CALCULATION OF PHASE INTEGRALS IN THE COVARIANT FORMULATION OF THE THEORY OF MULTIPLE PRODUCTION OF PARTICLES

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A method is proposed for the exact calculation of integrals over momentum space in the covariant statistical theory of the multiple production of particles. The method can be applied in other types of theory, though the calculations become more complicated. Approximate and exact calculations are carried out.

SRIVASTAVA and Sudarshan[1] have proposed a covariant formulation of the statistical theory of the multiple production of π mesons. In this formulation the probability of the production of n particles of arbitrary masses m_i is proportional to an integral over energy-momentum space,

$$I=\int\prod_{i=1}^{n} d^4k_i\,\delta(k_i^2+m_i^2)\,\delta^4\Big(\sum_{i=1}^{n} k_i-P\Big),\qquad(1)$$

where k is the four-momentum of a meson ($k_4 = i\epsilon$) and P is the momentum of the system (in the center-of-mass system $\mathbf{P} = 0$, $P_4 = i\mathscr{E}$).* We shall discuss a method for performing the integration of Eq. (1).

After integrating over all the k_{4i} we get

$$I=2^{-n}\int\prod_{i=1}^{n}\frac{d^3k_i}{\varepsilon_i}\,\delta^3\Big(\sum_1^n \mathbf{k}_i\Big)\,\delta\Big(\sum_1^n\varepsilon_i-\mathscr{E}\Big).\qquad(2)$$

(A similar form is found in the integral that appears in the quantum-field treatment of the process, if the wave functions are normalized to $\epsilon_i^{-1/2}$. We have under the integral sign the remaining part of the matrix element, but at high energies it is obviously almost constant.[2]) The integration in (2) can be carried out by a method similar to that devised by Lepore and Stuart.[3] Using the Fourier representation for the δ function, we bring (2) to the form†

$$I=2^{-n}(2\pi)^{-4}\int d^3r\,dt\,e^{i\mathscr{E}t}\times\Big[\int_0^\infty dk\int_\Omega d\varphi\,d(\cos\theta)\,\frac{k^2}{\varepsilon}\,e^{i(kr\cos\theta-\varepsilon t)}\Big]^n.\qquad(3)$$

After the integrations over the angles and some simple transformations we get

$$I=2^{-(2+n)}\pi^{n-3}\mathscr{E}^{2n-4}\int_0^\infty dz\int_{-\infty}^{\infty}d\alpha\cdot z^{2-n}e^{i\alpha}I_1^n,\qquad(4)$$

where

$$I_1=\int_\mu^\infty\sin\big(z\sqrt{y^2-\mu^2}\big)\,e^{i\alpha y}\,dy,$$

$$y^2=\mu^2(k^2+1),\quad \mu=1/\mathscr{E},\quad \alpha=t\mathscr{E},\quad z=r\mathscr{E}.\qquad(5)$$

Obviously y represents the total energy, and μ the rest energy, of a meson. Consequently $y \geq \mu$, and for $y \leq \mu$ the integrand can be set equal to zero, after which the integration can be extended from 0 to ∞, and we can use the formulas of the Laplace transformation,[4] defining the original function as

$$f(\mu)=\begin{cases}\sin\big(z\sqrt{y^2-\mu^2}\big), & y\geqslant\mu,\\ 0, & y\leqslant\mu.\end{cases}$$

Then

$$I_1=\frac{z\mu}{\sqrt{z^2-\alpha^2}}K_1\big(\mu\sqrt{z^2-\alpha^2}\big),\qquad(6)$$

where $K_1(x)$ is the Macdonald cylinder function $[K_1(x) = -\frac{1}{2}\pi H_1^{(1)}(ix)]$.

Thus the calculation of the quantity (1) reduces to the calculation of the integral

$$I_2=\int_0^\infty dz\int_{-\infty}^{\infty}d\alpha\,\frac{z^2e^{i\alpha}}{(z^2-\alpha^2)^{n/2}}\big[K_1\big(\mu\sqrt{z^2-\alpha^2}\big)\big]^n.\qquad(7)$$

The calculation of integrals of this form encounters difficulties[5] and is usually done approximately. In the extreme relativistic case the calculations can be carried through to the end. For this case we can set $\mu^2 = 0$; then

$$I_2=\frac{1}{2}\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}d\alpha\,\frac{z^2e^{i\alpha}}{(z^2-\alpha^2)^n}.\qquad(8)$$

*We are setting $\hbar = c = m_\pi = 1$ (m_π is the mass of the π meson).

†In (3)–(7) it is assumed for simplicity that all the particles are π mesons.

By displacing the poles in a suitable way, we can perform the integration by the method of residues. Integrating first over z and then over α, we get finally

$$I = [2^{1-n}\pi^{n-1}/(n-1)!(n-2)!]\,\mathscr{E}^{2n-4}. \quad (9)$$

Thus for n = 2 the integral I does not depend on the energy, for n = 3 we have $I \sim \mathscr{E}^2$, and so on. Exact computations show that for n = 2 the approximate value (9) is useful for practical purposes for arbitrary energy, and for n = 3 it is good for energies $\mathscr{E} > 2.5 \times 10^9$ ev (i.e., for kinetic energies in the laboratory system larger than 1.45×10^9 ev). For n = 4, 5, and so on, the threshold for the usefulness of (9) increases, so that exact computation is necessary.

The exact calculation can be carried out in the following way.[2] Integrating over $\mathbf{k}_1$ in (2) and taking account of the δ function of the momentum we get

$$I = 2^{-n}\int \varepsilon_1^{-1}\prod_{i=2}^{n}\varepsilon_i^{-1}\,d^3k_i\,\delta\Big(\sum_2^n \varepsilon_i + \varepsilon_1 - \mathscr{E}\Big). \quad (10)$$

Here

$$\varepsilon_1 = (k_1^2+1)^{1/2} = [p_{n-2} + k_2 + 2(\mathbf{p}_{n-2}\mathbf{k}_2) + 1]^{1/2},$$

where

$$\mathbf{p}_{n-r} = \sum_{r+1}^{n}\mathbf{k}_i, \qquad p_{n-r} = [p_{n-r-1}^2 + k_{r+1}^2 + 2(\mathbf{p}_{n-r-1}\mathbf{k}_{r+1})]^{1/2}.$$

For the further integration over $\mathbf{k}_2$ we go over to spherical coordinates, taking the polar axis along the momentum $\mathbf{p}_{n-2}$. After the integrations over the polar angle θ_2 and the azimuth φ_2 we get

$$I = 2^{-n}(2\pi)\int d\varepsilon_2\prod_{i=4}^{n}\frac{d^3k_i}{\varepsilon_i}\,\frac{d^3k_3}{\varepsilon_3 k_3 p_{n-2}}. \quad (11)$$

For the integration over $\mathbf{k}_3$ we take the axis of spherical coordinates along $\mathbf{p}_{n-3}$; using the fact that

$$p_{n-2} = \sqrt{p_{n-3}^2 + k_3^2 + 2p_{n-3}k_3\cos\theta_3},$$

we integrate first over cos θ_3 (we take the integral in the sense of the principal value, since p_{n-2} can pass through the value zero). It is not hard to show in this way that in any case the integral (11) reduces to the form

$$I = \frac{1}{4}(2\pi)^{n-1}\int\prod_{i=4}^{n}(\varepsilon_i^2 - m_i^2)^{1/2}\,d\varepsilon_i\,d\varepsilon_3\,d\varepsilon_2. \quad (12)$$

Here the π-meson mass ($m_\pi = 1$) has been replaced by masses m_i; thus the formula holds for the production of particles of arbitrary masses.

It is convenient to go over to the kinetic energies $x_i = \epsilon_i - m_i$. The expression (12) then takes the form

$$I = \frac{1}{4}(2\pi)^{n-1}\int\prod_{i=4}^{n}(x_i^2 + 2m_i x_i)^{1/2}\,dx_i\,dx_2\,dx_3. \quad (13)$$

In Eq. (13) we must perform (n − 1) successive integrations. Let us find the region of integration, taking account of the existence of a maximum energy for each of the particles.[6]

The kinetic energies of the particles lie in the ranges

$$0 \leqslant x_i \leqslant t_i, \quad (14)$$

where

$$t_i = \frac{\mathscr{E}}{2} - \frac{M^2 - m_i^2}{2\mathscr{E}} - m_i = \frac{T}{2}\Big(1 + \frac{M - m_i}{T + M + m_i}\Big). \quad (15)$$

Here M is the sum of the rest masses of all the particles except the one in question, and T is the kinetic energy of the system:

$$T = \mathscr{E} - \sum_{i=1}^{n} m_i = \sum_{i=1}^{n} x_i.$$

From these equations there follow the relations

$$T - t_i \leqslant \sum_2^n x_i \equiv T_{n-1} \leqslant T,$$

$$T - T_{n-3} - t_1 \leqslant x_2 + x_3 \leqslant T - T_{n-3}. \quad (16)$$

Let us consider for simplicity the case in which $m_1 = m_2 = m_3$. Simple arguments show that the integral over x_2, x_3 is equal to the area of the entire triangle shown in Fig. 1 for $T_{n-3} \geq T - t_1$,

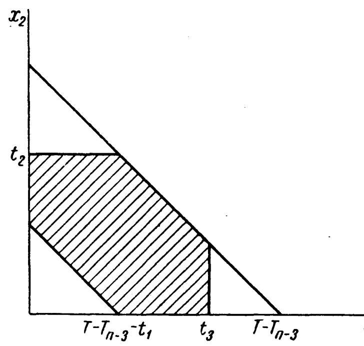

FIG. 1

and for $T_{n-3} \leq T - t_1$ it is equal to the area of the shaded region. The areas in question are easily calculated, and the further analysis of the limits of integration in terms of the relations (14) — (16) also presents no difficulty.

For n = 3 we have $T_{n-3} = 0$, and because $t_1 \leq T$ we find that I is simply equal to the area of the shaded region multiplied by π^2; that is,

$$I = \frac{1}{2}\pi^2\Big[T^2 - \sum_{i=1}^{3}(T - t_i)^2\Big] \quad (17)$$

[for $n = 2$ we have $I = \pi(\frac{1}{4} - \mathcal{E}^{-2})^{1/2}$].

For $n = 4$ we have $T_{n-3} = x_4$, and the expression (13) breaks up into two terms* ($x_4 \geq T - t$ or $x_4 \leq T - t$), in which the integrand contains the product of $(x_4^2 + 2m_4x_4)^{1/2}$ by the value of the corresponding area (as a function of x_4). Thus, on making some transformations, we get

$$I = \pi^3\Big[\int_0^t (T - x_4)^2 (x_4^2 + 2x_4)^{1/2} dx_4 - 3 \int_0^{T-t} (T - t - x_4)^2 (x_4^2 + 2x_4)^{1/2} dx_4\Big]$$

$$= \pi^3 \{[\tfrac{5}{4} (T - t)^3 + \tfrac{9}{4} (T - t)^2 - \tfrac{1}{2} (T - t) - \tfrac{15}{8}] \times [(T - t)^2 + 2 (T - t)]^{1/2} - [(\tfrac{2}{3} T - \tfrac{1}{4} t + \tfrac{5}{12}) (t^2 + 2t) - (\tfrac{1}{2} T^2 + T + \tfrac{5}{8}) (t + 1)][t^2 + 2t]^{1/2} + \tfrac{1}{2}\Big(T^2 + 2T + \tfrac{5}{4}\Big) \ln [t + 1 - (t^2 + 2t)^{1/2}] - \tfrac{3}{2} [(T - t)^2 + 2 (T - t) + \tfrac{5}{4}] \times \ln [T - t + 1 - ((T - t)^2 + 2 (T - t))^{1/2}]\}. \quad (18)$$

For $n = 5$, $T_{n-3} = x_4 + x_5$; the integral over x_2 and x_3 will be equal to the area of the triangle or the truncated triangle (Fig. 1), depending on whether $x_4 + x_5 \geq T - t$ or $x_4 + x_5 \leq T - t$. As can be seen from Fig. 2, in the first case the region of

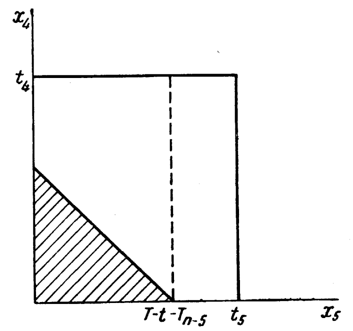

FIG. 2

integration over x_5 and x_4 is the unshaded part of the rectangle, and in the other case it is the shaded part (in this case $T_{n-5} = 0$; we always have $T - t - T_{n-5} < t_5$). For I we get the expression ($m_i = 1$):

$$I = 2\pi^4 \int_0^{t_5} dx_5 (x_5^2 + 2m_5x_5)^{1/2} \int_0^{t_4} (x_4^2 + 2m_4x_4)^{1/2} (T - x_4 - x_5)^2 dx_4 - 2\pi^4 \sum_{i=1}^{3} \int_0^{T-t} dx_5 (x_5^2 + 2m_5x_5)^{1/2} \times \int_0^{T-t-x_5} dx_4 (x_4^2 + 2m_4x_4)^{1/2} (T - x_4 - x_5 - t_i)^2. \quad (19)$$

It is obvious that a similar analysis of the limits of integration can be extended without difficulty to the case of large values of n.

The limits of the integrations can thus be found for arbitrary n and for arbitrary concrete types of particles. The integration itself, however, becomes steadily more complicated. This method of integration over the momentum space can also be applied in other theories of multiple production, but the integration itself is more complicated because of the changes of the forms of the integrands.

It must be noted that in the covariant statistical theory the angular distribution is symmetrical. The matrix element, which has not been taken into account, is obviously responsible for the experimentally observed unsymmetrical distribution.

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*We consider for simplicity the case $m_1 = m_2 = m_3 \neq m_4$ ($m_4 = 1$).

[1] P. Srivastava and G. Sudarshan, Phys. Rev. **110**, 765 (1958).

[2] L. Yakovlev, Dissertation. Moscow State University, 1958.

[3] J. V. Lepore and R. N. Stuart, Phys. Rev. **94**, 1724 (1954).

[4] I. Ryzhik and I. Gradshteĭn, Таблицы интегралов (Tables of Integrals), Moscow-Leningrad, 1951.

[5] Belen'kiĭ, Maksimenko, Nikishev, and Rozental', Usp. Fiz. Nauk **62**, No. 2, 1 (1957).

[6] L. Yakovlev, JETP **31**, 142 (1956), Soviet Phys. JETP **4**, 141 (1957).

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202