THE METHOD OF DISPERSION RELATIONS AND PERTURBATION THEORY

N. N. BOGOLYUBOV, A. A. LOGUNOV, and D. V. SHIRKOV

Joint Institute for Nuclear Research

Submitted to JETP editor April 17, 1959

J. Exptl. Theoret. Phys. (U.S.S.R.) **37**, 805-815 (September, 1959)

Both the idea and the results of the present investigation are related to Redmond's recent paper[1] on the exclusion of nonphysical poles from the Green's function. In contrast to that work based on the relation between the spectral representations of the Green's function and of the polarization operator we base ourselves on the principle of summing the information obtained from perturbation theory in the integrand of the Källen-Lehmann spectral integral. On summing in this way the contributions from the "principal logarithmic diagrams" we obtain expressions for the photon propagation function in quantum electrodynamics and for the meson propagation function in the symmetric theory which have all the essential properties of Redmond's result: the correct analytic behavior in the complex plane of the momentum variable p^2 and a singularity with respect to the variable of the square of the charge e^2 at the point $e^2 = 0$. However, in contrast to Redmond's result, which yields correctly only the lowest order of perturbation theory, the expressions obtained by us correspond to terms of arbitrarily high order in the perturbation theory expansions in the region of large p^2.

By taking into account the lowest order logarithmic terms, it is shown that the region of applicability of the new formulas coincides with the region of applicability of the old formulas containing the logarithmic singularities, since it is restricted by the condition of smallness of the invariant charge. The technique of reducing the expressions so obtained to the renormalization-invariant form is illustrated by the example of the photon Green's function. In conclusion some remarks are made with respect to the possible situation in nonrenormalizable theories.

1. INTRODUCTION

In Redmond's recent paper[1] an interesting result was obtained, that by requiring analyticity it is possible to obtain expressions for the Green's function which correspond to perturbation theory and at the same time do not contain the well-known logarithmic singularities.

Redmond's method consists of the following. By postulating, on the basis of considerations of correspondence with perturbation theory, a spectral representation for the polarization operator, Redmond writes the meson Green's function in the form

$$\Delta(p^2) = (\mu^2 - p^2 - i\varepsilon)^{-1}\left[1 + (\mu^2 - p^2)\int_{(3\mu)^2}^{\infty} \frac{\rho(m^2)dm^2}{m^2 - p^2 - i\varepsilon}\right]^{-1}. \tag{1.1}$$

By comparing this representation with the well known Källen-Lehmann representation

$$\Delta(p^2) = \frac{1}{\mu^2 - p^2} + \int_{(3\mu)^2}^{\infty} dm^2 \frac{I(m^2)}{m^2 - p^2 - i\varepsilon} \tag{1.2}$$

and by utilizing the obvious relation

$$\pi I(m^2) = \operatorname{Im}\Delta(m^2 + i\varepsilon), \tag{1.3}$$

Redmond expresses the spectral function I in terms of ρ and obtains

$$\Delta(p^2) = \frac{1}{\mu^2 - p^2} + \int_{(3\mu)^2}^{\infty} \frac{\rho(m^2)\,dm^2}{D(m^2)(m^2 - p^2 - i\varepsilon)}, \tag{1.4}$$

where

$$D(m^2) = R^2(m^2) + \pi^2(m^2 - \mu^2)^2\rho^2(m^2),$$

$$R(m^2) = 1 - (m^2 - \mu^2)\,\mathrm{P}\int_{(3\mu)^2}^{\infty} \frac{\rho(l^2)\,dl^2}{m^2 - l^2}.$$

Formula (1.3) corresponds to the expression for the boson Green's function obtained by Lehmann, Symanzik, and Zimmerman[2] if a subsidiary condition is imposed on the function ρ

$$\int_{(3\mu)^2}^{\infty} \rho(m^2)dm^2 \leqslant 1. \tag{1.5}$$

Then by utilizing for ρ the expression $\rho_0(m^2)$

which corresponds to the lowest order of perturbation theory he obtains for the meson Green's function the following expression

$$\Delta(p^2) = (\mu^2 - p^2)^{-1}\left[1 - (\mu^2 - p^2)\int_{4M^2}^{\infty}\frac{dm^2\rho_0(m^2)}{m^2 - p^2}\right]^{-1} - \frac{Z^{-1}}{p_0^2 - p^2}, \tag{1.6}$$

where p_0^2 is the value of the variable p^2 for which the first term in (1.6) has a pole, while Z^{-1} is the residue at the pole.

Expression (1.6) has the following interesting properties: 1) it does not have a nonphysical pole, since in the neighborhood of the pole the second term exactly compensates the singularity of the first term; 2) considered as a function of g^2, it has an essential singularity at $g^2 = 0$; 3) in the neighborhood of the point $g^2 = 0$ it admits an asymptotic expansion in powers of g^2 whose first term (of order g^2) coincides with the result of perturbation theory.

It should be noted that formula (1.6) does not lead to any correspondence with higher orders of perturbation theory, in particular, it gives incorrect principal logarithmic terms. This is not surprising since, as is well known,[3,4] as a result of summing the principal logarithmic terms for the meson Green's function the following expression is obtained

$$\Delta(p^2) = \frac{1}{(\mu^2 - p^2)\left[1 - (5g^2/4\pi)\ln(-p^2/\mu^2)\right]^{1/5}}, \tag{1.7}$$

which contains not a "false pole," but a nonphysical singularity of fractional power. It may be easily seen that Redmond's method based on the polarization operator is very inconvenient for the investigation of singularities of type (1.7). This is connected with the fact that a singularity of type (1.7) does not correspond to any simple approximation for the polarization operator.

In the following we present a somewhat different approach to the problem of the removal of nonphysical singularities from the approximate expressions for the Green's function in quantum field theory which, it seems to us, has a greater degree of generality. We base ourselves on the principle of the summation of the perturbation theory series in the integrand of the Källen-Lehmann spectral integral. In this way, by summing the principal logarithmic diagrams, we shall obtain expressions for the photon and the meson Green's functions which, on the one hand, will possess the required analytic properties, and, on the other, will correspond to the perturbation theory expansions in the ultra violet region. This principle of summation is very close to Symanzik's ideas[5] by means of which he studied the analytic properties of the Green's function.

2. ELIMINATION OF THE "LOGARITHMIC" POLE FROM THE PHOTON GREEN'S FUNCTION

The most natural method of studying the analytic properties of the Green's function is the method of dispersion relations. At the present time this method is the only approach to problems of quantum field theory which is apparently free of internal difficulties. Therefore, it appears to be quite natural that further progress in quantum field theory must be associated with this method.

The method of dispersion relations based on the most general principles of covariance, causality, unitarity, and spectrality* enables us to obtain expressions for quantities of the type of Green's functions and of transition-matrix elements in the form of spectral expansions. In this way the problem is reduced to the investigation of the properties of the corresponding spectral functions. These spectral functions, on being expanded in terms of a complete system of states, can be expressed in terms of Green's functions for more complicated processes. In this way a possibility appears in principle of obtaining a system of equations for the determination of the Green's functions. It should be noted that in contrast, for example, to the system of Schwinger's equations, no ultraviolet divergences arise in this case. However, a consistent development of such a program encounters a number of obstacles, since, for example, no spectral representations have yet been obtained for the higher Green's functions.

At this point a palliative possibility appears of obtaining the lacking information about the spectral functions with the aid of perturbation theory data. Symanzik[5] has adopted this particular approach. By considering the n-th perturbation-theory term for a certain vertex part he showed that this term can be represented in a definite spectral form. Then, by making use of the hypothesis of the possibility of summing the series for the spectral function, Symanzik concluded that the vertex part under investigation could be represented as a whole in the given spectral form.

Symanzik utilized this approach to obtain general theoretical conclusions leading to the proof of the dispersion relations. In our opinion it would be of considerable interest to investigate on the basis of the "principle of summation in the integrand of the spectral representation" the different possibilities for making approximations. As the simplest example we shall investigate by this method the Källen-Lehmann spectral formula for the boson Green's function, and instead of summing the whole pertur-

*In speaking of the principle of spectrality we have in mind the condition of the existence of a complete set of physical states of positive energy.

bation theory series we shall restrict ourselves to summing only that class of diagrams the study of which in the opinion of a number of authors[6] leads to the proof of the existence of nonphysical singularities.

In the case of the photon Green's function in quantum electrodynamics, such diagrams can be represented in the form of a photon line with an arbitrary number of simplest insertions — second order electron-positron loops. It is customary to call such diagrams "principal logarithmic diagrams." The contribution of the n-th term of diagrams of this class is of the form

$$-(e^2 F(k^2, m^2))^n / k^2, \tag{2.1}$$

where $F(k^2, m^2)$ corresponds to the second order loop. An explicit expression for F is given, for example, in Sec. 32.1 of reference 4.

In the region $|k^2| \gg m^2$ the function F has the form

$$F(k^2, m^2) = \frac{1}{3\pi} \ln \frac{4m^2 - k^2}{4m^2}. \tag{2.2}$$

We have introduced the term $4m^2$ into the argument of the logarithm in order to represent correctly the imaginary part of the function F

$$\operatorname{Im} F(k^2, m^2) = -\frac{1}{3} \theta(k^2 - 4m^2), \qquad \theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases},$$

at the same time retaining its normalization $F(0, m^2) = 0$.

We note that a direct summation of the terms of (2.2), which was for the first time carried out in reference 7, leads to the expression

$$-\frac{1}{k^2}\left[1 - \frac{e^2}{3\pi} \ln \frac{4m^2 - k^2}{4m^2}\right]^{-1}, \tag{2.3}$$

on the basis of which it was concluded[6] that a logarithmic pole exists, and that, consequently, there is an internal contradiction in the theory.

We shall now apply to the diagrams under investigation the principle of summing in the integrand of the spectral representation. It may be easily seen that the n-th term of (2.1) can be represented in the Källen-Lehmann spectral form. By restricting ourselves to the approximation (2.2) we obtain

$$D_n = -\frac{1}{k^2}\left(\frac{e^2}{3\pi} \ln \frac{4m^2 - k^2}{4m^2}\right)^n = \int_{4m^2}^{\infty} \frac{I_n(z)}{z - k^2} dz,$$

where the function $I_n(z)$ is defined by the imaginary part of the function D_n by means of (1.3). On carrying out the summation under the integral sign of the spectral representation for the function I

$$I(z) = \sum_{n=1}^{\infty} I_n(z),$$

we see that $I(z)$ as a whole is represented by the imaginary part of (2.3). On substituting (2.3) into (1.3) we obtain

$$I(z) = \begin{cases} \frac{e^2}{3\pi z}\left[\left(1 - \frac{e^2}{3\pi} \ln \frac{z - 4m^2}{4m^2}\right)^2 + \frac{e^4}{9}\right]^{-1}, & z \geqslant 4m^2 \\ 0 & z < 4m^2 \end{cases} \tag{2.4}$$

We thus obtain for the photon Green's function the following expression

$$D(k^2) = -\frac{1}{k^2} + \frac{e^2}{3\pi} \int_{4m^2}^{\infty} \frac{dz}{z(z - k^2 - i\varepsilon)\left[\left(1 - \frac{e^2}{3\pi} \ln \frac{z - 4m^2}{4m^2}\right)^2 + \frac{e^4}{9}\right]}. \tag{2.5}$$

We note that a formula of this type was recently investigated by Redmond and Uretsky.[10]

It is easily seen* that (2.5) represents the function

$$D(k^2) = -\frac{1}{k^2}\left(1 - \frac{e^2}{3\pi} \ln \frac{4m^2 - k^2}{4m^2}\right)^{-1} - \frac{3\pi / e^2}{[1 - 4\exp(-3\pi / e^2)][k^2 - 4m^2 + 4m^2 \exp(3\pi / e^2)]},$$

which, on taking into account the fact that e^2 is small, can be written for the case of $|k^2|$ much greater than m^2 in the form

$$D(k^2) = -\frac{1}{k^2[1 - (e^2 / 3\pi) \ln(-k^2 / 4m^2)]} - \frac{3\pi / e^2}{k^2 + 4m^2 \exp(3\pi / e^2)}. \tag{2.6}$$

The function (2.6) has the following remarkable properties: 1) it does not have a logarithmic pole; 2) in the neighborhood of the point $e^2 = 0$ it has, regarded as a function of e^2, a singularity of "superconducting" type $\exp(-3\pi/e^2)$; 3) in the neighborhood of the point $e^2 = 0$ it has an asymptotic expansion that coincides with the ordinary perturbation-theory expansion and can be represented in the form (2.3). It is also clear that the second term in the right-hand side of (2.6) cannot be obtained as a matter of principle from perturbation theory[8] because of its exponential order of smallness, as a result of which it does not correspond to any Feynman diagrams.

We also note that the final expression (2.6) is consistent with the initial approximation (2.3). In the region in which we have utilized (2.3) for the calculation of the spectral function, it differs from the final expression (2.6) by negligibly small terms of order $\exp(-3\pi/e^2)$.

We see thus that the procedure of summing the principal logarithmic diagrams is not a unique operation. A direct summation of the logarithmic

*Cf., for example, reference 10.

terms leads to (2.3). Summation in the integrand of the Källen-Lehmann spectral integral leads to (2.6). Assuming that quantum electrodynamics has physically sensible solutions which are in agreement with the basic principles of quantum field theory, and that consequently the photon Green's function satisfies the Källen-Lehmann theorem, we must choose from these two possibilities the formula (2.6). This equation, which does not contain any paradoxes such as the "zero-charge difficulty,"[6] is thus the result of summing the main logarithmic terms carried out in agreement with the basic physical foundations of the theory.

Naturally, (2.6) is not the only expression with correct analytic behavior in the complex plane of the variable k^2, whose asymptotic expansion into a series in powers of e^2 coincides with the usual perturbation theory. Examples of expressions of this type, other than (2.6), can be obtained by adding terms containing $\exp(-e^{-2})$ to the spectral function (2.4).

We have obtained (2.4) from (2.3), which corresponds to the main logarithmic terms of perturbation theory. We also could have started not with (2.3), but with (cf. Sec. 43.1 of reference 4)

$$-1/k^2[1-e^2F(k^2,\ m^2)],$$

which describes the sum of the main logarithmic diagrams for arbitrary values of k^2 and which reduces to (2.3) in the limit $|k^2| \gg m^2$. The corresponding spectral function has a more awkward form in comparison with (2.4) but at the same time retains all its essential properties.

We also note that (2.5) and (2.6) are very close to Redmond's formulas (1.4) and (1.6). This is due to the fact that the usual expression for the photon propagation function (2.3) has a nonphysical singularity in the form of a pole, which can be well represented by an approximate expression for the polarization operator.

An interesting feature of the spectral function (2.4) is its resonance nature. As already pointed out, this formula was obtained on the basis of summing the main logarithmic terms. It is therefore of interest to find out whether the resonance character will be retained in the spectral functions obtained on the basis of summing logarithmic terms of a higher order of smallness, and to discover in what range expressions of type (2.4) can be regarded as sensible approximations.

For this purpose we take for the initial expression for the photon propagation function a formula (Sec. 43.2 of reference 4) obtained by summing terms of the form $(e^2 \ln z)^n$ and $e^2(e^2 \ln z)^m$

$$D(z) = -\frac{1}{z}\Big[1-t+\frac{ie^2}{3}\theta(z-4m^2) + \frac{3e^2}{4\pi}\ln\Big(1-t+\frac{ie^2}{3}\theta(z-4m^2)\Big)\Big]^{-1},$$

where

$$t = \frac{e^2}{3\pi}\ln\frac{z-4m^2}{4m^2}.$$

From this expression we shall obtain in place of (2.4) (for $z \geq 4m^2$):

$$I(z) = \frac{e^2}{3\pi z} \times \frac{1+\dfrac{9}{4\pi}\tan^{-1}\dfrac{e^2/3\pi}{1-t}}{\Big\{1-t+\dfrac{3e^2}{2\pi}\ln\Big[(1-t)^2+\dfrac{e^4}{9}\Big]\Big\}^2+\dfrac{e^4}{9}\Big[1+\dfrac{9}{4\pi}\tan^{-1}\dfrac{e^2/3\pi}{1-t}\Big]^2}. \tag{2.7}$$

It can be seen from (2.7) that the resonance character of the spectral function is retained when higher logarithmic terms are taken into account. However, the effect of these higher terms on the behavior of the spectral function in the resonance region and above, is not small. By comparing (2.7) with (2.4) we see, for example, that in the resonance region (2.7) differs from (2.4) by the factor 8/17, while for very large values of z it differs by the factor 13/4. These factors do not depend on the degree of smallness of the parameter e^2.

It should be emphasized that the improvement of perturbation theory obtained by summing logarithmic terms of different orders of smallness is not a consistent operation. As is well known (cf. Sec. 42.4 of reference 4), the corresponding formulas can be trusted only in the region in which the quantity

$$e^2d(k^2) = -e^2k^2D(k^2)$$

is small compared to unity.

It follows from this that the expressions obtained above for the spectral functions represent sensible approximations only in the region below "resonance." In the resonance region and above these formulas cannot be trusted, since we are making use here of the initial approximation outside the region of its applicability, actually going outside the framework of weak coupling. Therefore, we can only make assumptions about the resonance character of spectral functions.

The inapplicability of the expressions obtained above for the spectral functions in the region of very large z is not surprising, since the problem of determining the true asymptotic behavior even for the single-particle Green's function requires the simultaneous investigation of the asymptotic behavior of other higher Green's functions and vertex parts.

3. REMOVAL OF NONPHYSICAL SINGULARITIES FROM THE MESON GREEN'S FUNCTION

A similar correction of the logarithmic summation formulas may be carried out also for other Green's functions. Let us consider, for example, the meson propagation function in the symmetric pseudoscalar theory of meson-nucleon interaction. An expression for this Green's function, obtained[3,4] by improving the ordinary perturbation theory, has the form (1.7). The corresponding spectral function arising as a result of the summation in the integrand of the spectral representation will be of the form

$$I(z)=\frac{1}{\mu^2-z}\,\frac{\sin\left(\frac{4}{5}\tan^{-1}\frac{5g^2/4}{1-t}\right)}{\pi\left[(1-t)^2+25g^4/16\right]^{2/5}}, \tag{3.1}$$

where

$$t=(5g^2/4\pi)\ln(z/\mu^2).$$

We now inquire as to what constitutes the difference between the meson Green's function calculated by means of (3.1) and the initial approximation (1.7). The expression (1.7) has a branch point at $p^2=-\mu^2\exp(4\pi/5g^2)$. Therefore in the present case the elimination of the nonphysical singularity is reduced to the subtraction from (1.7) of the integral along the cut which may be made from the point $-\mu^2\exp(4\pi/5g^2)$ along the negative real axis to $-\infty$. It can be easily seen that this integral will have an order of smallness of $\exp(-4\pi/5g^2)$, as a result of which it will not correspond to any perturbation-theory terms.

4. REDUCTION OF THE PHOTON GREEN'S FUNCTION TO THE RENORMALIZATION-INVARIANT FORM

It is not difficult to see that the expressions obtained above for the Green's functions are not renormalization-invariant. In this section using the photon Green's function as an example we shall show the manner in which they may be brought to invariant form.

We start with (2.6), rewritten in the form

$$\frac{e^2d}{3\pi}=-\frac{e^2k^2D}{3\pi}=\frac{1}{3\pi/e^2-\ln(k^2/m^2)}+\frac{1}{1-\exp\left[3\pi/e^2-\ln(|k^2|/m^2)\right]}. \tag{4.1}$$

The function e^2d, called the invariant charge, must be an invariant of the transformation of the renormalization group (cf. Sec. 42 in reference 4). However, (4.1) clearly does not satisfy this requirement.

The usual technique of bringing expressions to the renormalization-invariant form utilizes the apparatus of the Lie differential equations and also makes use of considerations of correspondence with the ordinary perturbation theory. Since expressions of the type (4.1) cannot be expanded in powers of e^2, it will be technically more convenient to start not from the Lie differential equations, but from the functional equations of the renormalization group.

With this object in view we shall seek the analogue of the usual function d normalized to unity for $|k^2|=\lambda^2$ in the form

$$\frac{e_\lambda^2}{3\pi}d\left(\frac{k^2}{\lambda^2},e_\lambda^2\right)=\frac{1}{\Phi(e_\lambda^2/3\pi)-\ln(k^2/\lambda^2)}+\frac{1}{1-\exp\left[\Phi(e_\lambda^2/3\pi)-\ln(k^2/\lambda^2)\right]}, \tag{4.2}$$

being guided by the fact that (as shown originally by Gell-Mann and Low[9] on the basis of group theoretic considerations) the invariant function can depend on e_λ^2 and λ^2 only through the argument

$$\varphi(e_\lambda^2)+\ln\lambda^2.$$

The requirement that the function e^2d be invariant has the form

$$e_\lambda^2 d(k^2/\lambda^2,e_\lambda^2)=e_0^2 d(k^2/m_0^2,e_0^2)=e_{\lambda_1}^2 d(k^2/\lambda_1^2,e_{\lambda_1}^2), \tag{4.3}$$

where m_0 is a quantity of the order of magnitude of the electron mass, while e_0 is the corresponding value of the charge. This requirement determines the relationship between the laws of transformation of charge and of the normalizing momentum by means of the function Φ

$$\Phi\left(\frac{e_\lambda^2}{3\pi}\right)-\ln\frac{k^2}{\lambda^2}=\Phi\left(\frac{e_0^2}{3\pi}\right)-\ln\frac{k^2}{m_0^2}=\Phi\left(\frac{e_{\lambda_1}^2}{3\pi}\right)-\ln\frac{k^2}{\lambda_1^2},$$

from where, in particular, it follows that

$$\Phi(e_\lambda^2/3\pi)-\Phi(e_0^2/3\pi)=-\ln(\lambda^2/m_0^2). \tag{4.4}$$

A qualitative restriction on the function Φ follows from (4.4). When $\lambda^2\to\infty$ the function $\Phi(e_\lambda^2/3\pi)$ must tend monotonically to $-\infty$. The explicit form of this function may now be determined from the condition of normalization of d. By setting $k^2=\lambda^2$ in (4.2) we obtain

$$\frac{e_\lambda^2}{3\pi}=\frac{1}{\Phi(e_\lambda^2/3\pi)}+\frac{1}{1-\exp\left[\Phi(e_\lambda^2/3\pi)\right]},$$

i.e.

$$x=1/\Phi(x)-1/\left[e^{\Phi(x)}-1\right].$$

From this equation it can be seen that Φ is indeed a monotonically decreasing function of its argument. For small x, $\Phi(x) \sim 1/x$ and (4.2) goes over into (4.1). At $x = \frac{1}{2}$ the function Φ goes through zero and reverses sign. As $x \to 1$ the function Φ tends to $-\infty$.

From the condition of normalization of the function d and from (4.3) and (4.4) it follows that

$$\Phi\left(\frac{e_0^2}{3\pi} d\left(\frac{\lambda^2}{m_0^2}, e_0^2\right)\right) = \Phi\left(\frac{e_0^2}{3\pi}\right) - \ln\frac{\lambda^2}{m_0^2},$$

from which we obtain

$$\lim_{\lambda\to\infty} \frac{e_0^2}{3\pi} d\left(\frac{\lambda^2}{m_0^2}, e_0^2\right) = 1. \qquad (4.5)$$

If now, following the generally accepted rules (cf., for example, reference 9), we determine from (4.5) the renormalization constant Z_3 for the photon Green's function, we obtain for it the finite value

$$Z_3 = e_0^2/3\pi. \qquad (4.6)$$

Naturally no final interpretation should be given to (4.6).

We have carried through the above arguments in order to show how expressions can be brought to the renormalization-invariant form without utiliz- the Lie equations.

5. ON ONE POSSIBILITY IN NONRENORMALIZABLE THEORIES

A model expression for the Green's function in a nonrenormalizable theory was given by Redmond and Uretsky.[10] We now attempt to obtain an expression of this type by making use of the method developed earlier of summing in the integrand of a spectral representation. We shall carry through the argument using as an example the nonlinear fermion theory with a fourth order interaction Lagrangian $g\bar{\psi}O\psi\bar{\psi}O\psi$, and we shall consider the 4-fermion vertex Green's function $\Gamma(p', q', p, q)$.

By basing ourselves on renormalization-invariant considerations, we can assume that the symmetric asymptotic expression in the ultraviolet region has for its principal approximation the form*

$$\Gamma(p, p, p, p) = \Gamma(p) = 1/[1 + gF(p)], \qquad (5.1)$$

where $F(p)$ is the contribution from the simplest fermion-antifermion loop. Perturbation theory calculations for $\Gamma(p)$ lead to the expansion

$$\Gamma(p) = 1 + \sum_n g^n \Gamma_n(p), \qquad (5.2)$$

*The equations of the renormalization group for this case are given in reference 11.

where, for example, for $|p^2| \gg m^2$ an expression of the type

$$\Gamma_1 = -p^2 \ln(p^2/m^2) + p^2 I_1 + I_0. \qquad (5.3)$$

holds for Γ_1.

Here I_0 and I_1 are divergent constants. In the general case the term Γ_n contains a polynomial in p^2 of degree n with divergent coefficients. By means of the usual subtraction methods it is possible to eliminate from Γ only the divergent term which does not depend on p^2 (of the type I_0) since the introduction of counter terms proportional to powers of p^2 leads in the final analysis to nonlocal effects (cf., for example, Sec. 28.3 in reference 4).

Therefore the formal subtraction of divergent terms of the type I_0 and $p^2 I_1$, leading to (5.1) when $F(p) = p^2 \ln(p^2/m^2)$, is not quite consistent.

Here we can make a hypothesis that Γ regarded as a function of g has an essential singularity* at $g = 0$ in the neighborhood of which there exists no asymptotic expansion for it in powers of g. An attempt to expand formally in powers of the coupling constant leads to divergent expressions in each order [the terms $I_0 + p^2 I_1$ in (5.3)].

On the basis of this hypothesis, and taking into account the fact that the finite terms in (5.2) have the form $[gp^2 \ln(p^2/m^2)]^n$, we can attempt to sum the series (5.2) with the aid of the following spectral representation

$$\Gamma(p) = 1 + p^2 \int_{m_0^2}^{\infty} \frac{\rho(z)\, dz}{z(z - p^2)}. \qquad (5.4)$$

At the present time we do not have for the 4-vertex function an analogue of the Källen-Lehmann spectral formula. Therefore we actually have to postulate the representation (5.4). However, it appears to be a fairly natural one, since, for example, it corresponds to the separate terms given by perturbation theory.

Naturally, the representation of the individual terms of the sum (5.2) in the spectral form (5.4) is purely formal, in view of the divergences contained in them. However, in accordance with the assumption which we have made, the complete expression of the form (5.4) for Γ obtained by means of summing the series

$$\rho(z) = \sum_n g^n \rho_n(z) \qquad (5.5)$$

for the spectral function ρ may turn out to be finite. The individual terms ρ_n in the sum (5.5) are

*In this connection see references 12 and 13.

determined by the imaginary components of the logarithm. Therefore, in order to evaluate the sum (5.5) we must make use of expression (5.1) with

$$F(p^2) = p^2 \{\ln(|p^2|/m^2) - i\pi\theta(p^2 - m_0^2)\}.$$

In this way we arrive at a spectral formula of the type

$$\Gamma(p) = 1 + gp^2 \int_{m_0^2}^{\infty} \frac{dz}{(z-p^2)\,[(1+gz\ln(z/m^2))^2 + \pi^2 g^2 z^2]}, \quad (5.6)$$

which is quite close to the model expression of reference 10. This formula has the following important property. The integral in the right hand side of (5.6) is convergent; however, when we attempt to expand it into a power series in the coupling constant, we obtain divergent expressions in each order. Thus, (5.6) is in complete agreement with our assumptions.

With the aid of Cauchy's theorem for the function $\Gamma(p)/p^2$, we can rewrite (5.6) in the form

$$\Gamma(p) = \frac{1}{1 + gp^2\ln(p^2/m^2)} - \frac{p^2}{gp_0^2(p^2 - p_0^2)\,[1 + \ln(-p_0^2/m^2)]}, \quad (5.7)$$

where p_0^2 is a root of

$$1 + gp_0^2\ln(-p_0^2/m^2) = 0.$$

In the limit of small $g > 0$, the root p_0^2 tends to $-1/g$, and the second term in (5.7) takes on the form

$$gp^2/(1 + gp^2)(1 - \ln gm^2). \quad (5.8)$$

The function $f(x) = [1 - \ln x]^{-1}$ has the following properties:

$$f(0) = 0,\ f'(0) = f''(0) = \ldots = f^{(n)}(0) = \ldots = \infty. \quad (5.9)$$

Therefore since the term (5.8) is small for small gm^2, it does not give an asymptotic expansion in powers of g. An attempt at a formal expansion leads in accordance with (5.9) to a series with diverging coefficients.

The following circumstance is curious. The second term in (5.7) turns out to be (for small g) quite small everywhere for p^2 not too close to p_0^2. In this region the function Γ is practically equal to the first term in (5.7). This fact may serve as the justification for the formal subtraction of a polynomial in powers of p^2 with divergent coefficients from the expansion (5.2) with a subsequent summation of the finite terms with the aid of (5.1).

We emphasize again that the argument given above does not pretend to be rigorous to any degree, and represents an attempt of describing one of the presently possible variants of the situation in nonrenormalizable theories.

6. CONCLUSION

Let us now take stock of the situation. As has been just demonstrated, even a very preliminary attempt at a synthesis of the method of dispersion relations and of perturbation theory allows us to obtain expressions for the Green's functions which do not contain nonphysical singularities. It should, of course, be emphasized that the range of applicability of the new formulas does not differ from the range of applicability of the old formulas, being limited by considerations of going outside the framework of weak coupling, and that in this region the new formulas practically do not differ from the old ones.

Therefore, the procedure of eliminating nonphysical singularities does not in itself provide an actual method of going outside the framework of the generally adopted approximations. However, a deeper synthesis of the approximation methods and of the dispersion relations initiated by the papers of Redmond and Symanzik may be of great significance in principle.

We shall clarify this remark by using the Lee model as an example. As is well known, the exact solution of the Lee model contains the difficulty of a logarithmic pole. On the other hand, the nonrelativistic Lee model has the property of causality with respect to time and, consequently, on the basis of the principle of spectrality the one-dimensional analogue of the Källen-Lehmann theorem, with respect to the variable E, holds for this model. Since the exact expression for the Green's function does not satisfy the Källen-Lehmann theorem, this means that the initial Hamiltonian contradicts the condition of spectrality, i.e., that (as is well known) the system of eigenfunctions of this Hamiltonian includes states with negative energy. Since such states are physically meaningless, this means that the initial Hamiltonian was not well chosen.

By applying the correction described above to the Lee model Green's function, we shall obtain for it an expression without the logarithmic pole, which will be equivalent to the subtraction from the Hamiltonian of terms corresponding to negative energy states, i.e., to its reduction to a physically sensible form.

In electrodynamics two hypotheses can be made with respect to the cause of the appearance of the logarithmic pole: 1) the initial Lagrangian is a nonphysical one, i.e., its complete system of eigenfunctions does not satisfy the requirement of spectrality; 2) the reason for the appearance of the logarithmic pole is contained in the inappropriate choice of the approximation method used.

From the point of view of the first possibility, which is equivalent to the situation in the Lee model, the correction described above for the photon Green's function corresponds to correcting the Lagrangian. In the second variant this procedure reduces to the automatic elimination of parasitic singularities which do not correspond to the physical content of the theory. Naturally, at present, it is not possible to choose between these two possibilities.

It can now be seen that the method of summation in the integrand of spectral representations appears as a certain new "super-subtraction" procedure which removes nonphysical singularities irrespectively of the reason for their appearance. It should be emphasized that this "second subtraction" procedure is a perfectly natural one, since it represents a mathematical formation of the requirement of the correspondence of the results to which it leads with the initial physical principles of the theory.

The authors express their gratitude to Prof. D. I. Blokhintsev, B. V. Medvedev, and M. K. Polivanov for discussion of the results.

[1] P. J. Redmond, Phys. Rev. **112**, 1404 (1958).

[2] Lehmann, Symanzik and Zimmerman, Nuovo cimento **2**, 425 (1955).

[3] Galanin, Ioffe, and Pomeranchuk, JETP **29**, 51 (1955), Soviet Phys. JETP **2**, 37 (1956).

[4] N. N. Bogolyubov and D. V. Shirkov. Введение в теорию квантованных полей (Introduction to the Theory of Quantized Fields) Moscow, Gostekhizdat, 1957 [English translation, New York, Interscience, 1959].

[5] K. Symanzik, Progr. Theoret. Phys. **20**, 690 (1958).

[6] L. D. Landau and I. Ya. Pomeranchuk, Dokl. Akad. Nauk SSSR **102**, 489 (1955); cf. also the article by L. D. Landau in the collection "Niels Bohr and the Development of Physics," London, 1955.

[7] Landau, Abrikosov, and Khalatnikov, Dokl. Akad. Nauk SSSR **95**, 773, 1177; **96**, 261 (1954).

[8] K. Ford, Phys. Rev. **105**, 320 (1957).

[9] M. Gell-Mann and F. Low, Phys. Rev. **95**, 1300 (1954).

[10] P. J. Redmond and J. L. Uretsky, Phys. Rev. Letters **1**, 147 (1958).

[11] M. É. Maĭer and D. V. Shirkov, Dokl. Akad. Nauk SSSR **122**, 45 (1958), Soviet Phys.–Doklady **3**, 931 (1959).

[12] R. Arnowitt and S. Deser, Phys. Rev. **100**, 349 (1955).

[13] L. N. Cooper, Phys. Rev. **100**, 362 (1955).

Translated by G. M. Volkoff
151