KINETIC THEORY OF IMPACT IONIZATION IN SEMICONDUCTORS

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The effect of impact ionization processes on the distribution function for electrons and holes in a strong electric field is studied. It is shown that the energy dependence of the impact ionization probability near the threshold is essentially different for crystals with small and high dielectric constants; the solution of the kinetic equation is considered in both these cases. Expressions are obtained for the equilibrium number of carriers in a strong field, the impact-ionization coefficient, the critical field, etc. The dependence of the breakdown field on temperature, on specimen thickness, and on the electron-lattice interaction law is found. The connection of the expressions obtained with the known breakdown criteria of Fröhlich and Hippel is established. It is shown that increasing the electric field causes a decrease in the recombination speed, as a result of which the equilibrium number of carriers starts growing as the field increases long before the appearance of impact ionization.

THE electric breakdown of semiconductors apparently takes place as a result of the unlimited growth of carrier concentration with increasing field strength.[1] In a stationary state the number of carriers, n, is determined by the relationship

$$n\{\overline{w_i(E,T)} - \overline{w_r(n,E,T)}\} + n_0(E,T) = 0, \qquad (1)$$

where $\overline{w_i(E, T)}$ and $\overline{w_r(n, E, T)}$ are the impact ionization and recombination probabilities averaged over the distribution function, $n_0(E, T)$ is the number of carriers of a given type created in unit volume of the semiconductor in unit time by thermal ionization and direct field extraction of valence electrons into the conduction band, E is the field strength and T the temperature.

With increasing field, as will be shown below, $\overline{w_r}$ decreases but $\overline{w_i}$ grows rapidly and, consequently, n increases. In the field E_c for which $\overline{w_i} = \overline{w_r}$ the carrier concentration tends to infinity, which is the breakdown criterion for the case given. Thus, quantitative consideration of the behavior of a semiconductor in the pre-breakdown region, as well as a study of the mechanism of breakdown itself, requires the solution of the kinetic equation taking into account the processes of impact ionization and recombination. This is the aim of the present work.

Following the usual method,[2] it is not difficult to obtain the following system of equations for determining the symmetric $f_0(\epsilon)$ and antisymmetric $f_1(\epsilon)$ parts of the distribution function, $f(\mathbf{P})$

$$f(\mathbf{P}) = f_0(x) + \frac{e\mathbf{EP}}{m} f_1(x) + \ldots = f_0(x) - \frac{e\mathbf{EP}}{\varepsilon_i m} \frac{\tau(x)}{1 + w_i(x)\tau(x)} \frac{df_0(x)}{dx}, \qquad (2)$$

$$\left[\eta(x) + \left(\frac{E}{E_i}\right)^2 \frac{\lambda^2(x)\,\delta}{x\delta(x)} \frac{1}{1 + w_i(x)\tau(x)}\right] \frac{df_0(x)}{dx} + f_0(x) - \frac{\tau(x)}{x^{3/2}\delta(x)} \frac{S(x)}{N_i} = 0, \qquad (3)$$

$$\frac{1}{N_i x^{1/2}} \frac{dS(x)}{dx} - [w_i(x) + w_r(x)] f_0(x) + n_0(x, E, T) + N_i \int_1^\infty w_i(x, x') f_0(x') x'^{1/2} dx' = 0, \qquad (4)$$

where $x = \epsilon/\epsilon_i$; $\epsilon = P^2/2m$ is the energy; $\mathbf{P}$ is the momentum; ϵ_i is the threshold ionization energy; τ^{-1} is the frequency of collision with phonons; $w_i(x)$ and $w_r(x)$ are the total probabilities of impact ionization and recombination; $w_i(x, x')$ is the probability of the creation by ionization of a carrier with energy x by a carrier with initial energy x', $\lambda(x) = l(x)/l(1)$; $l(x) = P\tau(x)/m$ is the mean free path; $S(x)$ is the carrier current through the surface $\epsilon(\mathbf{P}) = x\epsilon_i$, caused by the field and phonon interactions; $\frac{3}{2}N_i$ is the total number of states with energy $\epsilon < \epsilon_i$;

$$\delta(x) = 4m \int_0^{2P} B(q)\,\hbar\omega_q q\,dq \Big/ \int_0^{2P} B(q)(1 + 2N_q)\,q^3 dq \sim \frac{\hbar\omega_P}{\varepsilon(N_P + 1/2)}, \qquad (5)$$

$$\eta(x) = \int_0^{2P} B(q)(\hbar\omega_q)^2(2N_q+1)\,q\,dq \Bigg/ 2\varepsilon_i \int_0^{2P} B(q)\,\hbar\omega_q q\,dq,$$

$$\tau^{-1}(x) = \frac{Vm}{4\pi\hbar^4 P^3}\int_0^{2P} B(q)(2N_q+1)\,q^3 dq,\quad E_i = \frac{\sqrt{3\delta}\varepsilon_i}{el(\varepsilon_i)}. \quad (6)$$

Here, q, ω_q and N_q are the momentum, frequency, and number of phonons; B_q is the square of the matrix element of the interaction of an electron with a phonon; V is the normalization volume; E_i is the field in which the mean energy of the carriers becomes of the order ϵ_i; $\delta = \delta(1)$. The small value of δ, the average fraction of the energy lost by an electron in one collision with the lattice, is a condition for the applicability of the approximation considered. For the parameter values of interest to us, $\delta \sim 10^{-2}$. The small quantity $\eta(x)$ we will neglect henceforth. For those valence crystals in which the electrons interact mainly with acoustic phonons $B_{ac}(q) \sim q$ and $\hbar\omega_q^{ac} = cq$, where c is the velocity of sound. When the interaction is with optical phonons $B_{op}(q) = \text{const.}$ and $\hbar\omega_q^{op} = \hbar\omega_0 = \text{const.}$

From (5) and (6) it follows that

$$\delta_{ac}(x) = 4mc^2/kT, \qquad \lambda_{ac}(x) = 1,$$

$$E_i^{ac} = \sqrt{12mc^2/kT}\,\varepsilon_i/el,$$

$$\delta_{op}(x) = \hbar\omega_0/\varepsilon(N_P + 1/2), \qquad \lambda_{op}(x) = 1,$$

$$E_i^{op} = \sqrt{3\hbar\omega_0\varepsilon_i/(N_P+1/2)(el)^2}. \quad (7)$$

Before proceeding to the solution of Eqs. (2) — (4), we make some remarks on the choice of the probabilities $w_r(x)$, $w_i(x)$ and $w_i(x, x')$. The effect of recombination on the form of the distribution function is insignificant in view of the inequality $w_r\tau \ll 1$, which is well fulfilled. Therefore, the corresponding term in (4) can be considered as a small contribution and the fact that it is in general nonlinear has no effect. In the majority of cases, however, an important part is played by the so-called "radiationless" recombination associated with carrier capture into local states. Its probability for a sufficiently large number of carriers can be considered as independent of concentration. Such a process can only take place for very slow carriers ($\epsilon \lesssim \hbar\omega_m$, where ω_m is the maximum lattice vibration frequency), which in fact are not included in the conditions considered, since for them $\delta(x)$ is not small. Therefore, it is most natural to include radiationless recombination, not in (4), but in the boundary condition for $S(x)$ at $x = 0$. The last two terms in (4) can be taken into account in a similar way. In fact, the probability of creation of a carrier with energy ϵ by thermal ionization and by the field decreases exponentially with increase of ϵ. Likewise, the number of carriers with energies essentially exceeding the ionization threshold is exponentially small. Therefore, both $n_0(E, T)$ and $w_i(x, x')$ cause the creation of only very slow carriers $x \ll 1$, and we include them only in the boundary condition

$$\frac{S(0)}{N_i} - w_r^0 f_0(0) + \frac{n_0(E,T)}{N_i} + \sum_{e,h} N_i \int_0^\infty x^{1/2}dx \int_1^\infty w_i^{(1,2)}(x,x') f_0(x')\, x'^{1/2} dx' = 0,$$

$$w_r^0 = \int_0^\infty w_r(x)\, x^{1/2}\, dx, \quad (8)$$

where the summation takes into account the presence of two carrier types and the indices 1 and 2 refer to the creation probability of carriers of the same or opposite charge.

The quantity $w_i(x)$, as shown in Appendix 1, can increase near the threshold either linearly or quadratically, depending upon whether the value of the dielectric constant of the crystal is small or large, i.e.,

$$w_i(x) = p(x-1)^j k_j(x)/\tau(x), \qquad j = 1, 2,$$

$$k_j(x) = 1 + \sum_{n=1}^\infty k_{jn}(x-1)^n. \quad (9)$$

The dimensionless quantity p thus defined is rather large ($p \sim 10^2$).

Equations (3) and (4) take an essentially different form in the regions $x < 1$ and $x > 1$. It is natural, therefore, to solve them in each of these regions separately and then match the solutions obtained at $x = 1$. Below the ionization potential $w_i(x) = 0$, $w_r(x)$ is a small correction, and all the remaining terms in the right half of (4) do not depend on the value of $f_0(x)$ in this region. Equations (3) and (4) are consequently integrated in the general form. However, on the basis of the remarks made above, we will use a simplified form of the solution in which the value $\eta(x)$ is neglected, and $w_r(x)$, $w_i(x, x')$, and $n_0(x, E, T)$ are taken into account only in the boundary condition (8).

$$S(x) = \text{const} \equiv -\frac{N_i\delta}{\tau(1)}\sigma(E) f_0(1),$$

$$f_0(x) = f_0(1)\exp\left\{\left(\frac{E_i}{E}\right)^2 \int_x^1 \frac{x'\delta(x')}{\lambda^2(x')\delta}dx'\right\}$$

$$\times\left\{1 + \left(\frac{E_i}{E}\right)^2 \sigma(E)\int_x^1 \exp\left[-\left(\frac{E_i}{E}\right)^2 \int_{x'}^1 \frac{t\delta(t)}{\lambda^2(t)\delta}dt\right]\frac{dx'}{x'\lambda(x')}\right\}. \quad (10)$$

The constants of integration $f_0(1)$ and $\sigma(E)$ must be determined from the boundary conditions. The value of $\sigma(E) \equiv -S(1)\tau(1)/N_i\delta f_0(1)$, giving the mean probability of impact ionization, is simply determined, as will be shown below, by solving the equation in the region $x > 1$. Knowing this value is sufficient to determine all the characteristics of the semiconductor in the strong electric field. In fact, we show below that in the region $x > 1$ the distribution function falls practically to zero for $x - 1 \gtrsim (\delta/p)^{1/4}\sqrt{E/E_i} = \alpha$. Consequently, the contribution of this region to all the observed quantities (number of carriers, conductivity, mean energy) is of the order $\alpha \ll 1$. In other words, all these quantities can be evaluated using the function (10) by averaging over the region $x < 1$. The single quantity completely determined by the distribution of carriers over the ionization potential is the mean probability of impact ionization. But it is equal at the same time to $-S(1) = N_i\delta\sigma(E)f_0(1)/\tau(1)$, which is easy to see by integrating (4) from 1 to ∞, and omitting the last three terms, which are insignificantly small in this region. Thus, all the information which we must obtain from the solution of the kinetic equation in the region $x > 1$ is included in the function $\sigma(E)$, proportional to the ratio $S(x)/f_0(x)$ at $x = 1$.

The total number of carriers n and the mean impact ionization and recombination probabilities are determined from the following relationships

$$n = N_i\int_0^\infty f_0(x)\,x^{1/2}dx = f_0(1)\,N_i\exp\left[\left(\frac{uE_i}{E}\right)^2\right]\left\{\varphi_{1/2}^{(1)}\left(\frac{E}{E_i}\right) + \left(\frac{E_i}{E}\right)^2\sigma(E)\exp\left[-\left(\frac{uE_i}{E}\right)^2\right]\varphi_{1/2}^{(2)}\left(\frac{E}{E_i}\right)\right\}, \tag{11}$$

$$\overline{w_i(E)} = \frac{S(1)}{n} = \frac{\delta}{\tau(1)}\sigma(E) \times \frac{\exp[-(uE_i/E)^2]}{\varphi_{1/2}^{(1)}(E/E_i) + (E_i/E)^2\sigma(E)\exp[-(uE_i/E)^2]\,\varphi_{1/2}^{(2)}(E/E_i)}, \tag{12}$$

$$\overline{w_r(E)} = \frac{N_i}{n}w_r^0 f_0(0) = w_r^0\frac{1 + (E_i/E)^2\sigma(E)\exp[-(uE_i/E)^2]\,\zeta(E/E_i)}{\varphi_{1/2}^{(1)}(E/E_i) + (E_i/E)^2\sigma(E)\exp[-(uE_i/E)^2]\,\varphi_{1/2}^{(2)}(E/E_i)}, \tag{13}$$

$$u^2 = \int_0^1 \frac{x\delta(x)}{\lambda^2(x)\,\delta}dx,$$

$$\zeta(z) = \int_0^1 \exp\left[\frac{1}{z^2}\int_0^x \frac{x'\delta(x')}{\lambda^2(x')\,\delta}dx'\right]\frac{dx}{x\lambda(x)}, \tag{14}$$

$$\varphi_k^{(1)}(z) = \int_0^1 \exp\left[-z^{-2}\int_0^x \frac{x'\delta(x')}{\lambda^2(x')\,\delta}dx'\right]x^k dx,$$

$$\varphi_k^{(2)}(z) = \int_0^1 \frac{dx}{x\lambda(x)}\int_0^x \exp\left[-z^{-2}\int_x^{x'} \frac{x''\delta(x'')}{\lambda^2(x'')\,\delta}dx''\right]x'^k dx'. \tag{15}$$

The integral $\zeta(z)$ in general diverges at the lower limit. This is associated with the fact that we have considered in (8) the number of recombinations as proportional to $f_0(0)$. In fact, as follows from the reasons given, this number is determined by the mean value of $f_0(E)$ in the region of very small energies ($\epsilon \lesssim \hbar\omega_m$). Consideration of this fact should lead to the exclusion of the integral for $x \sim \hbar\omega_m/\epsilon_i$. The exact value of this limit for calculating $\zeta(z)$ has no significance, since the divergence of the integral is logarithmic.

Using the expressions (11) – (13) and the evident relationships

$$-S(1) = N_i\int_1^\infty w_i(x)\,f_0(x)\,x^{1/2}dx,$$

$$N_i\int_0^\infty w_i^{(1)}(x, x')\,x^{1/2}\,dx = 2N_i\int_0^\infty w_i^{(2)}(x, x')\,x^{1/2}\,dx = 2w_i(x'), \tag{16}$$

the boundary condition (8) can be rewritten in a form completely analogous to relation (1) (index e refers to electrons and h to holes),

$$[\overline{w_{re}(E)} - \overline{w_{ie}(E)}]\,n_e(E) - \overline{w_{ih}(E)}\,n_h(E) - n_{0e}(E.T) = 0,$$

$$[\overline{w_{rh}(E)} - \overline{w_{ih}(E)}]\,n_h(E) - \overline{w_{ie}(E)}\,n_e(E) - n_{0h}(E,T) = 0. \tag{17}$$

If w_r^0 can be taken as independent of n, then (see reference 4)

$$n_e(E) = \frac{n_{0e}(E,T)}{\overline{w_{re}(E)}}\,\frac{1 + (n_{0h}(E,T)/n_{0e}(E,T) - 1)\,r_h(E)}{1 - r_e(E) - r_h(E)},$$

where

$$r(E) = \frac{\overline{w_i(E)}}{\overline{w_r(E)}}. \tag{18}$$

We study the behavior of this expression on increasing the field. In the region $E \ll E_i$, we have $r(E) \ll 1$ and (18) takes the usual form $n(E) = n_0/\overline{w_r(E)}$. Thus, the growth of n is determined mainly by the function $\varphi_{1/2}^{(1)}(E/E_i)$ [we notice that $\sigma(E)$ becomes of the order of unity only when E is comparable with E_i]. In particular, for the cases mentioned above, involving acoustical and optical phonons in valence crystals, n is proportional to $E^{3/2}$ and E^3 respectively. This behavior is occasioned by the fact that when the field becomes sufficiently large ($E \gtrsim 10^3$ v/cm), the mean electron energy starts to increase, the relative number of slower electrons decreases, and there is an associated decrease in the recombination velocity. When the field approximates to $E_i \sim 10^5$ v/cm, $\overline{w_i(E)}$ begins to increase rapidly and at a field E_c, determined by the condition

$$r_e(E_c) + r_h(E_c) = 1, \tag{19}$$

breakdown occurs. As a rule, the values of E_i are different for electrons and holes, therefore one of the quantities $r(E)$ is much greater than the other and the condition of breakdown takes the form $r(E_c) = 1$. The carrier type for which the value E_i is smaller is taken here. A direct comparison of (12) and (13) shows that $r(E_i) \sim \delta/w_r^0\tau \gg 1$ and, consequently, $E_c < E_i$. But the field dependence of all the quantities entering into these formulae is small compared with the exponential, and in the zero-order approximation we have

$$E_c = uE_i \ln^{-1/2}\left\{\frac{\sigma(E_c)\,\delta}{w_r^0\,\tau\,(1)}\left[1 + \left(\frac{E_i}{E_c}\right)^2 \sigma(E_c)\exp\left[-\left(\frac{uE_i}{E_c}\right)^2\right]\right.\right.$$
$$\left.\left.\times\,\zeta\left(\frac{E_c}{E_i}\right)\right]^{-1}\right\} \approx uE_i \ln^{-1/2}\frac{\delta}{w_r^0\tau}. \tag{20}$$

The value of w_r^0 is of the order $(\hbar\omega_m/\epsilon_i)^{3/2}/\tau_r$ where τ_r is the recombination lifetime of carriers when the field is absent. Taking this lifetime as of the order of a microsecond, and $\tau \sim 10^{-12}$ sec, we obtain the following estimate of the critical field $E_c \approx E_i u/5$ (see reference 5).

If the region in which the field acts is sufficiently small ($\lesssim 1$ cm), then the lifetime of carriers is determined not by recombination but by their departure from this region. The carrier concentration depends in this case on the distance t from the boundary of the specimen and the number of carriers n_0 flowing through this boundary in unit time[1] ($n_{0h} = 0$):

$$n_e(E, t) = \frac{n_{oe}}{v_{ed}}\exp[\varkappa_e(E)\,t]$$
$$\times\frac{\varkappa_e(E)\exp[-\varkappa_e(E)L - \varkappa_h(E)t] - \varkappa_h(E)\exp[-\varkappa_h(E)L - \varkappa_e(E)\,t]}{\varkappa_e(E)\exp[-\varkappa_e(E)\,L] - \varkappa_h(E)\exp[-\varkappa_h(E)\,L]}. \tag{21}$$

Here v_d is the drift velocity of the carriers in the field E, L is the dimension of the region, and $\kappa(E)$ is the so-called impact-ionization coefficient.

$$\varkappa(E) \equiv \frac{\overline{w_i(E)}}{v_d} = \frac{\sqrt{\;}}{l\,(1)}\frac{E_i}{E}\sigma(E)$$
$$\times\frac{\exp[-(uE_i/E)^2]}{\varphi^{(1)}(E/E_i) + (E_i/E)^2\sigma(E)\exp[-(uE_i/E)^2]\,\varphi^{(2)}(E/E_i)}, \tag{22}$$

$$\varphi^{(1)}(z) = \int_0^1 \exp\left[-z^{-2}\int_0^x \frac{x'\delta(x')}{\lambda^2(x')\,\delta}\,dx'\right]\frac{d}{dx}[x\lambda(x)]\,dx,$$

$$\varphi^{(2)}(z) = \int_0^1 \frac{dx}{x\lambda(x)}$$
$$\times\int_0^x \exp\left[-z^{-2}\int_x^{x'}\frac{x''\delta(x'')}{\lambda^2(x'')\,\delta}\,dx''\right]\frac{d}{dx'}[x'\lambda(x')]\,dx'. \tag{23}$$

For the particular cases corresponding to (7)

$$\varphi_{ac}^{(1)}(z) = \frac{\sqrt{\pi}}{2z'}\Phi(z'),\quad \varphi_{ac}^{(2)}(z) = \frac{\sqrt{\pi}}{4z'}\int_0^{z'^2} e^t\Phi(\sqrt{t})\frac{dt}{t},$$

$$z' = \frac{1}{\sqrt{2}z},\qquad \varphi_{op}^{(1)}(z) = z^2(1 - \exp[-z^{-2}]),$$

$$\varphi_{op}^{(2)}(z) = z^2\{\mathrm{Ei}(z^{-2}) + 2\ln z - C\}, \tag{24}$$

where $\Phi(z)$ is the error integral, $\mathrm{Ei}(z)$ is the exponential integral function, and C is Euler's constant ($C \approx 0.577$).

The breakdown field for the limited region is determined according to (21) by the condition

$$[\varkappa_e(E_c) - \varkappa_h(E_c)]\,L = \ln[\varkappa_h(E_c)/\varkappa_e(E_c)] \tag{25}$$

and is thus a function of the ratio $\sqrt{3\delta}\,L/l(1)$. As long as this ratio is small we have

$$E_c(L) = uE_i\ln^{-1/2}\left\{\frac{\sqrt{3\delta}L}{l\,(1)}\frac{E_i}{E_c}\frac{\sigma(E_c)}{\ln[\varkappa_h(E_c)/\varkappa_e(E_c)]}\right.$$
$$\left.\times\left[\varphi^{(1)}\left(\frac{E_c}{E_i}\right) + \left(\frac{E_i}{E_c}\right)^2\sigma(E_c)\exp\left[-\left(\frac{uE_i}{E_c}\right)^2\right]\varphi^{(2)}\left(\frac{E_c}{E_i}\right)\right]^{-1}\right\}$$
$$\approx uE_i\ln^{-1/2}\frac{\sqrt{3\delta}L}{l\,(1)}. \tag{26}$$

The temperature dependence of E_c is determined mainly by the quantity $E_i \sim \sqrt{\delta}/l(1)$. For acoustical phonons $E_c \sim \sqrt{T}$, for optical $E_c \sim \coth^{1/2}(\hbar\omega_0/kT)$.

In conclusion, we make a series of remarks on the connection of the parameter we have introduced, u, with the known breakdown criteria of Fröhlich and Hippel. Since $\lambda(1) = 1$ the integrand of u^2 in (14) can be either of the order of unity if $\lambda(x)$ does not increase with increasing energy, or increases sufficiently slowly, or much greater than unity, if the mean free path increases rapidly with growth of x. In the first case, which apparently, obtains always in valence crystals, breakdown occurs in fields of the order E_i, i.e., when the mean carrier energy becomes of the order ϵ_i. In form this condition agrees with Fröhlich's criterion,[6] although the primary idea of this criterion was somewhat different. In the second case, which has been well studied and of which ionic crystals are an example, the integrand of u^2 in (14) attains a maximum for small energies and, therefore,

$E_c \sim uE_i \gg E_i$. In ionic crystals, for energies $\epsilon \gtrsim \hbar\omega_0$, the mean free path $\lambda(x)$ is proportional to x, and consequently $u \sim [\epsilon_i/\hbar\omega_0]^{1/2}$. The breakdown field is determined by the relationship

$$E_c \sim uE_i \ln^{-1/2}\frac{\delta}{w_r^0\tau(1)} \sim \frac{\hbar\omega_0}{el(\hbar\omega_0)}\ln^{-1/2}\frac{\delta}{w_r^0\tau}. \qquad (27)$$

In other words, breakdown occurs in fields for which $eE_c l \sim \hbar\omega_0$ for carriers with energy $\epsilon \sim \hbar\omega_0$, which agrees qualitatively with Hippel's criterion.[9]

Starting from (5) and (6), it is not difficult to verify that Fröhlich's criterion is applicable to crystals in which $B(q)$ for $q \to 0$ increases less rapidly than $[q(1+2N_q)]^{-1}$, and Hippel's criterion applies in the opposite case.

We proceed now to the solution of our basic problem — finding the distribution function in the region $x > 1$ where the process of impact ionization is important. It will be convenient here to introduce new units of energy and current

$$y = \frac{x-1}{\alpha_j}, \qquad s(y) = \frac{\tau(1)}{\delta}\gamma_j S(x)/N_i, \qquad (28)$$

$$\alpha_j = (\delta E^2/pE_i^2)^{1/(2+j)}, \qquad \gamma_j = (E_i/E)^2\alpha_j, \qquad \beta_j = p\alpha_j^j. \quad (29)$$

The system (3) and (4), expressed in these variables, takes the following form:

$$\frac{ds(y)}{dy} - \frac{xk_j(x)}{\lambda(x)}y^j f_0(y) = 0, \qquad (30a)$$

$$\frac{x\lambda(x)}{1+\beta_j k_j(x)y^j}\frac{df_0(y)}{dy} + \gamma_j\frac{x^2\delta(x)}{\lambda(x)\delta}f_0(y) - s(y) = 0. \quad (30b)$$

An essential fact, on which all further discussion is based, is the smallness of α, evident directly from (29). In the most interesting region of the field $\alpha \lesssim 0.1$. The functions $f_0(x)$ and $S(x)$ are essentially different from zero only in the narrow region $x - 1 \lesssim \alpha$, outside which they fall off exponentially. In this region the coefficients of (30), in the arguments of which the substitution of y for x has not been made, are very slowly changing functions of y and with great accuracy can be taken as the first terms of corresponding series of degree $x - 1 = \alpha y$. In the zero order approximation in α, which we mainly use, all these are unity. Further, in this approximation, (30) can be solved exactly only in particular — although perhaps the most interesting — cases. We therefore now describe a general method allowing us to determine with adequate accuracy the quantity $\sigma(E)$ of direct interest to us.

Equations (30) for any values of the parameters have two linearly independent solutions; one exponentially decreasing at infinity, the other increasing. Apparently, only the first of these is physically permissible. It is determined to within an arbitrary multiplier, but the ratio of $s(y)$ to $f_0(y)$ at any point, including at $y = 0$, is strictly defined. Therefore, the requirement of a solution decreasing at infinity is equivalent to the problem of determining the value of $\sigma(E)$. We eliminate from (30) the function $f_0(y)$. Then for the current $s(y)$ we obtain a second-order equation

$$\frac{d}{dy}\left\{\frac{\lambda(x)}{xk_j(x)y^j}e^{\gamma_j F(y)}\frac{ds(y)}{dy}\right\} - \frac{1+\beta_j k_j(x)y^j}{x\lambda(x)}e^{\gamma_j F(y)}s(y) = 0,$$

$$F(y) = \int_0^y \frac{x'\delta(x')}{\lambda^2(x')\delta}[1+\beta_j k_j(x')y'^j]\,dy'. \qquad (31)$$

The quantity $\sigma(E)$ which is sought is derived from its solution in the following manner:

$$\sigma(E) = -\frac{1}{\gamma_j}\lim_{y\to 0}\left\{\frac{xk_j(x)y^j}{\lambda(x)}\frac{s(y)}{ds(y)/dy}\right\} \qquad (32)$$

and is a function of the parameters β_j and γ_j. Instead, to find this function, we invert the problem, take fixed values of σ, and seek the inverse function $\beta_j = \beta_j(\sigma, \gamma_j)$. In these circumstances (31) appears as a typical eigenvalue problem; the given boundary conditions at $y = 0$ [Eq. (32)] and at infinity $[s(y) \to 0]$ require the finding of the value of the parameter β_j, for which the equation has a nontrivial solution. Solving then the expression obtained for σ, we find the relationship of importance to us $\sigma = \sigma(\beta_j, \gamma_j) = \sigma(E)$. The eigenvalue of interest to us must, apparently, be the smallest, since, from its physical meaning, the function $f_0(y)$ cannot tend to zero anywhere except at infinity. But, from (30a) and the conditions $xk_j(x)/\lambda(x) > 0$ and $s(\infty) = 0$ it follows that $s(y)$ also has no zeros in the interval $(0, \infty)$. These properties, by virtue of the oscillation theorem, are possessed only by the eigenfunction corresponding to the lowest eigenvalue.

One of the most accurate and at the same time simplest ways of finding the lowest eigenvalue is the variational method. Equation (31) with the boundary condition (32) is equivalent to the following variational problem:[10]

$$-\beta_j = \min\frac{\int_0^\infty\left\{\frac{\lambda(x)}{xk_j(x)y^j}\left[\frac{ds(y)}{dy}\right]^2 + \frac{s^2(y)}{x\lambda(x)}\right\}e^{\gamma_j F(y)}dy - \frac{s^2(0)}{\sigma\gamma_j}}{\int_0^\infty\frac{k_j(x)}{x\lambda(x)}e^{\gamma_j F(y)}y^j s^2(y)\,dy} \qquad (33)$$

with the additional condition that only functions satisfying (32) are permissible. The eigenvalue is obtained by this method rather accurately even when the variational functions are only rough approximations to the true solution. In the problem considered, however, one can also hope to obtain a reasonable approximation for the function (although this is not a necessity), since its quantitative behavior is very simple. Directly from (31) it is apparent that $s(y)$ is a smooth function monotonically decreasing with increase of y.

Evaluation of the integrals entering into (33) and the subsequent solution requires as a rule rather cumbersome expressions. Therefore we will utilize this method only in cases when an accurate solution cannot be found.

We proceed now to the actual solution of (30) in different cases.

I. $j = 1$. As already remarked above, this case corresponds to semiconductors with not very large dielectric constants ($\mu \sim 1$). In the null approximation with respect to α, the system (30) takes the following form

$$\frac{ds(y)}{dy} - yf_0(y) = 0,$$

$$\frac{1}{1+\beta_1 y}\frac{df_0(y)}{dy} + \gamma_1 f_0(y) - s(y) = 0. \tag{34}$$

The integration of these equations is carried out in Appendix II and leads to the following results:

$$f_0(y) = \frac{\mathrm{const}}{\sqrt{z}}\left\{W_+(z^2) + \rho W_-(z^2)\right\}\exp\left\{-\frac{1}{4}\beta_1\gamma_1\left(y+\frac{1}{2\beta_1}\right)^2\right\}, \tag{35}$$

$$s(y) = -\frac{1}{\sqrt{\beta_1}}\left\{\frac{W_+(z^2)-\rho W_-(z^2)}{W_+(z^2)+\rho W_-(z^2)} - \frac{\sqrt{\beta_1}\gamma_1}{2}\right\}f_0(y), \tag{36}$$

where

$$z = \beta_1^{1/4}\left\{y + \frac{1}{2\beta_1}\left(1 + \frac{\beta_1\gamma_1^2}{4}\right)\right\}, \qquad \rho = \frac{1}{4}\beta_1^{-3/4}\left(1 - \frac{\beta_1\gamma_1^2}{4}\right),$$

$W_\pm(x) = W_{\rho^2 \pm \frac{1}{4}, \frac{1}{4}}(x)$ is the so-called Whittaker function. From (35) and (36) it follows that

$$\sigma(E) = \frac{1}{\sqrt{\beta_1}}\left\{\frac{W_+(z_0^2)-\rho W_-(z_0^2)}{W_+(z_0^2)+\rho W_-(z_0^2)} - \frac{\sqrt{\beta_1}\gamma_1}{2}\right\},$$

$$\text{where } z_0 = 2\rho\,\frac{4+\beta_1\gamma_1^2}{4-\beta_1\gamma_1^2}. \tag{37}$$

The various limiting cases of these formulae are also treated in Appendix II. Here we only remark that the quantity $\beta_1\gamma_1^2 = \delta(E_i/E)^2$ for the fields of interest to us is small, which allows formulae (35) — (37) to be greatly simplified.

II. $j = 2$. The case of large values of the dielectric permittivity. In Appendix I it is shown that μ can be considered as large if the condition

$$\mu \gg e^2/\hbar\sqrt{2\alpha\varepsilon_i/m}, \tag{38}$$

is satisfied, where e is the electronic charge. By inspection this criterion can be interpreted as follows; the quantity μ is considered large in crystals in which the binding energy of Coulombic levels $me^4/\mu^2\hbar^2$ is smaller than the width of the region $\alpha\epsilon_i$ where impact ionization takes place. In the semiconductors of most interest — germanium and silicon — the binding energy of the Coulomb levels is about 10^{-2} ev, and the quantity $\alpha\epsilon_i \sim 0.1$ ev. It is natural, therefore, to suppose that they belong to the case $j = 2$. The intermediate case when

$$w_i(x) = \frac{p}{\tau(1)}\,(x-1) + c_1(x-1)^2] \text{ and } \alpha c_1 \sim 1,$$

also leads to the case $j = 2$ by the transformation $y' = y + 1/2\alpha c_1$ and, therefore, will not detain us.

Putting $j = 2$ in the original system (30) gives

$$\frac{ds(y)}{dy} - y^2 f_0(y) = 0, \qquad \frac{1}{1+\beta_2 y^2}\frac{df_0(y)}{dy} + \gamma_2 f_0(y) - s(y) = 0. \tag{39}$$

It is not possible to find the general solution of these equations for arbitrary values of the parameters. We proceed, therefore, in the following manner: we divide the integral of possible values of the field into two partially overlapping regions, in one of which $\beta_2 \ll 1$, and in the other $\gamma_2 \ll 1$. In the first region, which is apparently the one of greater interest, we find an analytical solution of (39), and in the second utilize the general method described above for determining the function $\sigma(E)$.

We will start with a proof that these regions do, in fact, overlap, and thus the combination of the solutions we obtain contains the solution of the problem for any values of the parameters E and p. To do this, we remark that the product $\beta_2^{3/2}\gamma_2 \sim \sqrt{p\delta^2} \ll 1$, and therefore for all values of E one of the quantities β_2 and γ_2 must be small. The regions overlap when $\beta_2 \sim \gamma_2 \sim (p\delta^2)^{1/5} < 1$, although the margin in the latter inequality is not very large. The considerations given have an obvious physical significance. The energy relaxation time due to collisions with phonons is much greater than the momentum relaxation time. In ionization collisions these times are of the same order. Therefore, if the ionization processes play an important role in establishing momentum equilibrium ($\beta_2 \gtrsim 1$), then the energy relaxation is determined only by them ($\gamma_2 \ll 1$). On the other hand, if phonons make a significant contribution to establishing energy equilibrium ($\gamma_2 \gtrsim 1$), then the momentum loss in ionization for time τ is insignificantly small ($\beta_2 \ll 1$). The region $\beta_2 \ll 1$ is of the greatest interest, since it corresponds to a field $E < E_i$, and in general this condition, as was shown above, is satisfied even by breakdown fields. We commence the discussion with this case.

a) $\beta_2 \ll 1$. Region of comparatively weak fields.

In the zeroth approximation with respect to α_2 and β_2 (the method used, taking into account the corresponding corrections, is given in Appendix III) a second-order equation for $f_0(y)$ can be obtained from (39):

$$d^2 f_0(y)/dy^2 + \gamma_2 df_0(y)/dy - y^2 f_0(y) = 0. \qquad (40)$$

By a series of elementary substitutions this can be reduced to the Whittaker equation and its solution takes the following form:

$$f_0(y) = \mathrm{const}\, y^{-1/2} \exp\{-{}^1/_2 \gamma_2 y\}\, W_{-(\gamma_2/4)^2, 1/4}(y^2). \qquad (41)$$

Using the known behavior of the Whittaker function, it is not difficult to discover the behavior of $f_0(y)$ at zero and at infinity:

$$y \gg 1: \; f_0(y) = \mathrm{const}\, y^{-(\gamma_2/4)^2 - 1/2} \exp\{-{}^1/_2 (y^2 + \gamma_2 y)\}, \qquad (42)$$

$$y \ll 1: \; f_0(y) \approx \mathrm{const}\left\{\frac{\Gamma(1/2)}{\Gamma(3/4 + \gamma_2^2/16)}\left(1 - \frac{1}{2}\gamma_2 y\right) + \frac{\Gamma(-1/2)}{\Gamma(1/4 + \gamma_2^2/16)}\, y + O(y^2)\right\}. \qquad (43)$$

With the help of the last formulae and the second of Eqs. (39) the quantity $\sigma(E)$ of interest to us can be determined:

$$\sigma(E) = -\frac{1}{\gamma_2}\left\{\frac{d}{dy} \ln f_0(y) + \gamma_2\right\}_{y=0} = \frac{2}{\gamma_2}\frac{\Gamma(3/4 + \gamma_2^2/16)}{\Gamma(1/4 + \gamma_2^2/16)} - \frac{1}{2}. \qquad (44)$$

In the limiting cases of small $(\gamma_2 \gg 1)$ and large $(\gamma_2 \ll 1)$ fields we have

$$\sigma(E) = \begin{cases} 2/\gamma_2^4 = (2p/\delta)(E/E_i)^6, & \gamma_2 \gg 1 \\ \dfrac{2}{\gamma_2}\dfrac{\Gamma(3/4)}{\Gamma(1/4)} = 2\dfrac{\Gamma(3/4)}{\Gamma(1/4)}\left(\dfrac{p}{\delta}\right)^{1/4}\left(\dfrac{E}{E_i}\right)^{3/2}, & \gamma_2 \ll 1. \end{cases} \qquad (45)$$

The distribution function (41) in the latter case $(\gamma_2 \ll 1)$ is close to that obtained by Heller.[11]

b) $\gamma_2 \ll 1$. Region of very strong fields. In the zero approximation with respect to α_2 and γ_2 we must solve the following variational problem:

$$-\beta_2 = \min \frac{\int_0^\infty \{[y^{-1} ds/dy]^2 + s^2(y)\} dy - s^2(0)/\gamma_2\sigma}{\int_0^\infty y^2 s^2(y)\, dy}$$

$$\lim_{y \to 0}\left\{\frac{1}{y^2 s(y)}\frac{ds(y)}{dy}\right\} = -\frac{1}{\gamma_2\sigma}. \qquad (46)$$

The detailed method of solution is given in Appendix III. Here we confine ourselves to a summary of the results. The connection between the quantity $g = (\gamma_2\sigma)^{-2/3}$ and β_2 is given in parametric form by the two relationships

$$\beta_2 = g\{g^2\psi_1(\nu) - \psi_2(\nu)\}, \qquad g^2 = \frac{d\psi_2(\nu)}{d\nu} \Big/ \frac{d\psi_1(\nu)}{d\nu}, \qquad (47)$$

where

$$\psi_1(\nu) = \frac{\nu \sin \pi\nu}{\pi}\frac{\Gamma^3(\nu)\Gamma(4\nu)}{\Gamma^2(2\nu)\Gamma(3\nu)},$$

$$\psi_2(\nu) = \left[\frac{\nu \sin \pi\nu}{3\pi}\Gamma^2(\nu)\right]^{2/3}\frac{\Gamma(4\nu)}{\Gamma(8\nu/3)}\frac{\Gamma(\nu/3)\Gamma^2(4\nu/3)\Gamma(7\nu/3)}{\Gamma(\nu)\Gamma^2(2\nu)\Gamma(3\nu)}. \qquad (48)$$

The corresponding function $s(y)$ takes the form

$$s(y) = z^\nu K_\nu(z), \qquad z = \left[\frac{2^{2\nu}\nu \sin \pi\nu}{3\pi\sigma}\Gamma^2(\nu)\right]^{1/2\nu} y^{3/2\nu}, \qquad (49)$$

where $K_\nu(z)$ is the Macdonald function. ν varies from $\frac{3}{4}$ to $\frac{1}{2}$. In the limiting case when $\beta_2 \to 0$, ν tends to $\frac{3}{4}$ and the solution (49) agrees with that which is obtained from (41) in the limit as $\gamma_2 \to 0$. Thus, the solutions we have obtained in fact join up in the region where the conditions $\beta_2 \ll 1$ and $\gamma_2 \ll 1$ are simultaneously satisfied. In the other limiting case when $\nu \to \frac{1}{2}$, g tends to infinity. Thus, $\beta_2 \sim g^3 \sim (\gamma_2\sigma)^{-2}$. In fact, for very large fields

$$\sigma(E) = \sqrt{\psi_1(1/2)/\delta}\, E/E_i. \qquad (50)$$

APPENDIX I

The probability of the creation of electrons with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ and a hole with momentum $\mathbf{p}_3$, due to an ionizing collision of an electron with an original momentum $\mathbf{p}_0$, can always be written in the form

$$(2\pi/\hbar)\,|M(\mathbf{p}_0;\, \mathbf{p}_1, \mathbf{p}_2)|^2\, \delta[\varepsilon_e(\mathbf{p}_0) - \varepsilon_e(\mathbf{p}_1) - \varepsilon_e(\mathbf{p}_2) - \varepsilon_h(\mathbf{p}_3)] \times \delta[(\mathbf{p}_0 - \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3)/\hbar].$$

Hence, the following expression is obtained for the total ionization probability:

$$w_i(\varepsilon_0) = \frac{2\pi}{\hbar}V^3 \times \int |M(\mathbf{p}_0;\, \mathbf{p}_1, \mathbf{p}_2)|^2\, \delta[\varepsilon_0 - \varepsilon_e(\mathbf{p}_1) - \varepsilon_e(\mathbf{p}_2) - \varepsilon_h(\mathbf{p}_3)] \times \delta\left(\frac{\mathbf{p}_0 - \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3}{\hbar}\right)\frac{d^3p_1 d^3p_2 d^3p_3}{(2\pi\hbar)^9}. \qquad (A.1)$$

All the conservation laws can be satisfied only for sufficiently large values of $\mathbf{p}_0$. The ionization threshold is determined by the condition

$$\varepsilon_i \equiv \varepsilon_e(\mathbf{p}_i) = \varepsilon_{min}(\mathbf{p}_i) \equiv \min\{\varepsilon_e(\mathbf{p}_1) + \varepsilon_e(\mathbf{p}_2) + \varepsilon_h(\mathbf{p}_i - \mathbf{p}_1 - \mathbf{p}_2)\}.$$

The condition of a minimum in the right-hand side of the equality means that at the threshold $\nabla\epsilon_e(\mathbf{p}_1) = \nabla\epsilon_e(\mathbf{p}_2) = \nabla\epsilon_h(\mathbf{p}_3) = \mathbf{v}$, i.e., the speeds of all the final particles are equal. Close to the threshold the argument of the energy δ-function can be developed in a power series of the departure of the momenta from their values $\mathbf{p}_m(\mathbf{p}_0)$ determined from the minimum condition written down above. The coefficients of the corresponding quadratic form, after transforming to principal axes, we will label $m_k^{*-1}(\mathbf{p}_0)$. After introducing new variables of integration according to the formulae

$$p_k - p_{km}(\mathbf{p}_0) = \sqrt{2m_k^*(\mathbf{p}_0)[\varepsilon_e(\mathbf{p}_0) - \varepsilon_{min}(\mathbf{p}_0)]}\,\pi_k$$

etc., (A.1) acquires the following form:

$$w_i(\mathbf{p}_0) = \frac{2\pi}{\hbar}\frac{m_1^*(\mathbf{p}_0)\, m_2^*(\mathbf{p}_0) m_3^*(\mathbf{p}_0)}{(2\pi)^3(2\pi\hbar)^6}[\varepsilon_e(\mathbf{p}_0) - \varepsilon_{min}(\mathbf{p}_0)]^2 \times V^3 \int |M(\mathbf{p}_0;\ \mathbf{p}_1,\ \mathbf{p}_2)|^2\, \delta\left(1 - \sum_{k=1}^{6}\pi_k^2\right) d^6\pi = \frac{1}{\hbar}\frac{m_1^*(\mathbf{p}_0)\, m_2^*(\mathbf{p}_0)\, m_3^*(\mathbf{p}_0)}{(2\pi)^2(2\pi\hbar)^6} V^3 \overline{|M(\mathbf{p}_0)|^2}[\varepsilon_e(\mathbf{p}_0) - \varepsilon_{min}(\mathbf{p}_0)]^2. \tag{A.2}$$

In the Born approximation $M(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2)$ is simply the matrix element of the corresponding interaction energy. Since the momenta which are exchanged by the particles participating in the reaction are of the order $\sqrt{m\epsilon_i}$, the collision parameters making the principal contribution to ionization at the threshold are of the order $\hbar/\sqrt{m\epsilon_i}$. In this region the interaction potential must be of Coulomb order e^2/r, since the polarization of the medium only occurs at large distances. Therefore

$$M \sim \frac{e^2}{\hbar}\sqrt{m\varepsilon_i}\frac{\hbar^3}{V(m\varepsilon_i)^{3/2}} = \frac{1}{V}\frac{e^2\hbar^2}{m\varepsilon_i},$$

so that

$$w_i(\mathbf{p}_0) \sim \frac{e^4 m}{\hbar^3} V\varepsilon_i^{-2}[(\nabla\varepsilon_e(\mathbf{p}_i) - \mathbf{v})(\mathbf{p}_0 - \mathbf{p}_i)]^2 \approx \frac{e^4 m}{\hbar^3} V\left(\frac{\varepsilon_0 - \varepsilon_i}{\varepsilon_i}\right)^2. \tag{A.3}$$

This expression is almost the same as the result obtained by Tevordt[12] from an exact analysis of a somewhat simplified model. In deriving (A.3) we have neglected the difference between the slowly varying quantities $m_k^*(\mathbf{p}_0)$ and their values at the ionization threshold, and have replaced all the m_k^* by some mean value m. Also we took the speed of the final particles v as small compared with the speed of the primary $\nabla\epsilon_e(\mathbf{p}_0)$.

When the Born approximation is inapplicable, the ionization cross-section for slow electrons differs from the Born multiplier $|\psi(0, 0)|^2$, where $\psi(\mathbf{r}_1, \mathbf{r}_2)$ is the wave function of the final state describing the motion of the two electrons relative to the hole.[13] When a long-range Coulomb interaction is present, this multiplier tends to infinity as $(\epsilon - \epsilon_i)^{-1}$.[14*] The evaluation of the matrix element given above is then correct only in the region $\epsilon - \epsilon_i \gtrsim e^4 m/\mu^2\hbar^2$ (criterion for applicability of the Born approximation). The dielectric permitivity μ enters into this criterion because the growth of $|\psi(0, 0)|^2$ is determined by the long-range part of the Coulomb interaction. In the region of small energies

$$w_i(\mathbf{p}_0) \sim \frac{e^4 m}{\hbar^3} V\left(\frac{\varepsilon_0 - \varepsilon_i}{\varepsilon_i}\right)^2 \frac{e^4 m}{\mu^2\hbar^2(\varepsilon_0 - \varepsilon_i)} = \frac{\varepsilon_i}{\hbar}\left(\frac{e^4 m}{\mu\hbar^2\varepsilon_i}\right)^2 V\frac{\varepsilon_0 - \varepsilon_i}{|\varepsilon_i}. \tag{A.4}$$

The energies of interest to us are of the order $\epsilon_i + \alpha\epsilon_i$. Consequently, if μ is large enough so that $\mu^2\alpha\epsilon_i\hbar^2/e^4 m \gg 1$ the Born approximation is applicable for them and Formula (A.3) can be used. In cases where $\mu \sim 1$, the situation is completely analogous to that which exists in gases and the ionization cross section close to the threshold increases linearly.

*The results of Geltman[14] cannot be considered as strictly proven, since one of the terms entering into the interaction of the final particles was considered as a small perturbation. More convincing from our point of view is the fact that the experimentally measured ionization cross section in gases close to the threshold depends linearly on energy.

APPENDIX II

We transform the system (34) by introducing new variables according to the formulae

$$z = \beta_1^{1/4}\left[y + \frac{1}{2\beta_1}\left(1 + \frac{\beta_1\gamma_1^2}{4}\right)\right], \quad \rho = \frac{1}{4}\beta_1^{-3/4}\left(1 - \frac{\beta_1\gamma_1^2}{4}\right),$$

$$\chi_1(z) = \left\{\left(1 - \frac{\sqrt{\beta_1}\gamma_1}{2}\right) f_0(y) + \sqrt{\beta_1}s(y)\right\} \times \exp\left[\frac{1}{4}\beta_1\gamma_1\left(y + \frac{1}{\beta_1}\right)^2\right];$$

$$\chi_2(z) = \left\{\left(1 + \frac{\sqrt{\beta_1}\gamma_1}{2}\right) f_0(y) - \sqrt{\beta_1}s(y)\right\} \times \exp\left[\frac{1}{4}\beta_1\gamma_1\left(y + \frac{1}{\beta_1}\right)^2\right]. \tag{A.5}$$

The functions $\chi_{1,2}(z)$ then satisfy the following equations:

$$d^2\chi_{1,2}(z)/dz^2 + (4\rho^2 \mp 1 - z^2)\chi_{1,2}(z) = 0, \tag{A.6}$$

which easily lead to the Whittaker equation. The corresponding solutions are given by (35) — (37). Here we retain in the analysis some limiting cases. It was remarked above that in the region of fields of interest to us the quantity $\beta_1\gamma_1^2 = \delta(E_i/E)^2$ is small compared with unity. Therefore we will retain only terms of the order $\sqrt{\beta_1}\,\gamma_1$, and will neglect terms of the order $\beta_1\gamma_1^2$.

a) $\rho \ll 1$. Region of large fields. In this case the quantity $z_0 = 2\rho(4 + \beta_1\gamma_1^2)/(4 - \beta_1\gamma_1^2)$ is also small. In the zero approximation in ρ^2 the solution has the following form:

$$f_0(y) \approx z^{-1/2}\exp\left\{-\frac{1}{4}\beta_1\gamma_1\left(y + \frac{1}{\beta_1}\right)^2\right\} W_{1/4,\,1/4}(z^2) \approx \exp\left\{-\frac{\sqrt{\beta_1}}{2}\left(y + \frac{1}{2\beta_1}\right)^2 - \frac{1}{4}\beta_1\gamma_1\left(y + \frac{1}{\beta_1}\right)^2\right\}, \tag{A.7}$$

$$s(y) = -\{1 - \sqrt{\beta_1}\gamma_1/2\} f_0(y)/\sqrt{\beta_1}, \quad \sigma(E) = 1/\sqrt{\beta_1}\gamma_1 - {}^1/_2. \tag{A.8}$$

b) $\rho \gg 1$. Region of relatively small fields. For this case it is most convenient to start directly from (A.6). We transform afresh to the independent variable $y = (4\rho)^{1/3}(z - 2\rho)$ and neglect the small quantity $\rho^{-4/3}y^2$. The functions $\chi_{1,2}(y)$ then satisfy the equation

$$d^2\chi_{1,2}(y)/dy^2 - (y \pm \sqrt{\beta_1})\chi_{1,2}(y) = 0. \tag{A.9}$$

Consequently,

$$\chi_{1,2}(y) = (y \pm \sqrt{\beta_1})^{1/2} K_{1/3}[{}^2/_3(y \pm \sqrt{\beta_1})^{3/2}], \tag{A.10}$$

$$f_0(y) \approx \sqrt{y} K_{1/3}({}^2/_3 y^{3/2}) \exp[-{}^1/_4 \beta_1\gamma_1 (y + 1/\beta_1)^2] + O(\beta_1),$$

$$s(y) = -[\sqrt{y} K_{2/3}({}^2/_3 y^{3/2}) / K_{1/3}({}^2/_3 y^{3/2}) - \gamma_1/2] f_0(y); \tag{A.11}$$

$$\sigma(E) = -3^{1/3}\Gamma({}^2/_3)/\gamma_1\Gamma({}^1/_3) - {}^1/_2. \tag{A.12}$$

If γ_1 is not small, then

$$\sigma(E) = -{}^1/_2[K_{2/3}(\gamma_1^3/12)/K_{1/3}(\gamma_1^3/12) - 1]. \tag{A.13}$$

This solution is easily obtained also from (34) under the condition $\beta_1 \ll 1$.

APPENDIX III

1. The evaluation of the corrections to the solution of (30) for $j = 2$ which are proportional to β_2, α_2, and α_2^2, can be performed in the following way. We introduce the new independent variable

$$z = \left\{2\int_0^y y\sqrt{\frac{k_2(x)[1+\beta_2 k_2(x)y^2]}{\lambda^2(x)}}\,dy\right\}^{1/2}$$

and then by substitution

$$f_0(y) = \left(\frac{dz}{dy}\right)^{-1} \exp\left\{-\frac{1}{2}\int_0^z \left[\gamma_2 \frac{x^2\delta(x)}{\lambda(x)\delta} + \frac{d}{dy}\left(\frac{x\lambda(x)}{1+\beta_2 k_2(x)y^2}\right)\right]\frac{\lambda(x)}{xk_2(x)}\frac{dz}{dy}\,dz\right\}\varphi(z) \tag{A.14}$$

arrive at the equation in the normal form

$$d^2\varphi(z)/dz^2 - \{{}^1/_4\gamma_2^2\Psi(z) + z^2\}\varphi(z) = 0. \tag{A.15}$$

We develop $\Psi(z)$ in a power series of z: $\Psi(z) = \Psi(0) + \Psi_1(z) + \Psi_2(z^2) + \ldots$. In essence this series is an expansion of $\Psi(z)$ in powers of α_2 and β_2: $\Psi \sim \alpha_1$, Ψ_2 contains terms proportional to α_2^2 and β_2, etc. By the transformation

$$z' = \left(1 + \frac{1}{4}\gamma_2^2\Psi_2\right)^{1/4}\left(z + \frac{1}{2}\gamma_2^2\frac{\Psi_1}{4+\gamma_2^2\Psi_2}\right)$$

Equation (A.15) leads to the previous form. We limit ourselves here to this preliminary treatment and shall not proceed to explicit expressions for $\Psi(z)$, Ψ_0, Ψ_1 and Ψ_2 in view of their cumbersomeness.

2. As the variational function for the problem formulated in Eq. (46), we will choose the function $s(y) = z^\nu K_\nu(z)$, where $z = (\xi y)^{3/2\nu}$. This function has the correct behavior at zero, $s(y) - s(0) \sim y^3$, and decreases monotonically with increase of y, i.e., it satisfies the basic qualitative requirements for $s(y)$. Also in the limiting cases of small ($\nu \to \frac{3}{4}$) and large ($\nu \to \frac{1}{2}$) values of β_2, it gives an accurate solution of (39) for $\gamma_2 = 0$. Of the two parameters ξ and ν, only one is disposable by virtue of the additional condition (46). We will take ν as the independent variable. The parameter ξ is expressed in terms of ν and σ in the following way:

$$\xi^3 = (2^{2\nu}/3\pi\sigma)\nu\Gamma^2(\nu)\sin\pi\nu. \tag{A.16}$$

The evaluation of all the integrals entering into (46) is most conveniently carried out using the known integral forms of the Macdonald functions

$$K_\nu(z) = \frac{1}{2}z^\nu\int_0^\infty t^{-\nu-1}\exp\left\{-\frac{t}{2} - \frac{z^2}{2t}\right\}dt,$$

$$\int_0^\infty z^\mu K_\nu(z)K_{\nu'}(z)\,dz = \frac{2^{\mu-2}}{\Gamma(\mu+1)}\Gamma\left(\frac{\mu+\nu+\nu'+1}{2}\right)\Gamma\left(\frac{\mu-\nu+\nu'+1}{2}\right) \times \Gamma\left(\frac{\mu+\nu-\nu'+1}{2}\right)\Gamma\left(\frac{\mu-\nu-\nu'+1}{2}\right). \tag{A.17}$$

As a result, (46) is reduced to

$$-\beta_2 = g\min\{-g^2\psi_1(\nu) + \psi_2(\nu)\}, \tag{A.18}$$

where the functions $\psi_1(\nu)$ and $\psi_2(\nu)$ are determined by (48). The condition for the minimum of this expression leads to the connection (47) between β_2 and g.

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