dissertation, Physics Inst., Acad. Sci., 1956.

[8] Logunov, Ovchinnikov, and Rusanov, J. Tech. Phys. (U.S.S.R.) **27**, 1135 (1957), Soviet Phys.–Tech Phys. **2**, 1032 (1958).

[9] Yu. S. Korobochko, J. Tech. Phys. (U.S.S.R.) **27**, 1603 (1957), Soviet Phys.–Tech Phys. **2**, 1486 (1958).

[10] Yu. N. Lobanov and V. A. Petukhov, Ускорители элементарных частиц (Elementary Particle Accelerators) Atomizdat, Moscow, 1957, p. 73.

[11] L. M. Korvizhnykh and A. N. Lebedev, JETP **34**, 984 (1958), Soviet Phys. JETP **7**, 679 (1958).

[12] A. N. Matveev, JETP **35**, 372 (1958), Soviet Phys. JETP **8**, 259 (1959).

Translated by H. Lashinsky
137