CONVECTIVE PINCH INSTABILITY

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An investigation is made of the stability with respect to axially symmetric perturbations, including entropy-wave perturbations, of a pinch with a distributed current.

IT is well known that plasma configurations with closed lines of force are particularly susceptible to convection or transposition instabilities, corresponding to the transposition of neighboring lines of force.[1] In this report we investigate the simplest configuration of this type: a straight, axially symmetric pinch which is contained by the magnetic field produced by the current through the pinch. We are interested chiefly in perturbations of the convection type, i.e., perturbations that are constant along the lines of force. The equilibrium velocity distribution is assumed to be Maxwellian; this assumption is valid if the time during which equilibrium is maintained in the pinch exceeds the time between collisions. The conductivity of the plasma is assumed to be infinite.

1. HYDRODYNAMIC APPROXIMATION

As has been shown in references 2 and 3, the magnetohydrodynamic equations for small oscillations reduce to a single self-adjoint equation for the displacement from equilibrium $\eta(\mathbf{r}, t)$. This equation can be obtained[2,3] by means of a variational principle and it is found that a necessary and sufficient condition for stability is that the potential energy of the small oscillations be positive.

We assume that the plasma is inside a conducting wall and that the radial component η_r vanishes at the interface. In this case the potential energy is

$$V = \frac{1}{2}\int\Big\{\gamma p(\operatorname{div}\eta)^2 + \frac{1}{4\pi}\,(\operatorname{curl}[\eta\times\mathbf{H}])^2 + \eta\nabla p\operatorname{div}\eta - \frac{1}{4\pi}[\eta\operatorname{rot}\times\operatorname{curl}\mathbf{H}]\operatorname{curl}[\eta\times\mathbf{H}]\Big\}\,dr, \qquad (1)$$

where the displacement along the field due to the last two terms in (1) vanishes because the equation is self-adjoint. In Eq. (1) p and **H** are the equilibrium pressure and magnetic field and $\gamma = 5/3$ is the exponent of the adiabat.

In the case being considered $(H_z = 0)$ the potential energy (1) for axially symmetric perturbations is a quadratic form in the two independent variables η_r and $\operatorname{div}\eta$. The condition which must be satisfied if this form is to be positive definite is

$$-d\ln p/d\ln r < 4\gamma/(2+\gamma\varepsilon), \qquad (2)$$

where $\epsilon = 8\pi p/H^2$. It is apparent that this condition also follows from one of the two conditions for convection stability.[1]

We now consider perturbations that depend on azimuth φ. Without loss of generality this dependence can be written in the form

$$\eta_r = \eta_r(r, z)\cos m\varphi, \qquad \eta_\varphi = \eta_\varphi(r, z)\sin m\varphi,$$
$$\eta_z = \eta_z(r, z)\sin m\varphi.$$

In this case, the factors $\cos^2 m\varphi$ and $\sin^2 m\varphi$ appear in Eq. (1); when averages are taken these terms yield $1/2$. Varying Eq. (1) with respect to η_φ gives $\operatorname{div}\eta = 0$, whence

$$V = \frac{1}{4}\int\Big\{\frac{1}{4\pi}\frac{m^2H^2}{r^2}(\eta_r^2+\eta_z^2) + \frac{1}{4\pi}\Big[H\frac{\partial\eta_z}{\partial z} + \frac{\partial}{\partial r}(H\eta_r)\Big]^2 + \eta_r\frac{dp}{dr}\Big[\frac{1}{r}\frac{\partial}{\partial r}(r\eta_r) + \frac{\partial\eta_z}{\partial z}\Big] + \eta_r\Big[\frac{\partial\eta_z}{\partial z} + \frac{1}{H}\frac{\partial}{\partial r}(H\eta_r)\Big]\frac{dp}{dr}\Big\}\,dr. \qquad (3)$$

Equation (3) is a quadratic form in η_r, η_z and $\partial\eta_r/\partial r + \partial\eta_z/\partial z$. If this form is to be positive definite the following condition must be satisfied*

$$-d\ln p/d\ln r < m^2/\varepsilon. \qquad (4)$$

If $\epsilon < 2/3\,\gamma$, the condition in (2) is stronger; if $\epsilon > 2/3\,\gamma$ the condition in (4) is stronger (with $m = 1$). It is interesting to note that in the many photographs of contracting pinches it is possible to distinguish two regions clearly: an inner region where twist perturbations develop ($m = 1$), and an outer region where the perturbations are essentially axially symmetric. This behavior is in complete agreement with the conditions given in (2) and (4).

Writing

$$\frac{d\ln p}{d\ln r} = \begin{cases} -1/\varepsilon & \text{for} \quad \varepsilon > 2/3\,\gamma \\ -4\gamma/(2+\gamma\varepsilon) & \text{for} \quad \varepsilon < 2/3\,\gamma \end{cases} \qquad (5)$$

*This relation is given in a different form in reference 4.

and taking account of the equilibrium condition, which can be written in the form

$$\frac{d\ln p}{d\ln r}=\frac{1}{1+\varepsilon}\left(\frac{d\ln\varepsilon}{d\ln r}-2\right), \tag{6}$$

it is easy to obtain the limiting stable pressure distribution. When $\gamma = 5/3$ this distribution is of the form

$$p = p_0\frac{\varepsilon}{1-\varepsilon}, \qquad r = a(1-\varepsilon) \quad \text{for } \varepsilon > 0.4,$$
$$p = 18.1\, p_0\left(\frac{\varepsilon}{1+{}^5/_4\varepsilon}\right)^{5/2}, \qquad r = 3.94a\,\frac{1+{}^5/_4\varepsilon}{\varepsilon^{3/4}} \quad \text{for } \varepsilon < 0.4, \tag{7}$$

where p_0 and a are constants of integration. This distribution has a singularity as $r \to 0$ and cannot be obtained experimentally. However, if there is a current carrying metal conductor along the z axis it is possible to produce a situation in which the plasma is stable close to the axis. In this case, as follows from (2), (4) and (6), the quantity ϵ must be smaller than unity everywhere and the rate at which the plasma pressure falls off in the radial direction must be small.

2. KINETIC ANALYSIS

In a high-temperature plasma the collision time may become extremely large; in this case the hydrodynamic approximation no longer applies. In a strong magnetic field it is possible to use the drift-approximation equations; these are the hydrodynamic equations for motion transverse to the field lines, and the kinetic equation for the longitudinal motion.[5] In the case of axial symmetry these equations are of the form

$$Mn\frac{\partial^2\eta}{\partial t^2} = -\nabla p_\perp' - \frac{\mathbf{r}}{r^2}(p_\parallel' - p_\perp')$$
$$+\frac{1}{4\pi}\,\mathbf{curl\ curl}\ [\eta\times\mathbf{H}]\times\mathbf{H}] + \frac{1}{4\pi}\,\mathbf{curl\ H}\times\mathbf{curl}\,[\eta\times\mathbf{H}]], \tag{8}$$

$$\frac{\partial f'}{\partial t}+\frac{u}{r}\frac{\partial f'}{\partial\varphi}+\frac{\partial\eta}{\partial t}\nabla f_0+\frac{u}{r}\frac{\partial\eta}{\partial\varphi}\nabla f_0$$
$$-\left\{\frac{u}{T}eE_\varphi' - \frac{Mu^2}{T}\frac{1}{r}\frac{\partial\eta_r}{\partial t} + \frac{Mw^2}{2T}\operatorname{div}\frac{\partial\eta}{\partial t} - \frac{Mw^2}{2T}\frac{1}{r}\frac{\partial\eta_r}{\partial t}\right\} \tag{9}$$

Here f_0 is the ion Maxwellian equilibrium distribution function, T is the temperature, E_φ is the azimuthal component of the electric field, f' is the perturbation on the distribution function, $p_\perp'$ and $p_\parallel'$ are the perturbations of the transverse and longitudinal pressures respectively. M is the mass, n is the ion density, u is the longitudinal ion velocity, and w is the transverse ion velocity.

We consider perturbations whose aximuthal dependence is of the form $e^{im\varphi}$ ($m \neq 0$) and assume that the oscillation frequency satisfies the condition $\omega \ll v_T/r$, where v_T is the mean ion thermal velocity. Under these conditions we can neglect the time derivatives in Eq. (9) and

$$f' = -\eta\nabla f_0 - ireE_\varphi' f_0/MT.$$

Similar considerations apply for the electrons. The field E_φ' is found from the charge neutralization condition; it is apparent, that $E_\varphi' = 0$. Thus, $f' = -\eta\nabla f_0$, whence $p_\parallel' = p_\perp' = -\eta\nabla p_0$. If this relation is substituted in Eq. (8) we obtain an equation which coincides exactly with the magnetohydrodynamic equation with $\operatorname{div}\eta = 0$, i.e., the stability criterion is (4), the same as in the hydrodynamic approximation.[4]

We now consider perturbations with axial symmetry ($m=0$). In this case Eq. (9) can be integrated with respect to time and we obtain the relations

$$p_\perp' = -\eta_r dp/dr - 2p\operatorname{div}\eta + p\eta_r/r,$$
$$p_\parallel' = -\eta_r dp/dr - p\operatorname{div}\eta - 2p\eta_r/r, \tag{10}$$

which coincide with the adiabatic relations obtained by Chew, Goldberger, and Low.[6] As has been shown in reference 3, hydrodynamic equations with this kind of pressure anisotropy can also be obtained by means of a variational principle for the potential energy which, in the present case, is

$$V_k = \frac{1}{2}\int\left\{\eta_r\operatorname{div}\eta\frac{dp}{dr} + 2p(\operatorname{div}\eta)^2 - \frac{2p}{r}\eta_r\operatorname{div}\eta + 3p\frac{\eta_r^2}{r^2}\right.$$
$$+\frac{1}{4\pi}\left[H\frac{\partial\eta_z}{\partial z}+\frac{\partial}{\partial r}(H\eta_r)\right]^2$$
$$\left.+\frac{1}{H}\frac{dp}{dr}\eta_r\left[H\frac{\partial\eta_z}{\partial z}+\frac{\partial}{\partial r}(H\eta_r)\right]\right\}d\mathbf{r}. \tag{11}$$

The condition that the quadratic form in η_r and $\operatorname{div}\eta$ (11) be positive definite is

$$-\frac{d\ln p}{d\ln r} < \frac{7/2 + 5\varepsilon/4}{1+\varepsilon}. \tag{12}$$

This condition is somewhat weaker than (2). For this reason it is of interest to delineate the effect of collisions between particles which lead to equilibration of the longitudinal and transverse pressures. In order to avoid complications we neglect the difference between the equilibration times for the electron and ion pressures and introduce an average relaxation time τ. If we assume that the total energy of the particles is conserved in collisions it is apparent that the pressure relaxation is described by the relations

$$-\frac{\partial p_\perp'}{\partial t} = -\frac{1}{3\tau}(p_\perp' - p_\parallel'), \quad \frac{\partial p_\parallel'}{\partial t} = -\frac{2}{3\tau}(p_\parallel' - p_\perp'). \tag{13}$$

We assume that all quantities have time factors of the form $e^{\lambda t}$. Then, taking account of collisions we can write Eq. (10) in the form

$$p'_{\perp} + \frac{1}{3\lambda\tau}(p'_{\perp} - p'_{\parallel}) = -\eta_r \frac{dp}{dr} - 2p \operatorname{div} \eta + \frac{p}{r}\eta_r,$$
$$p'_{\parallel} + \frac{2}{3\lambda\tau}(p'_{\parallel} - p'_{\perp}) = -\eta_r \frac{dp}{dr} - p \operatorname{div} \eta - \frac{2p}{r}\eta_r. \quad (14)$$

If the quantities $p'_{\parallel}$ and $p'_{\perp}$ are determined in this way and substituted in Eq. (8), neglecting the inertia term on the left side we obtain an equation which can be regarded as the Euler equation for the variational problem $\delta(V_k + V/\lambda\tau) = 0$, with a characteristic value of zero, i.e., the solution satisfies the equation

$$V_k + V/\lambda\tau = 0. \quad (15)$$

Here V_k is given by Eq. (11) and V is given by Eq. (1). It follows from Eq. (15) that $\lambda\tau$ is real.

Now suppose that the energy V_k is positive, that is to say, the plasma is stable in the absence of collisions. Then, if V is positive, (15) has only the damped solution with $\lambda < 0$; if V is negative, of the extremum values of the functional $F = V_k + V/\lambda\tau$ there is one which vanishes when the parameter $\lambda\tau$ changes from 0 to ∞. This means that Eq. (15) has a solution with the increment $\lambda \sim 1/\tau > 0$. Thus, the two modes (the dynamic mode and the one associated with collisions) given by the kinetic equations for the small oscillations yield a stability condition which is exactly the same as the hydrodynamic stability condition.

3. DRIFT INSTABILITY OF A PINCH

Tserkovnikov[7] has shown that under certain conditions there can be an increasing solution for oscillations whose phase velocity is of the order of the drift velocity of the particles. Since it is difficult to delineate the physical significance of the drift instability in the kinetic analysis given by this author, we consider the problem here by means of the hydrodynamic equations, in which quantities of order ρ/r are retained, where ρ is the ion Larmor radius. Thus, in the heat transfer equations we introduce the thermal drift of the electrons ($\alpha = e$) and ions ($\alpha = i$)[8]

$$\mathbf{q}_\alpha = -(5n_\alpha T_\alpha / 2e_\alpha H^2)[\mathbf{H}\nabla T_\alpha],$$

but neglect the usual thermal conductivity and viscosity, assuming that the frequency of collisions between particles is much lower than the cyclotron frequency of the ions and that the wavelength of the perturbations is much larger than ρ. We also assume that the Debye radius is much smaller than all the characteristic lengths so that the charge neutralization condition is satisfied ($n_i = n_e = n$). We also assume that all quantities vary as $\exp(-i\omega t + ikz)$ so that the linearized equations of continuity are written in the form

$$n' = -\operatorname{div}(\eta n) = -\operatorname{div}(n\xi) + (kv_0/\omega)n'. \quad (16)$$

Here n' is the perturbed density, η is the ion displacement, ξ is the displacement of the electrons from the equilibrium position

$$v_0 = (c/eHn)\,dp/dr$$

and the equilibrium drift velocity of the electrons results in a current flow along the axis of the pinch.

The equations of motion are the Euler equation

$$-\omega^2 nM\eta + \nabla p' = \frac{1}{4\pi}[\operatorname{curl}\mathbf{H}'\times\mathbf{H}] + \frac{1}{4\pi}[\operatorname{curl}\mathbf{H}\times\mathbf{H}'] \quad (17)$$

and the equation of momentum transfer for the electrons, in which we neglect the inertia term is:

$$\frac{1}{n}\nabla p'_e - \frac{n'}{2n^2}\nabla p = -e\mathbf{E}' - \frac{e}{c}[\mathbf{v}_0\times\mathbf{H}'] + \frac{i\omega e}{c}[\xi\times\mathbf{H}]. \quad (18)$$

The linearized heat transfer equations are[8]

$$\omega\left(T'_e + \xi_r\frac{dT}{dr} + \frac{2}{3}T\operatorname{div}\xi\right) - kv_0T'_e + \frac{i}{2\tau}(T'_e - T'_i)$$
$$= -\frac{5}{3}\frac{kT_e'}{n}\frac{1}{r}\frac{d}{dr}\left(\frac{rncT}{eH}\right)$$
$$+\frac{5}{3}\frac{kcT}{eH}\frac{dT}{dr}\left(\frac{n'}{n} + \frac{T_e'}{T} - \frac{H'}{H}\right), \quad (19)$$

$$\omega\left(T'_i + \eta_r\frac{dT}{dr} + \frac{2}{3}T\operatorname{div}\eta\right) + \frac{i}{2\tau}(T'_i - T'_e)$$
$$= \frac{5}{3}\frac{kT'_i}{n}\frac{1}{r}\frac{d}{dr}\left(\frac{rncT}{eH}\right) - \frac{5}{3}\frac{kcT}{eH}\frac{dT}{dr}\left(\frac{n'}{n} + \frac{T_i'}{T} - \frac{H'}{H}\right). \quad (20)$$

Here the last term on the left side takes account of the heat exchange between electrons and ions and τ is the characteristic time for temperature equilibration.

Equations (16) – (20), together with Maxwell's equations

$$\operatorname{curl}\mathbf{H}' = (4\pi e/c)\{-n'\mathbf{v}_0 + i\omega(\xi - \eta)n\}, \quad (21)$$

$$\operatorname{curl}\mathbf{E}' = (i\omega/c)\mathbf{H}' \quad (22)$$

represent the complete system of equations for the small oscillations.

If we neglect the magnetoacoustic oscillations the inertia term in the z component of (17) can be neglected so that

$$p' + H'H/4\pi = 0. \quad (23)$$

Equation (23) and the radial component of (17) yield

$$\eta_r = -2p'/\omega^2 Mrn. \quad (24)$$

From (21), taking account of (16) and (23), we have

$$\xi_r - \eta_r = \frac{kc}{e\omega nH}p',$$

$$\operatorname{div}(\xi - \eta) = \frac{kv_0}{\omega}\frac{n'}{n} - \frac{kc}{\omega eHn^2}\frac{dn}{dr}p'. \quad (25)$$

Using (23) — (25), we reduce the remaining equations to a system of three algebraic equations

$$(\nu - q)\frac{T'}{T} + \left\{2 + \left(\frac{2}{5} + \frac{\varepsilon}{2}\right)q - s\right\}\frac{\theta}{T} + \left\{-\frac{2}{5}\nu + s + \frac{4}{\nu k^2\rho^2}\left(\frac{2}{5}q - s\right)\right\}\frac{p'}{p} = 0; \quad (26)$$

$$\frac{5}{2}\left(2 + \frac{\varepsilon}{2}q\right)\frac{T'}{T} + \left(\nu + \frac{i}{\tau^*} - q\right)\frac{\theta}{T} - \frac{5}{3}\frac{\varepsilon}{2}s\frac{p'}{p} = 0; \quad (27)$$

$$(\nu - q)\frac{T'}{T} + (q - s)\frac{\theta}{T} + \left\{-\nu\left(1 + \frac{\varepsilon}{2}\right) - 4 + \varepsilon + \frac{4}{\nu k^2\rho^2}\left[2 + \left(1 + \frac{\varepsilon}{2}\right)q - s\right]\right\}\frac{p'}{p} = 0. \quad (28)$$

Here

$$T' = (T_e' + T_i')/T, \quad \theta = (T_e' - T_i')/T, \quad q = d\ln p/d\ln r,$$

$$s = d\ln T/d\ln r, \quad \nu = \omega eHr/kcT,$$

$$\tau^* = \tau kcT/eHr, \quad \rho = \sqrt{TMc^2/e^2H^2},$$

and ρ is the mean ion Larmor radius. Equation (26) is half the sum, and (27) is half the difference, of the heat transfer equations (19) and (20). Equation (28) is obtained from (22) by substitution of the electric field from (18). The requirement that the determinant of the system (26) — (28) must vanish gives a dispersion equation of the fourth degree. Since the coefficients of this equation are functions of r while the frequency can be considered independent of r, the perturbations must be localized with respect to r; strictly speaking they are δ-functions. We assume that the wavelength of the perturbation is much smaller than the ion Larmor radius $k\rho \ll 1$; thus in the dispersion equation there is a large coefficient $1/k^2\rho^2$. In this case the two roots are large (of order $\nu \sim 1/k\rho$), and we obtain the values

$$\omega^2 = \frac{2p}{Mnr}\left\{\frac{d\ln p}{d\ln r} + \frac{4\gamma}{2 + \gamma\varepsilon}\right\}, \quad (29)$$

where $\gamma = 5/3$. This is obviously the frequency of the convection oscillations and the condition $\omega^2 > 0$ coincides exactly with (2).

The other two roots are of order unity and are determined by the equation

$$\left(\nu - q + \frac{i}{\tau^*}\right)(\nu - q) = \gamma^2\frac{(4 + \varepsilon q)^2}{4\gamma + q(2 + \gamma\varepsilon)} \times \left\{1 + \left(\frac{7}{10} + \frac{\varepsilon}{4}\right)q - s\right\}. \quad (30)$$

If (2) is satisfied, the following condition must be satisfied if the imaginary parts of the roots of (3) are to be positive:

$$d\ln T/d\ln r < 1 + (7/10 + \varepsilon/4)\,d\ln p/d\ln r. \quad (31)$$

In the case of small curvature ($r \to \infty$) this condition becomes the condition $d\ln T/d\ln p < 7/10 + \epsilon/4$, which has been obtained earlier by Tserkovnikov and represents the hydrodynamic analog of one of the conditions obtained in reference 7 by means of the kinetic equation. The condition in (31) shows that for a given value of $d\ln p/d\ln r$, the temperature cannot increase too rapidly with r and that for a given temperature gradient the pressure must not fall off too rapidly with r.

It is interesting to note that when $d\ln T/d\ln r = 0$, the condition in (31) becomes (2) (with $\gamma = 5/7$). Thus, the drift thermal conductivity leads to a more stringent stability condition than the usual condition; when this is taken into account we arrive at (2) with, $\gamma = 1$ (the isothermal condition).

According to (26) — (28), when $\nu \sim 1$ we have $p'p \sim (k\rho)^2 T'/T \ll T'/T \sim \theta T$, i.e., the oscillations represent entropy waves, or a perturbation of the temperatures T_e and T_i at constant pressure. Because of the particle drift these waves propagate along the axis of the pinch and the plasma, in accordance with Eqs. (24) and (25), executes radial oscillations $\eta_r \approx \xi_r \sim rT'/T$ which compensate for the change in pressure by virtue of the drift thermal conductivity.

If (31) is not satisfied these oscillations increase without limit and this causes heat transfer along the radius until the temperature gradient is reduced to values given by (31).

It should be noted that the temperature difference between the electron temperature and ion temperature is of importance in these oscillations because $\theta \to 0$ when $\tau^* \to 0$ and (30) has only one root beside the real root $\nu = q$.

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