SOVIET PHYSICS JETP VOLUME 37(10), NUMBER 4 APRIL, 1960

ON THE THEORY OF THE RELATIVISTIC TRANSFORMATIONS OF THE WAVE FUNCTIONS AND DENSITY MATRIX OF PARTICLES WITH SPIN

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Submitted to JETP editor May 28, 1959

J. Exptl. Theoret. Phys. (U.S.S.R.) **37**, 1116-1126 (October, 1959)

Expansions are obtained for the wave functions of a particle with spin s and for those of a system of two particles with arbitrary spins s_1 and s_2, in terms of the irreducible representations of the homogeneous Lorentz group; this makes possible a relativistically invariant classification of the states. For the invariant description of the polarization of free particles an expansion of the density matrix is found in terms of the irreducible representations of the Lorentz group.

1. The finite-dimensional representations (tensor and spinor representations) of the Lorentz group are widely used in quantum field theory. Since, however, the Lorentz group is not compact, there exists for it an entirely different class of representations, namely the infinite-dimensional representations. The application of the infinite-dimensional representations in the theory of elementary particles was first achieved in a paper by Ginzburg and Tamm.[1]

The unitary representations of the Lorentz group were discovered almost simultaneously by several authors — Gel'fand and Naĭmark,[2] Dirac,[3] and Harish-Chandra[4] — but did not find any applications in physics for a long time thereafter.

In 1955 Shapiro[5] proposed and treated the problem of the expansion of the wave function of a free particle in terms of the irreducible representations of the Lorentz group. Such an expansion makes possible a relativistically invariant classification of the states of the particle in terms of the eigenvalues of the invariants of the homogeneous group: the scalar $\hat{F} = \frac{1}{2} M_{\mu\nu}M_{\mu\nu}$ and the pseudoscalar $\hat{G} = \frac{1}{16}\epsilon_{\mu\nu\rho\sigma}M_{\mu\nu}M_{\rho\sigma}$, where $M_{\mu\nu}$ is the four-dimensional angular-momentum tensor:

$$M_{\mu\nu} = -i(p_\mu \partial/\partial p_\nu - p_\nu \partial/\partial p_\mu) + S_{\mu\nu}.$$

$S_{\mu\nu}$ is the spin part of the four-dimensional angular momentum. Dolginov[6] also treated this problem by another method.

Later Chou Kuang-Chao and Zastavenko[7] made improvements in Shapiro's expansion for particles with nonzero spin. It must be remarked, however, that the method they used involves extremely cumbersome calculations, since they did not use to its full extent the theory of the representations of the Lorentz group.

From the mathematical point of view the formulas obtained in references 5 and 7 make up an integral transformation, which, following reference 7, we shall call the "Shapiro transformation." In the present paper we begin with a simpler derivation of the Shapiro transformation for a free particle with arbitrary spin s; we then use an analogous method to study a system of two interacting particles. In the concluding part of the paper an expansion in irreducible parts is carried out for the density matrix of the free particle and the system of two particles; this is of importance for the description of polarization in the relativistic case.

2. Let us consider a free particle with arbitrary spin s and mass κ ($\kappa \neq 0$). The amplitude of the one-particle state (after second quantization) has the form

$$\Phi_1 = \int \frac{\kappa d^3p}{p_0} \sum_{\sigma=-s}^{s} \varphi_\sigma(\mathbf{p})\, a_\sigma^+(\mathbf{p})\, \Phi_0, \tag{1}$$

where, as usual, $a_\sigma^+(\mathbf{p})$ is the operator for creation of a particle with momentum p and spin component σ along the z axis, Φ_0 is the vacuum amplitude, and $\varphi_\sigma(\mathbf{p})$ are the corresponding Fock amplitudes. Here we must be exact about what we mean by the spin: for example, for particles with spin $\frac{1}{2}$ that obey the Dirac equation the spin operator is ordinarily taken to be $\frac{1}{2}\boldsymbol{\sigma}$.[8] But then there do not exist states with prescribed momentum $\mathbf{p}$ and definite spin component in any direction in space, unless the direction is that of the momentum itself. It will be more convenient for us to go over to the spin in the rest system, i.e., to introduce the operator $\hat{S}_i = L(\mathbf{p})\, S_i^0 L^{-1}(\mathbf{p})$, where $\hat{S}_i^0$ is the spin operator in the rest system, which has the same properties as the spin in the nonrelativistic theory, and $L(\mathbf{p})$ is the pure Lorentz transformation* that gives the change from the rest system of the particle to the reference system in which it has the momentum $\mathbf{p}$: $L(\mathbf{p})p_0 = p$, $p_0 = (0, 0, 0, \kappa)$. It is obvious that in the rest system the spin of the particle can be oriented in an arbitrary direction, and there exist states with momentum $\mathbf{p}$ and the spin component σ along the z axis:

$$\psi_{\mathbf{p}\sigma}(x) = (2\pi)^{-3/2} v_\sigma(\mathbf{p}) e^{i(px)}, \quad v_\sigma(\mathbf{p}) = L(\mathbf{p}) v_\sigma(0), \quad (2)$$

where $v_\sigma(0)$ is the eigenfunction of the ordinary operator $\hat{S}_z^0$ in the rest system: $\hat{S}_z^0 v_\sigma(0) = \sigma v_\sigma(0)$. The field operator $\psi(x)$ is expanded in terms of the system of functions $\psi_{\mathbf{p}\sigma}(x)$, $\psi^*_{\mathbf{p}\sigma}(x)$:

$$\psi(x) = \int \frac{\varkappa d^3p}{p_0} \sum_{\sigma=-s}^{s} (a_\sigma(\mathbf{p}) \psi_{\mathbf{p}\sigma}(x) + b_\sigma^+(\mathbf{p}) \psi^*_{\mathbf{p}\sigma}(x)),$$

after which we can find the operators of the dynamical variables: the energy and momentum, and the angular momentum. Comparing them with the usual expressions, we find the commutation rule of the operators $a_\sigma(\mathbf{p})$:

$$[a_\sigma(\mathbf{p}), a_{\sigma'}^+(\mathbf{p}')] = \varkappa^{-1} p_0 \delta_{\sigma\sigma'} \delta(\mathbf{p} - \mathbf{p}') \quad (3)$$

(the left member must be the commutator for an integer spin s and the anticommutator for half-integral s).

We need to find the law of transformation of the Fock amplitudes $\varphi_\sigma(\mathbf{p})$. For this purpose we apply an arbitrary transformation g of the proper Lorentz group to the probability amplitude for finding the particle at the point x

$$f(x) = (\Phi_0, \psi(x)\Phi_1) = (2\pi)^{-3/2} \int \frac{\varkappa d^3p}{p_0} \sum_\sigma \varphi_\sigma(\mathbf{p}) v_\sigma(\mathbf{p}) e^{i(px)}. \quad (4)$$

We here take into account the fact that $v_\sigma(\mathbf{p})$ transforms according to a finite-dimensional representation τ_{s0} ($P = s$, $Q = 0$; for the notation see reference 9):

$$T_g v_\sigma(\mathbf{p}) = T_g \tau_{s0}(L_p) v_\sigma(0) = \tau_{s0}(gL_p) v_\sigma(0)$$
$$= \tau_{s0}(L_{gp}) \tau_{s0}(R(g, p)) v_\sigma(0) = \sum_{\sigma'=-s}^{s} D_{\sigma\sigma'}^{(s)} \{R(g, p)\} v_{\sigma'}(gp),$$

where we have introduced the rotation $R(g, p)$:

$$gL_p = L_{gp} R(g, p) \quad (5)$$

(this is the so-called Thomas precession, if g itself is a pure Lorentz transformation), and where we use the condition that under rotations in the rest system the wave functions $v_\sigma(0)$ transform according to the irreducible representation of weight s of the rotation group (fundamental property of the spin s). From this we find how the Fock amplitudes transform:*

$$T_g \varphi_\sigma(\mathbf{p}) = \sum_{\sigma'=-s}^{s} D_{\sigma\sigma'}^{(s)} \{R(g, g^{-1}p)\} \varphi_{\sigma'}(g^{-1}p). \quad (6)$$

It is clear from physical considerations that Eq. (6) must be a representation of the Lorentz group; formally this follows from the group property of the rotations R:

$$R(g_1, g_1^{-1}p) R(g_2, (g_1g_2)^{-1}p) = R(g_1g_2, (g_1g_2)^{-1}p). \quad (7)$$

It is also obvious that Eq. (6) is a unitary representation, because there is conservation of the total number of particles

$$\int \frac{\varkappa d^3p}{p_0} \sum_{\sigma=-s}^{s} |\varphi_\sigma(\mathbf{p})|^2;$$

therefore it can be expanded in terms of the unitary irreducible representations of the Lorentz group, which are all infinite-dimensional[10,11] and are characterized by two numbers m, ρ, which are connected with the eigenvalues of the invariants: $\hat{F} = -[1 + \frac{1}{4}(\rho^2 - m^2)]$, $\hat{G} = -\frac{1}{4}im\rho$. To find this expansion, let us go over from the wave function $\varphi_\sigma(\mathbf{p})$ to a new function $\chi_\sigma(g)$ of the Lorentz transformation g:

$$\chi_\sigma(g) = \sum_{\sigma'=-s}^{s} D_{\sigma\sigma'}^{(s)} \{R(g, p)\} \varphi_{\sigma'}(p), \quad (8)$$

where

$$p = g^{-1}p_0, \quad p_0 = (0, 0, 0, \varkappa).$$

It follows from (6) that $\chi_\sigma(g)$ transforms by the regular representation of the Lorentz group:

$$T_{g_0} \chi_\sigma(g) = \chi_\sigma(gg_0) \quad (9)$$

(Here T_{g_0} denotes the operator that acts on the function $\chi_\sigma(g)$ when the Lorentz transformation g_0 is applied to the coordinates), and the expansion of the regular representation in terms of the irreducible representations is known.[10] Let us now find how $\chi_\sigma(g)$ depends on the left unitary operator. Since an arbitrary spatial rotation does not change the momentum in the rest system ($up_0 = p_0$), it follows from (7) that

$$R(ug, (ug)^{-1}p_0) = R(u, p_0) R(g, g^{-1}p_0);$$

the rotation $R(u, p_0)$ is determined from the equation $uL_{p_0} = L_{up_0}R(u, p_0)$, and since $up_0 = p_0$, $L_{p_0} = 1$, we have $R(u, p_0) = u$. Using the definition (8), we get

*Every transformation of the proper Lorentz group can be represented in the form $g = u_2 \epsilon u_1$, where u_2 and u_1 are spatial rotations and ϵ is the change to a reference system moving along the z axis. If $u_2 = u_1^{-1}$ we call g a pure Lorentz transformation.

*This formula was given without proof in reference 7.

$$\chi_\sigma(ug) = \sum_{\sigma'=-s}^{s} D^{(s)}_{\sigma\sigma'}\{R(ug, (ug)^{-1}p_0)\}\varphi_{\sigma'}(p)$$
$$= \sum_{\sigma'=-s}^{s} D^{(s)}_{\sigma\sigma'}(u)\chi_{\sigma'}(g). \quad (10)$$

The properties (9) and (10) are sufficient for the expansion of $\chi_\sigma(g)$ in terms of irreducible representations. To accomplish it we use the analog of Plancherel's formula from reference 10. If $x(g)$ is the regular representation of the Lorentz group, its expansion in terms of irreducible representations has the form

$$K_{m\rho}(z_1, z_2) = \int x(z_1^{-1}kz_2)\alpha^*_{m\rho}(k)\,d\mu_l(k),$$
$$x(g) = (2\pi)^{-4}\sum_{m=-\infty}^{\infty}\int_0^\infty d\rho\,(m^2+\rho^2)\int d\mu(z)\,K_{m\rho}(z, z_1)\,\alpha_{m\rho}(k),$$

where $zg = kz_1$; here (and often in what follows) g denotes both some transformation of the proper Lorentz group and also the corresponding two-rowed matrix b of its spinor representation; $\alpha_{m\rho}(b)$ is a function defined on the group of two-rowed matrices b:

$$\alpha_{m\rho}(b) = |b_{22}|^{m+i\rho-2}b_{22}^{-m},$$
$$\text{for } b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}, \ \det b = 1.$$

Under transformations of the Lorentz group the function $K_{m\rho}(z_1, z_2)$ transforms by the irreducible $(m\rho)$ representation with respect to its first argument z_2, and its first argument z_1 remains unchanged [the space of the representations consists of the functions on the group of matrices $z = \begin{pmatrix} 1 & 0 \\ z & 1\end{pmatrix}$; for further details see references 5, 11]. The representations (m, ρ) and $(-m, -\rho)$ are equivalent, and therefore the expansion of $x(g)$ involves only one of them $(\rho > 0)$. It will be more convenient for us to take as the space of the representation the manifold of functions on the unitary group u; for this purpose let us introduce the matrices u_1, u_2 by the formula* $z_1 = k_1u_1$, $z_2 = k_2u_2$, and the new function

$$\hat{K}_{m\rho}(u_1, u_2) = \pi\alpha^*_{m\rho}(u_1)\,\alpha_{m\rho}(u_2)\,K_{m\rho}(z_1, z_2)$$
$$= \pi\int x(u_1^{-1}ku_2)\,\alpha^*_{m\rho}(k)\,d\mu_l(k).$$

It transforms in the following way:

$$T_{g_0}\hat{K}_{m\rho}(u_1, u) = \alpha_{m\rho}(k)\,\hat{K}_{m\rho}(u_1, u'), \qquad ug_0 = ku'. \quad (11)$$

This indeed means that $\hat{K}_{m\rho}(u_1, u)$ transforms by the irreducible representation $(m\rho)$ (with respect to its argument u). The expansion of the regular representation takes the form

$$\hat{K}_{m\rho}(u_1, u) = \pi\int x(u_1^{-1}ku)\,\alpha^*_{m\rho}(k)\,d\mu_l(k),$$
$$x(g) = \sum_{m=-\infty}^{\infty}\int_0^\infty d\rho\,\frac{m^2+\rho^2}{(2\pi)^4}\int du_1\,\hat{K}_{m\rho}(u_1, u)\,\alpha_{m\rho}(k), \quad (12)$$

where $u_1g = ku$. Let us now substitute in (12) the function $\chi_\sigma(g)$ instead of $x(g)$ [cf. Eq. (8)]. It is well known that every matrix of the type k can be represented in the form $k = \gamma\epsilon\xi$ (for further details see reference 5), where

$$\gamma = \begin{pmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{i\gamma/2}\end{pmatrix}, \quad \varepsilon = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda\end{pmatrix}, \quad \zeta = \begin{pmatrix} 1 & \zeta \\ 0 & 1\end{pmatrix}$$

(in the spinor representation γ corresponds to a rotation through the angle γ around the z axis, and ϵ to a Lorentz transformation along the z axis). Furthermore $d\mu_l(k) = 2\pi d\mu(\gamma)\,d\mu(\epsilon)\,d\xi$, where $d\mu_l(k)$, $d\mu(\gamma)$, $d\mu(\epsilon)$, $d\xi$ are invariant measures* on the groups of matrices k, γ, ϵ, ξ. Substituting this in Eq. (12), we get

$$\hat{K}_{m\rho}(u_1\ u;\ \sigma) = 2\pi^2\int\chi_\sigma(u^{-1}\gamma\varepsilon\zeta u)\alpha^*_{m\rho}(\gamma\varepsilon\zeta)\,d\mu(\gamma)\,d\mu(\varepsilon)\,d\zeta$$
$$= \pi^2\sum_{\sigma'}\int d\mu(\gamma)\,D^{(s)}_{\sigma\sigma'}(u_1^{-1}\gamma)\,\alpha^*_{m\rho}(\gamma)\int\chi_{\sigma'}(\varepsilon\zeta u)\,\alpha^*_{m\rho}(\varepsilon)\,d\mu(\varepsilon)\,d\zeta.$$

Since $\alpha^*_{m\rho}(\gamma) = \exp\{im\gamma/2\}$, the integration over $d\mu(\gamma)$ gives $\delta_{\sigma', m/2}D^{(s)}_{\sigma, m/2}(u_1^{-1})$, from which there follows the selection rule: the wave function of a particle with spin s has as its components the irreducible representations (m, ρ) in which $m = -2s, -2s+2, \ldots, 2s-2, 2s$. In the remaining integral we transform $d\mu(\epsilon)\,d\xi$.

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*Here k is a triangular matrix of the type

$$\begin{pmatrix} k_{11} & k_{12} \\ 0 & k_{22}\end{pmatrix}, \ \det k = k_{11}k_{22} = 1;$$

u is a unitary matrix:

$$u = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha}\end{pmatrix}, \ |\alpha|^2 + |\beta|^2 = 1.$$

*Let $k = \begin{pmatrix} \lambda^{-1} & \mu \\ 0 & \lambda\end{pmatrix}$, where λ, μ are arbitrary complex numbers; let us set $\lambda = \lambda' + i\lambda''$, $\mu = \mu' + i\mu''$, where λ', λ'', μ', μ'' are real. A left invariant measure on the subgroup of matrices k is an expression

$$d\mu_l(k) = \omega(\lambda, \mu)\,d\lambda'd\lambda''d\mu'd\mu'',$$

that satisfies the condition

$$\int f(k)\,d\mu_l(k) = \int f(k_0k)\,d\mu_l(k)$$

for any function $f(k)$ that is integrable on the group of matrices k, and any fixed matrix k_0. Right invariant measure is defined analogously. It can be shown[11] that $d\mu_l(k) = d\lambda'd\lambda''d\mu'd\mu''$. For the subgroups γ, ϵ, ζ the left and right invariant measures are the same. We define them as follows:

$$d\mu(\gamma) = \frac{d\varphi}{2\pi} \text{ for } \gamma = \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2}\end{pmatrix},$$
$$d\mu(\varepsilon) = \frac{d\lambda}{\lambda} \text{ for } \varepsilon = \begin{pmatrix} 1/\lambda & 0 \\ 0 & \lambda\end{pmatrix},$$
$$d\zeta = d\zeta'd\zeta'' \text{ for } \zeta = \begin{pmatrix} 1 & \zeta \\ 0 & 1\end{pmatrix}, \ \zeta = \zeta' + i\zeta''.$$

Here λ is a real number, and ζ means simultaneously a definite type of two-rowed matrix and the complex number appearing in such a matrix.

To do so we set $p = (\epsilon\zeta u)^{-1} p_0$; by a change of variables it can be shown that $d\mu(\epsilon)\, d\zeta = (\lambda^4/2\kappa^2) \times d^3p/p_0$, where $\lambda = \epsilon_{22}$. Since $\alpha^*_{m\rho}(\epsilon) = \lambda^{-i\rho-2}$, we must express λ in terms of $\mathbf{p}$ and the parameters of the rotation u. It is easy to verify that for an arbitrary two-rowed matrix b of the spinor representation we have: $|k_{22}|^2 = |b_{21}|^2 + |b_{22}|^2$, if $b = ku$. Let us introduce a vector n_0 of zero length with its space part directed along the z axis, $n_0 = (0, 0, 1, 1)$, and calculate the scalar product:

$$(n_0,\ gp_0) = (gp_0)_3 - (gp_0)_0 = \varkappa\,(g_{30} - g_{00}),$$

since $p_0 = (0, 0, 0, \kappa)$. The elements of the matrix g are expressed in a known way in terms of the parameters of the corresponding matrix b of the spinor representation (see reference 11):

$$g_{30} = \tfrac{1}{2}\,(|b_{11}|^2 + |b_{12}|^2 - |b_{21}|^2 - |b_{22}|^2),$$

$$g_{00} = \tfrac{1}{2}\,(|b_{11}|^2 + |b_{12}|^2 + |b_{21}|^2 + |b_{22}|^2),$$

from which we get

$$|b_{21}|^2 + |b_{22}|^2 = -\varkappa^{-1}(n_0,\ gp_0) = |k_{22}|^2.$$

In our case $b = uL_p$, $k = (\epsilon\zeta)^{-1}$, from which we have $k_{22} = \lambda^{-1}$. Therefore $\lambda^{-2} = -\kappa^{-1}(n_0, uL_p p_0) = -\kappa^{-1}(u^{-1}n_0, p) = \kappa^{-1}[p_0 - (\mathbf{p}\cdot\mathbf{n})]$. Here the three-dimensional vector $\mathbf{n} = u^{-1}\mathbf{n}_0$ has been introduced. We still have to express the auxiliary function $\chi_{\sigma'}(\epsilon\zeta u)$ in terms of the Fock amplitude $\varphi_\sigma(\mathbf{p})$. For this purpose let us determine the rotation u from the condition $uL_p = (\epsilon\zeta)^{-1}\tilde{u}$. It is well known that the representation of an arbitrary matrix b of the spinor representation in the form $b = ku$ is ambiguous (see references 5 and 11). In the present case, however, $\arg\{(\epsilon\zeta)^{-1}\}_{22} = 0$ independently of the complex number ζ, and the rotation $\tilde{u}$ is determined uniquely. $\chi_{\sigma'}(\epsilon\zeta u)$ is connected with $\varphi_\sigma(\mathbf{p})$ by the formula (8), in which $R(g, g^{-1}p_0)$ appears, with $g = \epsilon\zeta u$. We have:

$$g^{-1}p_0 = (\varepsilon\zeta u)^{-1}p_0 = p, \qquad gL_p = \varepsilon\zeta u L_p = \tilde{u},$$

that is,

$$R\,(\varepsilon\zeta u,\ (\varepsilon\zeta u)^{-1}p_0) = \tilde{u}$$

and

$$\chi_{\sigma'}(\varepsilon\zeta u) = \sum_{\sigma=-s}^{s} D^{(s)}_{\sigma'\sigma}(\tilde{u})\,\varphi_\sigma(\mathbf{p}).$$

As the result we get the following formula:

$$\hat{K}_{m\rho}(u_1,\ u;\ \sigma) = \frac{\pi^2}{\varkappa^2} D^{(s)}_{\sigma,m/2}(u_1^{-1})$$

$$\times \sum_{\sigma'=-s}^{s} \int \frac{d^3p}{p_0}\left(\frac{p_0 - (\mathbf{pn})}{\varkappa}\right)^{-1+i\rho/2} D^{(s)}_{m/2,\sigma'}(\tilde{u})\,\varphi_{\sigma'}(\mathbf{p}).$$

Since u_1 does not change under Lorentz transformations, instead of $\hat{K}_{m\rho}(u_1, u;\ \sigma)$ we can introduce a function of only one argument u, which transforms according to the irreducible representation $(m\rho)$:

$$\hat{K}_{m\rho}(u_1,\ u;\ \sigma) = \frac{4\pi^3}{\varkappa} D^{(s)}_{\sigma,m/2}(u_1^{-1})\,c_{m\rho}(u),$$

$$c_{m\rho}(u) = \frac{1}{4\pi\varkappa}\sum_{\sigma=-s}^{s}\int\frac{d^3p}{p_0}\left(\frac{p_0-(\mathbf{pn})}{\varkappa}\right)^{-1+i\rho/2} D^{(s)}_{m/2,\sigma}(\tilde{u})\,\varphi_\sigma(\mathbf{p}). \tag{13}$$

To find the inverse transformation we use the second of the equations (12)

$$\chi_\sigma(g) = (2\pi)^{-4}\sum_{m=-\infty}^{\infty}\int_0^\infty d\rho\,(m^2+\rho^2)\int du_1 \hat{K}_{m\rho}(u_1, u;\ \sigma)\,\alpha_{m\rho}(k)$$

$$= \frac{1}{4\pi\varkappa}\sum_{m=-\infty}^{\infty}\int_0^\infty d\rho\,(m^2+\rho^2)\int du\, c_{m\rho}(u)\, D^{(s)*}_{m/2,\sigma}(u_1)\,\alpha^*_{m\rho}(k).$$

Here we have gone over from integration over du_1 to integration over du ($ug^{-1} = ku_1$); using (8), we easily get the final formula:

$$\varphi_\sigma(\mathbf{p}) = \frac{1}{4\pi\varkappa}\sum_{m=-\infty}^{\infty}\int_0^\infty d\rho\,(m^2+\rho^2)$$

$$\times\int du\left(\frac{p_0-(\mathbf{pn})}{\varkappa}\right)^{-1-i\rho/2} D^{(s)*}_{m/2,\sigma}(\tilde{u})\,c_{m\rho}(u). \tag{14}$$

Equations (13) and (14) give the Shapiro transformation for the wave function of a free particle with spin s. Conservation of the norm also follows from (13) and (14):

$$\sum_{\sigma=-s}^{s}\int\frac{d^3p}{p_0}\left|\varphi_\sigma(\mathbf{p})\right|^2 = \sum_{m=-\infty}^{\infty}\int_0^\infty d\rho\,(m^2+\rho^2)\int du\,|c_{m\rho}(u)|^2. \tag{15}$$

The rotation $\tilde{u}$ that appears in (13) and (14) is defined by

$$uL_p = k\tilde{u},\quad \arg k_{22} = 0. \tag{16}$$

(in the spinor representation). It is clear that $\tilde{u}$ is a function of u and $\mathbf{p}$. Formulas for the explicit expression of $\tilde{u}$ in terms of u and $\mathbf{p}$ are given in the Appendix.

3. Let us go on to the treatment of a system of two particles with spins s_1 and s_2, which can interact with each other in some way. The Fock expansion of the state amplitude contains terms describing the motion of an arbitrary number of free particles. We consider only the very first of these (as to number of particles):

$$\Phi_{11} = \int\frac{\varkappa_1 d^3p_1}{E_1}\frac{\varkappa_2 d^3p_2}{E_2}\sum_{\sigma_1\sigma_2}\varphi_{s_1s_2}(\mathbf{p}_1\sigma_1,\ \mathbf{p}_2\sigma_2)\,a^+_{\sigma_1}(\mathbf{p}_1)\,a^+_{\sigma_2}(\mathbf{p}_2)\,\Phi_0.$$

By the same method as in Sec. 2, we find the transformation law of the two-particle amplitude $\varphi_{s_1s_2}$:

$$T_g\varphi_{s_1s_2}(\mathbf{p}_1\sigma_1;\ \mathbf{p}_2\sigma_2) = \sum_{\sigma_1'\sigma_2'} D^{(s_1)}_{\sigma_1\sigma_1'}\{R(g,\ g^{-1}p_1)\}\, D^{(s_2)}_{\sigma_2\sigma_2'}\{R(g, g^{-1}p_2)\}$$

$$\times\ \varphi_{s_1s_2}(g^{-1}p_1, \sigma_1';\ g^{-1}p_2, \sigma_2'). \tag{17}$$

It is clear that (17) is a unitary transformation of the Lorentz group. We shall expand it in terms of irreducible representations, and thus obtain a relativistically invariant classification of the states of a system of two particles.

Let us go over from the momenta p_1, p_2 of the individual particles to new variables:

$$p = p_1 + p_2, \quad q = (\varkappa_2 p_1 - \varkappa_1 p_2)/(\varkappa_1 + \varkappa_2).$$

The vector p is timelike, and in the case of two free particles is the energy-momentum of the center of mass. As an invariant variable we introduce $\kappa^2 = p^2 = (p_1 + p_2)^2$; κ is the total mass in the center-of-mass system. Since the masses κ_1 and κ_2 of the particles are fixed, the wave function $\varphi_{s_1s_2}(\mathbf{p}_1\sigma_1, \mathbf{p}_2\sigma_2)$ depends on six variables: $\mathbf{p}_1$, $\mathbf{p}_2$. Instead of them let us introduce the following six independent variables: $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2$, κ, $\boldsymbol{\nu} = \mathbf{q}/|\mathbf{q}|$ (unit vector). Thus $\varphi_{s_1s_2}$ is converted into a function of $\mathbf{p}$, κ, $\boldsymbol{\nu}$:

$$\varphi_{s_1s_2}(\mathbf{p}_1\sigma_1, \mathbf{p}_2\sigma_2) = \Phi_{s_1s_2}(\mathbf{p}, \varkappa, \boldsymbol{\nu}, \sigma_1, \sigma_2), \tag{18}$$

and

$$\int \frac{\varkappa_1 d^3p_1}{E_1}\frac{\varkappa_2 d^3p_2}{E_2}\sum_{\sigma_1\sigma_2} |\varphi_{s_1s_2}(\mathbf{p}_1\sigma_1, \mathbf{p}_2\sigma_2)|^2 = \int_{\varkappa_1+\varkappa_2}^{\infty} a(\varkappa)\, d\varkappa \int \frac{\varkappa d^3p}{E} \int d\Omega_\nu \sum_{\sigma_1\sigma_2} |\Phi_{s_1s_2}(\mathbf{p}, \varkappa, \boldsymbol{\nu}, \sigma_1, \sigma_2)|^2, \tag{19}$$

where

$$a(\varkappa) = \frac{\varkappa_1^2 + \varkappa_2^2}{4(\varkappa_1+\varkappa_2)^2\varkappa^2}[\varkappa^2 - (\varkappa_1+\varkappa_2)^2]^{1/2}[\varkappa^2 - (\varkappa_1-\varkappa_2)^2]^{1/2},$$

$$E = \sqrt{\mathbf{p}^2 + \varkappa^2}.$$

Now let us use the same approach as in Sec. 2. Instead of $\varphi_{s_1s_2}(\mathbf{p}, \kappa, \boldsymbol{\nu}, \sigma_1, \sigma_2)$ we introduce a function on the Lorentz group g:

$$\chi_{s_1s_2}(g, \varkappa, \boldsymbol{\nu}, \sigma_1, \sigma_2) = \sum_{\sigma_1'\sigma_2'} D^{(s_1)}_{\sigma_1\sigma_1'}\{R(g, g^{-1}p_{1c})\} \times D^{(s_2)}_{\sigma_2\sigma_2'}\{R(g, g^{-1}p_{2c})\}\, \Phi_{s_1s_2}(\mathbf{p}, \varkappa, \boldsymbol{\nu}, \sigma_1, \sigma_2). \tag{20}$$

Here $p = g^{-1}p_0$; $p_0 = (0, 0, 0, \kappa)$; p_{1c} and p_{2c} are the momenta of the first and second particles in the center-of-mass system:

$$p_{1c} = [p_c\boldsymbol{\nu}, (\varkappa^2 + \varkappa_1^2 - \varkappa_2^2)/2\varkappa], \quad p_{2c} = [-p_c\boldsymbol{\nu}, (\varkappa^2 - \varkappa_1^2 + \varkappa_2^2)/2\varkappa],$$

$$p_c = \tfrac{1}{2}\varkappa^{-1}[\varkappa^2 - (\varkappa_1+\varkappa_2)^2]^{1/2}[\varkappa^2 - (\varkappa_1-\varkappa_2)^2]^{1/2},$$

p_{1c} and p_{2c} are completely determined if κ and $\boldsymbol{\nu}$ are given. From (17) we find that $\chi_{s_1s_2}$ transforms by the regular representation:

$$T_{g_0}\chi_{s_1s_2}(g, \varkappa, \boldsymbol{\nu}, \sigma_1, \sigma_2) = \chi_{s_1s_2}(gg_0, \varkappa, \boldsymbol{\nu}, \sigma_1, \sigma_2). \tag{21}$$

We can also determine how $\chi_{s_1s_2}$ depends on a left unitary factor in g:

$$\chi_{s_1s_2}(ug, \varkappa, \boldsymbol{\nu}, \sigma_1, \sigma_2) = \sum_{\sigma_1'\sigma_2'} D^{(s_1)}_{\sigma_1\sigma_1'}(u) D^{(s_2)}_{\sigma_2\sigma_2'}(u)\, \chi_{s_1s_2}(g, \varkappa, u^{-1}\boldsymbol{\nu}, \sigma_1', \sigma_2'). \tag{22}$$

The properties (21) and (22) are sufficient for the expansion in irreducible representations. The calculations are again based on the Plancherel formula (12), and are analogous to those carried out in Sec. 2, though more cumbersome. We give only the final formulas:

$$c_{n\varkappa}(m\rho u) = \sum_{\substack{M\mu\sigma\\ \sigma_1\sigma_1'\sigma_2\sigma_2'}} \int \frac{d^3p}{E} Y_J(m\rho u;\, \mathbf{p}, \varkappa, M)\, C^{JM}_{l\mu;s\sigma} C^{s\sigma}_{s_1\sigma_1;s_2\sigma_2} \times \int d\Omega_\nu Y^*_{l\mu}(\boldsymbol{\nu}) D^{(s_1)}_{\sigma_1\sigma_1'}\{R(L_p^{-1}, L_p p_{1c})\} D^{(s_2)}_{\sigma_2\sigma_2'}\{R(L_p^{-1}, L_p p_{2c})\} \times \Phi_{s_1s_2}(\mathbf{p}, \varkappa, \boldsymbol{\nu}, \sigma_1', \sigma_2'),$$

$$\Phi_{s_1s_2}(\mathbf{p}, \varkappa, \boldsymbol{\nu}, \sigma_1, \sigma_2) = \sum_{\substack{nM\mu\sigma\\ \sigma_1'\sigma_2'}} D^{(s_1)}_{\sigma_1\sigma_1'}\{R(L_p^{-1}, L_p p_{1c})\} D^{(s_2)}_{\sigma_2\sigma_2'} \times \{R(L_p^{-1}, L_p p_{2c})\}\, C^{JM}_{l\mu;s\sigma} C^{s\sigma}_{s_1\sigma_1';s_2\sigma_2'} Y_{l\mu}(\boldsymbol{\nu}) \times \sum_{m=-\infty}^{\infty} \int_0^{\infty} d\rho\, (m^2 + \rho^2) \int du\, Y^*_J(m\rho u;\, \mathbf{p}, \varkappa, M)\, c_{n\varkappa}(m\rho u). \tag{23}$$

Here the index n on $c_{n\kappa}(m\rho u)$ denotes a set of invariants: $n = (l, s, J)$; l is the orbital angular momentum in the center-of-mass system, s is the total spin, and J is the total angular momentum in this same system. $C^{JM}_{l\mu;s\sigma}$, etc., are Clebsch-Gordan coefficients; $D^{(s)}_{\sigma\sigma'}(u)$, as usual, are the matrices of the irreducible representation of weight s of the three-dimensional rotation group; and finally,

$$Y_J(m\rho u;\, \mathbf{p}, \varkappa, M) = \frac{1}{4\pi\varkappa}\left(\frac{E - (\mathbf{p}\mathbf{n})}{\varkappa}\right)^{-1+i\rho/2} D^{(J)}_{m/2, M}(\tilde{u}),$$

where $\tilde{u}$ is defined by (16) and $E = (\mathbf{p}^2 + \kappa^2)^{1/2}$. Besides m and ρ, κ, l, s, and J are also invariants of the Lorentz group. We note that if the Fock amplitude $\varphi_{s_1s_2}$ describes a system of two free particles with definite momenta $\mathbf{p}_1$ and $\mathbf{p}_2$, then the center-of-mass system exists and the quantities l, s, and J refer just to it. In the general case there does not exist a reference system such that in it the total momentum of the two particles is zero, because generally speaking $\varphi_{s_1s_2}$ is different from zero in the entire momentum space. Therefore we cannot indicate a reference system in which l, s, and J have intuitive physical meaning, and must simply regard them as invariant variables.

4. Let us now go on to the description of polarization. We recall that in nonrelativistic quantum mechanics the state of polarization of a particle is

completely described by the spin density matrix. It is convenient to take as the parameters characterizing the polarization the coefficients in the expansion of the density matrix in terms of tensor operators that are multipole moments of various ranks and transform according to the irreducible representations of the three-dimensional rotation group.[12,13] We wish to introduce quantities that describe the polarization but transform according to irreducible representations of the Lorentz group. For one free particle it is natural to define the multipole-moment operator by the relation

$$T_{JM}=\int\frac{\varkappa d^3p}{p_0}\sum_{\sigma\sigma'}\sqrt{2J+1}\,C^{s\sigma}_{s\sigma';JM}\,a^+_\sigma(\mathbf{p})\,a_{\sigma'}(\mathbf{p}).$$

The polarization of a beam of free particles is determined by the average values of these operators for the one-particle state (1):

$$\xi_{JM}=(\Phi_1,\ T_{JM}\Phi_1)=\int\frac{\varkappa d^3p}{p_0}\sum_{\sigma\sigma'}\sqrt{2J+1}\,C^{s\sigma}_{s\sigma';JM}\,\varphi^*_\sigma(\mathbf{p})\,\varphi_{\sigma'}(\mathbf{p}). \tag{24}$$

It is obvious that the quantity

$$\zeta_{JM}(\mathbf{p})=\sum_{\sigma\sigma'}\sqrt{2J+1}\,C^{s\sigma}_{s\sigma';\,JM}\varphi^*_\sigma(\mathbf{p})\,\varphi_{\sigma'}(\mathbf{p})$$

is the density of the multipole moment of rank J. The transformation law of ζ_{JM} follows from Eq. (6):

$$T_{g_0}\zeta_{JM}(\mathbf{p})=\sum_{M=-J}^{J}D^{(J)*}_{MM'}\{R(g_0,\ g_0^{-1}p)\}\,\zeta_{JM'}(g_0^{-1}p). \tag{25}$$

From this formula it can be seen that under the Lorentz transformation g_0 all the multipole moments $\zeta_{JM}(\mathbf{p})$ of a free particle with momentum $\mathbf{p}$ undergo the same rotation $R(g_0,\mathbf{p})$. Recalling that $\varphi_\sigma(\mathbf{p})$ is the amplitude of the state with momentum p and spin projection σ along the z axis in the rest system, we easily find that the rotation of the polarization vector relative to the momentum $\mathbf{p}$ is determined by the rotation

$$\widetilde{R}(g_0,p)=R^{-1}(\mathbf{p}'/p')\,R(g_0,p)\,R(\mathbf{p}/p),\quad p'=gp,$$

where $R(\mathbf{n})\mathbf{n}_0=\mathbf{n}$. From this it follows that for $g_0=u$, $\widetilde{R}(u,p)=1$, i.e., under space rotations the spin turns along with the momentum; if, on the other hand, $g_0=L_{p'}$, the angle of rotation of the polarization vector relative to the momentum is given by

$$\tan\theta=\frac{(u_0u_0'+1+uu'\cos\alpha)\,u'\sin\alpha}{uu_0'(u_0u_0'+1)+u'\,[u_0'(u_0^2+u^2)+u_0]\cos\alpha+uu_0u'^2\cos^2\alpha}, \tag{26}$$

where $(\mathbf{u},u_0)=(\mathbf{v}(1-v^2)^{-1/2},\ (1-v^2)^{-1/2})$, with $\mathbf{v}$ the original velocity of the particle; u', u_0' are expressed in an analogous way in terms of $\mathbf{v}'$, the speed of the new reference system relative to the original system; α is the angle between $\mathbf{v}$ and $\mathbf{v}'$. For $\alpha=\pi/2$, $\tan\theta=(v'/v)(1-v^2)^{1/2}$ (cf. Wigner's article[14]). It also follows from Eq. (26) that if the original velocity $v=1$ (i.e., is equal to the velocity of light), then $\theta=0$, and a longitudinal polarization remains longitudinal after an arbitrary Lorentz transformation.

The tensor $\xi_{JM}=\int\kappa p_0^{-1}d^3p\,\zeta_{JM}(\mathbf{p})$ does not transform in terms of itself, except in the case in which the particle has a definite momentum $\mathbf{p}$; in this case ξ_{JM} also transforms by Eq. (25), as was already pointed out in a paper by Shirokov.[15] The representation of the Lorentz group given by (25) is, however, reducible; namely, it is the complex conjugate of the representation (6) with s replaced by J. We cannot, however, apply the expansion in irreducible representations given by (13) and (14), since in the general case $\int\kappa p_0^{-1}d^3p\times\sum_M|\zeta_{JM}(\mathbf{p})|^2$ does not have to be bounded.

Furthermore, we shall not assume that the particle has a wave function, and shall treat the general case, in which the density matrix is given in the $|\mathbf{p}\sigma\rangle$ representation: $\rho(\mathbf{p}\sigma;\mathbf{p}'\sigma')$. The density matrix transforms like the product of two wave functions $\varphi_s(\mathbf{p}\sigma)\varphi_s^*(\mathbf{p}'\sigma')$, so that by means of (13) we can change from the momentum to the $(m\rho)$ representation:

$$\rho(m\rho u;\ m'\rho'u')=\sum_{\sigma\sigma'}\int\frac{d^3p}{p_0}\frac{d^3p'}{p_0'}Y_s(m\rho u;\ \mathbf{p}\sigma)$$
$$\times Y^*_s(m'\rho'u';\ \mathbf{p}'\sigma')\,\rho(\mathbf{p}\sigma;\ \mathbf{p}'\sigma'). \tag{27}$$

The Y_s have been defined above.

The matrix ρ transforms by the direct product $\gamma_{m_1\rho_1}\times\gamma^*_{m_2\rho_2}$ of two irreducible representations of the Lorentz group. It is easy to show that $\gamma^*_{m\rho}$ is equivalent to the representation $\gamma_{-m,-\rho}$. For the final expansion of the density matrix we need the expansion of the direct product of two unitary representations $\gamma_{m_1\rho_1}\times\gamma_{-m_2-\rho_2}$ in terms of irreducible representations. This problem has been solved by Naĭmark.[16] We use his result and get

$$c_{m\rho}(u)=\int du_1du_2\rho(m_1\rho_1u_1;\ m_2\rho_2u_2)$$
$$\times T(m_1\rho_1u_1;-m_2,-\rho_2,u_2;\,m\sigma u), \tag{28}$$

where T is the kernel of this integral transformation:

$$T(m_1\rho_1u_1;-m_2,-\rho_2,\ u_2;\ m\rho u)$$
$$=\pi^{3/2}\gamma_{n\sigma}(u_1u_2^{-1})\,\gamma_{n_1\sigma_1}(u_1u^{-1})\,\gamma_{n_2\sigma_2}(u_2u^{-1}),$$

$$n=-\tfrac{1}{2}(m+m_1-m_2),\qquad n_1=\tfrac{1}{2}(m-m_1-m_2),$$
$$n_2=\tfrac{1}{2}(m+m_1+m_2),\qquad \sigma=-\tfrac{1}{2}(\rho+\rho_1-\rho_2),$$
$$\sigma_1=\tfrac{1}{2}(\rho-\rho_1-\rho_2),\qquad \sigma_2=\tfrac{1}{2}(\rho+\rho_1+\rho_2),$$

and $\gamma_{n\sigma}(u)$ is the following function on the unitary group of matrices u:

$$\gamma_{n\sigma}(u) = |u_{21}|^{n-1+i\sigma}u_{21}^{-n} = |u_{21}|^{-1+i\sigma}\exp\{-in\arg u_{21}\}. \quad (29)$$

In Eq. (28) m runs through all the integral values for which n, n_1, and n_2 are integers. Thus the expansion of the density matrix ρ in terms of irreducible unitary representations of the Lorentz group is accomplished in two steps:

$$\rho(\mathbf{p}\sigma;\ \mathbf{p}'\sigma') \to \rho(m_1\rho_1u_1;\ m_2\rho_2u_2) \to c_{m\rho}(u).$$

Each of the steps here is reversible. The transformation inverse to the first step is obtained from (14). We write out only the inverse transformation for the second step:

$$\rho(m_1\rho_1u_1;\ m_2\rho_2u_2) = (2\pi)^{-4}\sum_m\int_{-\infty}^{\infty}d\rho\,(m^2+\rho^2)\int du\,c_{m\rho}(u)$$
$$\times T^*(m_1\rho_1u_1;\ -m_2,\ -\rho_2,\ u_2;\ m\rho u). \quad (28a)$$

We remark that in the general case the system of two particles also does not have a definite wave function and is characterized by a density matrix $\rho(\mathbf{p}_1\sigma_1,\ \mathbf{p}_2\sigma_2;\ \mathbf{p}_1'\sigma_1',\ \mathbf{p}_2'\sigma_2')$, which transforms like the product of two Fock amplitudes $\varphi_{s_1s_2}(\mathbf{p}_1\sigma_1,\ \mathbf{p}_2\sigma_2)\,\varphi^*_{s_1s_2}(\mathbf{p}_1'\sigma_1',\ \mathbf{p}_2'\sigma_2')$. To expand it in irreducible parts one must first change the density matrix ρ from the momentum representation $|\mathbf{p}\sigma\rangle$ into the representation $|m\rho u\rangle$ by means of the transformation (23), and then apply the expansion (28) — (28a). Because of their cumbersomeness we shall not present the final formulas, and shall only remark that the function $c_{m\rho}(u)$ finally obtained, which transforms by the irreducible representation $(m\rho)$, is characterized by specifying, in addition to m and ρ, the following invariants: $m_1\rho_1l_1s_1J_1$ and $m_2\rho_2l_2s_2J_2$.

The writer is deeply grateful to I. S. Shapiro for suggesting this subject and for his constant interest in the work.

APPENDIX

The rotation $\tilde{u}$ is determined from the condition (16)

$$uL_p = k\tilde{u},\ \arg k_{22} = 0 \quad (A.1)$$

and is thus a function of u and $\mathbf{p}$. Let us set $\tilde{u} = u_1u$; then $uL_pu^{-1} = ku_1$. It is clear that uL_pu^{-1} is a pure Lorentz transformation: $uL_pu^{-1} = L_{p'}$. We find the corresponding momentum p′ from the equation $p' = L_{p'}p_0 = uL_pu^{-1}p_0 = up$. There remains for us to find the rotation u_1 such that

$$L_{p'} = ku_1,\quad \arg k_{22} = 0. \quad (A.2)$$

Substituting here the explicit form of $L_{p'}$ in the spinor representation, $L_{p'} = (p_0 + \kappa + (\mathbf{p}\cdot\boldsymbol{\sigma}))/\sqrt{2\kappa(p_0' + \kappa)}$, and making some simple calculations, we get

$$u_1 = u_z(\psi_1)\,u_x(\theta_1)\,u_z(\varphi_1),\quad \psi_1 = -\varphi_1 = \pi/2 + \tan^{-1}(p_y'/p_x'),$$

$$\tan(\theta_1/2) = \sqrt{p_x'^2 + p_y'^2}/(p_0' + \varkappa - p_z'). \quad (A.3)$$

Since the rotation u_1 depends on two independent variables, it is completely determined when the unit vector $\mathbf{n}_1 = u_1^{-1}\mathbf{n}_0$, $\mathbf{n}_0 = (0, 0, 1)$, is prescribed. It can be shown that

$$u^{-1}\mathbf{n}_1 = \frac{\varkappa(p_0+\varkappa)\,\mathbf{n} - [p_0 + \varkappa - (\mathbf{pn})]\,\mathbf{p}}{(p_0+\varkappa)(p - (\mathbf{pn}))}, \quad (A.4)$$

where $\mathbf{n} = u^{-1}\mathbf{n}_0$.

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Translated by W. H. Furry
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