SCATTERING OF ELECTRONS BY LIGHT NONSPHERICAL NUCLEI

E. V. INOPIN and B. I. TISHCHENKO

Physico-technical Institute, Academy of Sciences, Ukrainian S.S.R.

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The scattering of electrons by nonspherical nuclei is treated in the Born approximation. Expressions for the elastic and inelastic cross sections have been derived for the general case of oriented nuclei with arbitrary deformations. The theory is compared with the experiments on the inelastic scattering of electrons by light nuclei.

1. INTRODUCTION

A number of recently published papers[1-3] are devoted to the analysis of the experimental data on light nuclei ($4 \leq A \leq 30$). This analysis indicates that many of these nuclei are appreciably nonspherical. In particular, the data on the quadrupole moments in this region and also on the Coulomb excitation of the low-lying levels suggest that the nonsphericity of some light nuclei is more pronounced than in the well known rare earth region of deformed nuclei. It is highly probable that further investigation of the properties of the low-lying levels reveals the presence of rotational states in other light nuclei as well.

In this connection it is of interest to study in detail the effect of the nonsphericity of the light nuclei on the scattering of electrons with high energies. The investigation of this effect should add to our knowledge of the size of the light nuclei and of the distribution of the electric charge in them. Furthermore, the study of the excitation of the low-lying levels by electrons will be helpful in deciding which type of motion corresponds to any level under consideration (single particle, rotational, or vibrational motion).

The investigation of the nonsphericity of light nuclei by the scattering of electrons is less difficult than in the case of heavy nuclei, owing to the following circumstances: a) the nonsphericity of light nuclei is more pronounced, as already noted; b) the energy of the excited rotational levels is of the order 1 Mev (greater by an order of magnitude than in heavy nuclei), so that it is possible to separate the nonelastically scattered group of electrons, i.e., to observe the nonelastic scattering in its pure form; c) we can use the Born approximation, which greatly simplifies all calculations and makes it possible to obtain a number of simple and rather general results concerning the process under consideration.

In the present paper we study the effect of the nonsphericity of the nucleus on the elastic scattering of electrons, as well as the excitation of the rotational levels by electrons. A few results obtained earlier by Schiff[4] will be re-derived in the interest of clarity of presentation. To analyze the inelastic scattering we used a method which is a generalization of the usual method for the analysis of the elastic scattering of electrons. We also investigate further possibilities of studying the nonsphericity by using oriented nuclei.

2. GENERAL EXPRESSIONS FOR THE CROSS SECTIONS FOR ELASTIC AND INELASTIC SCATTERING OF ELECTRONS

In the calculation of the cross sections for the scattering of electrons by light nuclei we shall use the Born approximation. As is known, the condition of applicability of the Born approximation for a point charge is $Z/137 \ll 1$, which is well satisfied in the case of light nuclei ($Z = 10$ to 15). The finite extent of the charge removes the singularity of the Coulomb potential and therefore leads, generally speaking, to even better conditions for the applicability of perturbation theory. An exception to this are the neighborhoods of those scattering angles for which the cross section, as calculated in the Born approximation, reduces to zero. The comparison of the results of the exact calculations performed by Ravenhall for C^{12} and O^{16} (see reference 5) with those of the Born approximation permits a rough estimate of the extent of the region in which the Born approximation is not valid. It appears that this region is characterized by the quantity $\Delta q/q_0 \approx \pm 5\%$ (q is the momentum transfer, and q_0 is that value of the momentum transfer for which the form factor goes to zero).

The cross section for a process in which the electron is scattered into the direction $\mathbf{n}$ and the nucleus goes from the state i to the state f is, in Born approximation, given by the formula

$$\sigma_{if}(\mathbf{n}) = \frac{1}{(2\pi)^2}\frac{p^2}{v^2}\left|\int \Psi_f^* V \Psi_i \, d\tau\right|^2. \tag{2.1}$$

The wave functions Ψ_i and Ψ_f have the form

$$\Psi_i = u(\mathbf{p}_0) e^{i\mathbf{p}_0\mathbf{r}} \chi_K(\mathbf{X}_1, \ldots \mathbf{X}_A, \theta_s) \sqrt{(2I_0+1)/8\pi^2}\, D^{I_0}_{\mu_0 K}(\theta_s),$$

$$\Psi_f = u(\mathbf{p}) e^{i\mathbf{p}\mathbf{r}} \chi_K(\mathbf{X}_1, \ldots \mathbf{X}_A, \theta_s) \sqrt{(2I+1)/8\pi^2}\, D^{I}_{\mu K}(\theta_s), \tag{2.2}$$

where $\mathbf{p}_0$ and $\mathbf{p}$ are the initial and final momenta of the electron, $\mathbf{r}$ is the coordinate of the electron, $u(\mathbf{p}_0)$ and $u(\mathbf{p})$ are the spinor amplitudes, χ_K is the wave function describing the internal state of the nucleus, $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_A$ are the coordinates of the nucleons, θ_s are the Eulerian angles defining the orientation of the nucleus, I_0 and μ_0 are the total angular momentum of the nucleus and its projection in the ground state, I and μ are the same quantities in the excited rotational state, and K is the projection of the total angular momentum on the symmetry axis of the nucleus. It is assumed that the internal state of the nucleus, described by χ_K, does not change in the considered process; only its rotational state, described by the functions $D^{I_0}_{\mu_0 K}$ and $D^{I}_{\mu K}$, changes.

Substituting for V in (2.1) the expression for the Coulomb interaction of the electron with the protons of the nucleus and summing over the polarizations of the electron, we obtain

$$\sigma^{I\mu}_{I_0\mu_0}(\mathbf{n}) = S_{II_0}\sigma_c(\theta)\left|\int D^{I*}_{\mu K} F(\mathbf{q}, \theta_s) D^{I_0}_{\mu_0 K}\, d\theta_s\right|^2, \tag{2.3}$$

where $\sigma_c(\theta)$ is the cross section for scattering from a point nucleus with charge Z;

$$S_{II_0} = (2I_0+1)(2I+1)/64\pi^4,$$

$$F(\mathbf{q}, \theta_s) = \int e^{i\mathbf{q}\mathbf{r}} \rho(\mathbf{r}, \theta_s)\, d\mathbf{r}$$

is a form factor corresponding to the density distribution (normalized to unity) of the protons in the nucleus,

$$\rho(\mathbf{r}, \theta_s) = \int |\chi_K(\mathbf{r}, \mathbf{X}_2, \ldots \mathbf{X}_A, \theta_s)|^2 d\mathbf{X}_2, \ldots d\mathbf{X}_A$$

$$\mathbf{q} = \mathbf{p}_0 - \mathbf{p}.$$

According to the unified model of the nucleus, this distribution will have an axis of symmetry, whose orientation is given by the angle θ_s, and also a center of symmetry. It can therefore be expressed in the form of a sum,

$$\rho(\mathbf{r}, \theta_s) \equiv \rho(\mathbf{r}, \boldsymbol{\omega}) = \sum_L \rho_L(r) P_L\left(\frac{\mathbf{r}\boldsymbol{\omega}}{r}\right), \tag{2.4}$$

which goes only over the even values of L; $\boldsymbol{\omega}$ is a unit vector defining the direction of the axis of symmetry of the nucleus.

In the general case the wave function of the nucleus in the initial state will be a superposition of the form

$$\varphi^l_{I_0} = \sum_m a^l_m D^{I_0}_{mK}. \tag{2.5}$$

Furthermore, the ensemble of nuclei which make up the target will be a mixture of states of the type (2.5), in which each of these states will be present with a probability w_l. Replacing $D^{I_0}_{\mu_0 K}$ in (2.3) by the expression (2.5), summing over l with the weight w_l and also over the final states with different values μ, we obtain

$$\sigma^I_{I_0}(\mathbf{n}) = S_{I_0 I}\sigma_c(\theta)\sum_{mm'\mu}\rho_{mm'}F_{\mu m}(F_{\mu m'})^*, \tag{2.6}$$

where

$$F_{\mu m} = \int D^{I*}_{\mu K} F(\mathbf{q}, \theta_s) D^{I_0}_{mK}\, d\theta_s, \qquad \rho_{mm'} = \sum_l w_l a^l_m a^{l*}_{m'}.$$

It is obvious that the quantities ρ_{mm} are elements of a $(2I_0+1)$-rowed density matrix describing the spin state of the ensemble of nuclei in the target.

It is known[6] that the density matrix ρ can be written in the form

$$\rho = \sum_{M=-J}^{J}\sum_{J=0}^{2I_0}\langle T^{JM+}\rangle T^{JM}. \tag{2.7}$$

The matrices T^{JM} are given by their matrix elements in the following way:

$$T^{JM}_{mm'} = (-1)^{I_0+m'}(I_0 I_0 m - m' \mid JM), \tag{2.8}$$

The quantities $\langle T^{JM+}\rangle$, the average values of the T^{JM+} (the plus sign denotes Hermitian conjugation), are parameters which define the orientation of the nuclei. In particular,

$$\langle T^{20}\rangle = (-1)^{2I_0}\left[\frac{5I_0(2I_0-1)}{(I_0+1)(2I_0+1)(2I_0+3)}\right]^{1/2} f_2, \tag{2.9}$$

where f_2 is the quadrupole polarization:

$$f_2 = \frac{3\overline{m^2} - I_0(I_0+1)}{I_0(2I_0-1)}. \tag{2.10}$$

It is obvious that $f_2 = 0$ for non-oriented nuclei, and $f_2 = 1$ for maximal orientation ($\overline{m^2} = I_0^2$).

After some simple transformations, which consist mainly in expanding the products of functions D^I_{MK} into a Clebsch-Gordan series and summing over the products of Clebsch-Gordan coefficients, we obtain finally

$$\sigma^I_{I_0}(\mathbf{n}) = (2I+1)\,\sigma_c(\theta) \sum_{MJL_1L_2} (-1)^{I_0+I+1/2\,(J+L_1-L_2)} \langle T^{JM+}\rangle$$
$$\times (I_0IK-K\,|\,L_1 0)(I_0IK-K\,|\,L_2 0)\; Z(L_1 I_0 L_2 I_0;\, IJ)$$
$$\times F_{L_1}(q)\,F_{L_2}(q)\sqrt{4\pi/(2J+1)}\,Y_{JM}(\mathbf{q}/q), \qquad (2.11)$$

where

$$F_L(q) = \frac{1}{4\pi}\int P_L\left(\frac{\omega \mathbf{q}}{q}\right) F(\mathbf{q}, \omega)\, d\omega,$$
$$F(\mathbf{q}, \omega) = \int e^{i\mathbf{q}\mathbf{r}} \rho(\mathbf{r}, \omega)\, d\mathbf{r}. \qquad (2.12)$$

The most important case in practice is that of an axially symmetric external field which orients the nucleus. If the direction of this field, $\mathbf{e}$, is taken as the quantization axis, the only $\langle T^{JM}\rangle$ different from zero are those with $M = 0$. We then have

$$\sigma^I_{I_0}(\mathbf{n}) = (2I+1)\,\sigma_c(\theta) \sum_{JL_1L_2} (-1)^{I_0+I+1/2\,(J+L_1-L_2)} \langle T^{J0}\rangle$$
$$\times (I_0IK-K\,|\,L_1 0)(I_0IK-K\,|\,L_2 0)$$
$$\times Z(L_1 I_0 L_2 I_0;\, IJ)\, F_{L_1}(q)\, F_{L_2}(q)\, P_J(\mathbf{e}\mathbf{q}/q). \qquad (2.13)$$

It is known[7] that $Z(L_1I_0L_2I_0;\, IJ) = 0$ if $L_1 + L_2 + J$ is odd. The sum (2.13) contains only even values of L_1 and L_2; it follows from the property just mentioned that only terms with even J give a contribution to the cross section. In particular, the polarization of the target (which corresponds to $\langle T^{10}\rangle \neq 0$) does not affect the process under consideration at all.

If the nuclei are not oriented, then

$$\langle T^{J0}\rangle = \overline{(-1)^{I_0+m}(I_0 I_0 m - m\,|\,00)}\,\delta_{J0}$$
$$= (-1)^{2I_0}(2I_0+1)^{-1/2}\delta_{J0},$$
$$Z(L_1 I_0 L_2 I_0;\, I0) = \delta_{L_1L_2}(-1)^{I_0-I}(2I_0+1)^{1/2}$$

and (2.13) leads to

$$\sigma^I_{I_0}(\mathbf{n}) = \sigma_c(\theta)(2I+1)\sum_L |(I_0IK-K\,|\,L0)|^2 F_L^2(q). \qquad (2.14)$$

3. FORM FACTORS FOR ELASTIC AND INELASTIC SCATTERING

The cross sections for elastic and inelastic scattering are expressed in terms of form factors $F_L(q)$ which are defined by formulas (2.12). By a simple transformation we obtain

$$F_L(q) = i^L \int \rho(\mathbf{r}, \omega)\, j_L(qr)\, P_L\left(\frac{\mathbf{r}\omega}{r}\right) d\mathbf{r}\, d\omega \qquad (3.1)$$

or, using (2.4),

$$F_L(q) = \frac{4\pi i^L}{2L+1}\int_0^\infty \rho_L(r)\, j_L(qr)\, r^2\, dr. \qquad (3.2)$$

The problem of the scattering of electrons by nonspherical nuclei consists in the determination of the functions $\rho_L(r)$ from the experimentally known functions $F_L(q)$. The problem is solved, in principle, by applying the Hankel transformation to (3.2), which leads to

$$\rho_L(r) = \frac{2L+1}{2\pi^2 i^L}\int_0^\infty F_L(q)\, j_L(qr)\, q^2\, dq. \qquad (3.3)$$

However, as is known from the analysis of the elastic scattering $(L = 0)$, this expression is practically of no use, since the value of $F_L(q)$ is given with considerable error, and only in a limited interval of values q. A more practical method is therefore to take different expressions for ρ_L, substitute these in (3.2), and compare the calculated values of $F_L(q)$ with experiment. One then chooses that function ρ_L which gives best agreement with the experimental data.

After the well studied form factor for elastic scattering, $F_0(q)$, and the density $\rho_0(r)$, the most important quantities are the form factor $F_2(q)$ and the density $\rho_2(r)$.

Let us consider several general properties of the quantities $\rho_2(r)$ and $F_2(q)$. We note, first of all, that the quantity $\rho_2(r)$ cannot be given in an entirely arbitrary way, but has to satisfy definite requirements. Indeed, the obvious condition $\rho(\mathbf{r}, \omega) > 0$ leads to $\rho_0(r) + \rho_2(r)\, P_2(\mathbf{r}\omega/r) > 0$, if we neglect the small terms with $\rho_4(r)$, $\rho_6(r)$, etc. We then have the inequalities

$$\rho_2(r) < 2\rho_0(r) \quad \text{for} \quad \rho_2(r) > 0, \qquad (3.4)$$
$$|\rho_2(r)| < \rho_0(r) \quad \text{for} \quad \rho_2(r) < 0. \qquad (3.5)$$

The quantity $\rho_2(r)$ can be related to the internal quadrupole moment of the nucleus,

$$Q_0 = Ze\int \rho(\mathbf{r}, \omega)\,(3z^2 - r^2)\, d\mathbf{r} = Ze\frac{8\pi}{5}\int_0^\infty r^4 \rho_2(r)\, dr. \qquad (3.6)$$

It follows from (3.2) and (3.6) that for small q

$$F_2(q) \approx -\frac{1}{30}\frac{Q_0}{Ze}q^2. \qquad (3.7)$$

It can be shown in an entirely analogous fashion that the form factors with $L = 4, 6$, etc. are related to the higher multipole moments, so that the quantities $F_L(q)$ vanish rapidly with increasing L.

We now list the expressions for six densities of the simplest form and the corresponding form factors, using the following notation:

$$a^2 = \int \rho_2(r)\, r^2\, dr \Big/ \int \rho_2(r)\, dr$$

is the mean square radius of the nonspherical part of the charge distribution,

$$x = qa,\ y = r/a,\ \Lambda(y) = 4\pi a^5 Ze\rho_2(r)/125\,Q_0,$$
$$\Phi(x) = -30a^2 ZeF_2(q)/Q_0.$$

1. Uniform Model:

$$\Lambda(y) = \begin{cases} {}^1/_{10}\,({}^3/_5)^{5/2}, & y \leqslant ({}^5/_3)^{1/2} \\ 0, & y = ({}^5/_3)^{1/2} \end{cases},$$

$$\Phi(x) = \frac{45}{z^2}\left\{\cos z - \frac{4\sin z}{z} + \frac{3\,\mathrm{Si}\,z}{z}\right\}, \qquad z = ({}^5/_3)^{1/2} x$$

2. Shell Model:

$$\Lambda(y) = {}^1/_{50}\,\delta(y-1), \qquad \Phi(x) = 15\, j_2(x).$$

3. Gaussian "Wine Bottle" Model:

$$\Lambda(y) = {}^1/_3\,(5/2\pi)^{1/2} y^2 \exp(-{}^5/_2\, y^2),$$
$$\Phi(x) = x^2 \exp(-x^2/10).$$

4. Exponential "Wine Bottle" Model I:

$$\Lambda(y) = {}^4/_3\, y \exp(-\sqrt{20}\, y),$$
$$\Phi(x) = x^2(1 + x^2/20)^{-3}.$$

5. Exponential "Wine Bottle" Model II:

$$\Lambda(y) = (3\sqrt{30}/4)\, y^2 \exp(-\sqrt{30}\, y),$$

$$\Phi(x) = x^2(1 + x^2/30)^{-4}.$$

6. Parabolic Model:

$$\Lambda(y) = \begin{cases} {}^1/_{10}\,({}^5/_7)^{5/2} y^2, & y \leqslant ({}^7/_5)^{1/2} \\ 0, & y \geqslant ({}^7/_5)^{1/2} \end{cases},$$

$$\Phi(x) = 75\, j_3(z)/z, \qquad z = ({}^7/_5)^{1/2} x.$$

In the case of the shell-like density the conditions (3.4) and (3.5) can, of course, not be fulfilled. However, the δ function may be interpreted as a function which is different from zero in a region of width Δ, with the constant value Δ^{-1}. This is possible if $q\Delta \ll 1$ in the considered region of momentum transfers q.

4. ELASTIC SCATTERING

Let us consider the effect of the nonsphericity of the nucleus on the elastic scattering. In the simplest case of elastic scattering from a nucleus with spin zero $(I = I_0 = 0)$ we obtain

$$\sigma_0^0(\theta) = \sigma_c(\theta)\, F_0^2(q). \tag{4.1}$$

Since the quantity $\rho_0(r)$, which defines $F_0(q)$, is, according to (2.4), equal to

$$\rho_0(r) = \frac{1}{4\pi}\int \rho(\mathbf{r}, \omega)\, d\omega, \tag{4.2}$$

it is obvious that the elastic scattering from a nonspherical nucleus with spin zero and charge density $\rho(\mathbf{r}, \omega)$ is the same as the scattering from a spherical nucleus with density $\rho_0(r)$. In particular, if we take for the distribution a uniformly charged ellipsoid of revolution with semi-axes a and b, then

$$\rho_0(r) = \begin{cases} \rho_0, & 0 \leqslant r \leqslant b \\ \rho_0[1 - \sqrt{(r^2 - b^2)/(a^2 - b^2)}], & b \leqslant r \leqslant a, \\ 0, & a \leqslant r \end{cases} \tag{4.3}$$

i.e., the nonsphericity appears in experiment as a smoothing out of the nuclear boundary.

Let us consider now a nonspherical nucleus with a density $\rho(\mathbf{r}, \omega)$ such that the surfaces of equal density are similar ellipsoids of revolution with the ratio of half-axes $a/b = \eta$. After a coordinate transformation which takes these ellipsoids into spheres of the same volume, $\rho(\mathbf{r}, \omega)$ goes over into $\rho_{\mathrm{sph}}(r)$. Let us compare the mean square radii of the nonspherical nucleus with density $\rho(\mathbf{r}, \omega)$ and of the spherical nucleus with density $\rho_{\mathrm{sph}}(r)$. It is easily seen that

$$\overline{r^2_{\mathrm{nonsph}}} = \frac{2 + \eta^2}{3\eta^{2/3}}\,\overline{r^2_{\mathrm{sph}}}. \tag{4.4}$$

The factor $(2+\eta^2)/3\eta^{2/3}$ is equal to unity for $\eta = 1$ and greater than unity for any other value of η. Hence the nonsphericity leads to an apparent increase in the nuclear radius as compared to the spherical nucleus.

The elastic scattering from non-oriented nuclei with non-zero spin is given by the formula

$$\sigma_{I_0}^{I_0}(\theta) = \sigma_0^0(\theta) + \Delta_{I_0}(\theta), \tag{4.5}$$

where

$$\Delta_{I_0}(\theta) = \sigma_c(\theta)\,(2I_0 + 1) \sum_{L=2}^{\infty} |(I_0 I_0 K - K \mid L0)|^2 F_L^2(\theta). \tag{4.6}$$

Obviously, $\Delta_{I_0} = 0$ not only for $I_0 = 0$, but also for $I_0 = \frac{1}{2}$.

We restrict ourselves to the first term in the sum (4.6) and assume $K = I_0$ (as is known, this relation can only be invalid for $K = \frac{1}{2}$), and obtain

$$\Delta_{I_0}(\theta) = \frac{5I_0(2I_0 - 1)}{(I_0 + 1)(2I_0 + 3)}\,\sigma_c(\theta)\, F_2^2(\theta). \tag{4.7}$$

The additional contribution to the cross section in the presence of nuclear spin is connected with the fact that in this case elastic scattering accompanied by a transfer of angular momentum L from the electron to the nucleus (according to the scheme $\mathbf{I}_0 + \mathbf{L} = \mathbf{I}_0$) becomes possible. Owing to the presence of a center of symmetry in the nucleus the odd values of L are excluded, and the lowest possible value is $L = 2$. This effect could possibly be observed in measurements of the ratio of the scattering cross sections for the nuclei Mg^{24} and Mg^{25}.

Let us consider now the effect of the orientation of nuclei on the elastic scattering. We restrict ourselves to the case of an axially-symmetric orienting field. In view of what was said above about the fast decrease of $F_L(\theta)$ with growing L, we may keep only terms containing F_0^2 and F_0F_2 in expression (2.4). With the help of (2.9) we then obtain

$$\sigma_{I_0}^{I_0}(\mathbf{n}) = \sigma_{I_0(\mathrm{nonorient})}^{I_0}(\theta) + \bar{f}_2\,\gamma_{I_0}\,\sigma_c(\theta)\, F_0(\theta)\, F_2(\theta)\, P_2\left(\frac{\mathbf{eq}}{q}\right), \tag{4.8}$$

where $\sigma^{I_0}_{I_0\,(\text{nonorient})}$ is the cross section for scattering from non-oriented nuclei, given by (4.5), and

$$\gamma_{I_0} = 10\frac{3K^2 - I_0(I_0+1)}{(I_0+1)(2I_0+3)}. \qquad (4.9)$$

The most striking feature of expression (4.8) is its dependence on the azimuthal angle, i.e., its azimuthal asymmetry. Evidently we observe the greatest asymmetry effect if we first measure the scattering with $\mathbf{e}\cdot\mathbf{q}/q = 1$ (the directions of the orienting field and of the momentum transfer are identical) and then with $\mathbf{e}\cdot\mathbf{q} = 0$ (the two directions are perpendicular to each other). We can therefore take the following quantity as a measure of the azimuthal asymmetry:

$$\delta_{I_0}(\theta) = \frac{\sigma^{I_0}_{I_0}(\mathbf{eq}/q=1) - \sigma^{I_0}_{I_0}(\mathbf{eq}/q=0)}{\sigma^{I_0}_{I_0}(\text{nonorient})(\theta)}. \qquad (4.10)$$

From (4.8) and (4.5) we obtain for the azimuthal asymmetry

$$\delta_{I_0}(\theta) = \frac{3}{2} f_2 \gamma_{I_0} \frac{F_0(\theta)F_2(\theta)}{F_0^2(\theta) + C^2_{I_0}F_2^2(\theta)},$$

$$C_{I_0} = \sqrt{5}(I_0 2K0 \mid I_0 K). \qquad (4.11)$$

The function $\delta_{I_0}(\theta)$ has extremal values at those angles θ for which $F_0 = \pm C_{I_0}F_2$. Since the quantity C_{I_0} is always close to unity, this condition can only be fulfilled in the neighborhood of the minimum of the elastic scattering. In particular, if the elastic scattering amplitude goes through zero at the angle θ_0, there will be a maximum and a minimum of the azimuthal asymmetry near θ_0. In the extremal points we obtain for δ_{I_0}

$$\delta_{I_0\,\text{extr}} = \pm\frac{3}{2}\left[\frac{5I_0(2I_0-1)}{(I_0+1)(2I_0+3)}\right]^{1/2}\nu(K) f_2. \qquad (4.12)$$

The sign in this expression has to be chosen such as to correspond to $F_0 = C_{I_0}F_2$ or $F_0 = -C_{I_0}F_2$, respectively. We see that the investigation of the scattering from oriented nuclei gives us the possibility to determine the sign of F_2 and, hence, of the quadrupole moment Q_0.

We have $\nu(K) = 1$ for $K = I_0$ and $\nu(K) = -1$ for $K = 1/2$. Hence the sign of the asymmetry for $K = 1/2$ is the opposite of that in the usual case, $K = I_0$. The factor in front of f_2 in (4.12) is of order unity and depends rather weakly on the spin of the nucleus. If $f_2 = 10\%$, the maximal asymmetry will be equal to 15% for a nuclear spin of $3/2$, and 20% for $I_0 = 5/2$.

5. INELASTIC SCATTERING. COMPARISON WITH EXPERIMENT

In the consideration of the excitation of the rotational levels of non-oriented nuclei we keep in (2.14) only the terms containing F_2, and obtain

$$\sigma^I_{I_0}(\mathbf{n}) = 5\sigma_c(\theta)\,|(I_0 2K0 \mid IK)|^2 F_2^2. \qquad (5.1)$$

It follows from this formula that only the levels with the spins I_0+2, I_0+1, I_0-1, and I_0-2 can be excited. The levels with the spins I_0-1 and I_0-2 can, evidently, occur only if the level order in the rotational band is reversed. The levels with spins which differ from I_0 by more than two units can only be excited through the terms with $L \geq 4$ in the expression (2.14). These terms are smaller than the term with $L = 2$ by $1 \div 2$ orders of magnitude. The observation of the excitation of these levels is therefore very difficult.

From (5.1) we obtain for the relative probability of the excitation of the levels belonging to the same rotational band and having the spins I and I′, the usual relation of the theory of the Coulomb excitation:

$$\sigma^I_{I_0}(\mathbf{n})/\sigma^{I'}_{I_0}(\mathbf{n}) = |(I_0 2K0 \mid IK)|^2/|(I_0 2K0 \mid I'K)|^2. \qquad (5.2)$$

If the nuclei are oriented, we obtain from (2.13), again keeping only the terms with $L = 2$,

$$\sigma^{\text{orient}}_{I_0 I}(\mathbf{n}) = \sigma^{\text{nonorient}}_{I_0 I}(\mathbf{n})\left\{1 + \frac{2}{3}\delta_{I_0 I}P_2(\mathbf{eq}/q)\right\}, \qquad (5.3)$$

where $\sigma^{\text{nonorient}}_{I_0 I}(\mathbf{n})$ is given by expression (4.1), and

$$\delta_{I_0 I} = -\frac{3}{2}\left[\frac{250\,I_0(2I_0-1)(2I_0+1)}{7(I_0+1)(2I_0+3)}\right]^{1/2} W(I_0 I 22 \mid 2I_0) f_2. \qquad (5.4)$$

It is obvious that the quantity $\delta_{I_0 I}$ determines the magnitude of the azimuthal asymmetry in a fashion analogous to (4.10). It is interesting to note that, in the inelastic scattering, this quantity is independent of the scattering angle and of the parameters describing the nuclear density. The values of the quantity $\delta_{I_0 I}/f_2$ are listed in Table I. We see that this quantity is in most cases close to unity, but can be either negative or positive. In particular, it is always negative for the levels with $I = I_0 + 2$, and positive for the levels with $I = I_0 + 1$.

TABLE I. Values of the quantity $\delta_{I_0 I}/f_2$

$I_0 \backslash I$	I_0+2	I_0+1	I_0-1	I_0-2
3/2	−3/7	15/14	−3/2	—
5/2	−75/98	255/196	−75/98	−15/7
7/2	−1	19/14	−5/14	−15/7
9/2	−90/77	15/11	−15/154	−15/7

Let us now turn to the available experimental data on the inelastic scattering of electrons from light nuclei. The excitation of the levels with spin 2^+ in the even-even nuclei Mg^{24}, Si^{28}, and S^{32} was studied by Helm;[8] Fregeau[9] investigated the C^{12}

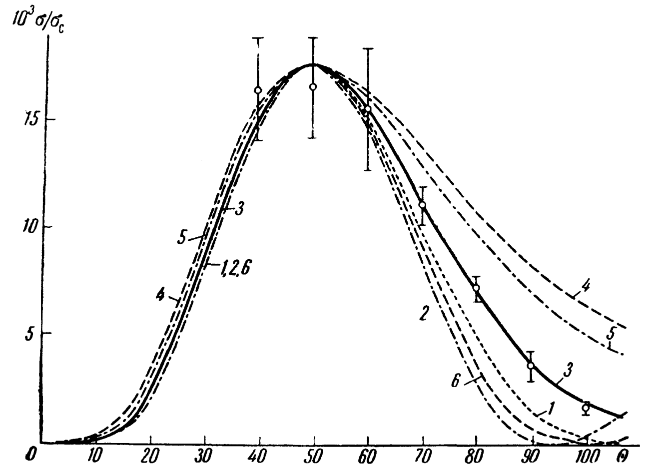

FIG. 1. Cross section for the inelastic scattering of electrons by the Mg^{24} nucleus with excitation of the 1.37-Mev level. The numbers of the calculated curves in this and the following figures correspond to the numbers of the models (see section 3).

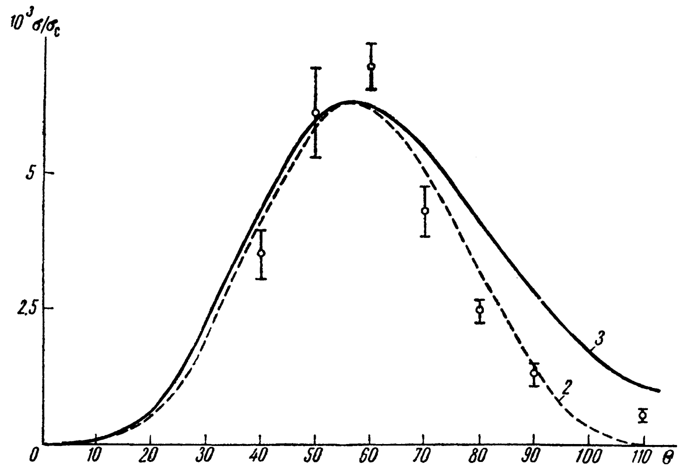

FIG. 2. Cross section for the inelastic scattering of electrons by the Si^{28} nucleus with excitation of the 1.79-Mev level.

nucleus. The levels Mg^{24} 0^+ ($E_0 = 0$), 2^+ ($E_2 = 1.37$ Mev), and 4^+ ($E_4 = 4.17$ Mev) can be regarded as a rotational band ($K = 0$), since they have the correct spins and parities, and also $E_4/E_2 \approx 3$, which is close to the value 3.33 required by the unified model. For the remaining of the above-mentioned nuclei the levels with spin and parity 4^+ have not yet been observed. But it is entirely possible that further investigation of the parities and spins of these nuclei will lead to their discovery. We shall regard these levels as rotational and attempt to draw conclusions on the distribution $\rho_2(r)$ in these nuclei by comparing the experimental data on the angular distribution of the inelastically scattered electrons with the results of the calculations.

TABLE II

Nucleus	Number of model	$a \cdot 10^{13}$cm^{-1}	$Q \cdot 10^{24}$cm^{-2}	η
C^{12}	3	2.73	0.200	1.5
Mg^{24}	1	3.93	0.822	
	2	4.16	0.801	
	3	3.94	0.898	1.8
	4	3.94	1.12	
	5	3.94	1.04	
	6	4.06	0.814	
Si^{28}	2	3.68	0.440	
	3	3.49	0.492	1.3
S^{32}	2	4.80	0.736	
	3	4.54	0.825	1.4

In Fig. 1 we make this comparison for Mg^{24}. We see, first of all, that widely different density distributions $\rho_2(r)$ lead to quite similar angular distributions. The parameters of these density distributions, listed in Table II, show that the values of the root mean square radii and of the quadrupole moments are also nearly identical for the different distributions. Apparently, the Gaussian "wine bottle" distribution gives the best agreement. Looking now at the criteria expressed by the relations (3.4) and (3.5), we see that these are not satisfied at all by the density distributions 1, 2, and 6, so that these types of distribution have to be discarded in the present case. For the remaining distribution types listed in Table I, relation (3.4), which corresponds to a positive quadrupole moment of the Mg^{24} nucleus, is well satisfied, but not relation (3.5). We have to except those large distances r, for which the values of $\rho_0(r)$ and $\rho_2(r)$ are so small that in this region the functions can be replaced by others (for which the above-mentioned relations are fulfilled) without affecting appreciably the value of the form factors in this region.

The value $Q_0 = 0.898$ for Mg^{24} obtained here is in satisfactory agreement with the value $Q_0 = 0.75$ which follows from the data of Alkhazov et al.[10] on the Coulomb excitation, and with the value $Q_0 = 0.70$ from the paper of Delyagin and Shpinel'[11] on the resonance scattering of γ rays.

The comparison of experiment and calculation for the remaining three nuclei is pictured in Figs. 2, 3, and 4. We see that the Gaussian "wine bottle" distribution is in good agreement with the experimental data for these nuclei as well. It gives a somewhat poorer fit in the case of the Si^{28} nucleus. The parameters of the distributions shown are given in Table II. The value $Q_0 = 0.492$ for Si^{28} obtained here is in agreement with the results of Alkhazov et al.[10] on the Coulomb excitation, which indicate that $Q_0 \leq 0.61$. The value $Q_0 = 0.2$ for

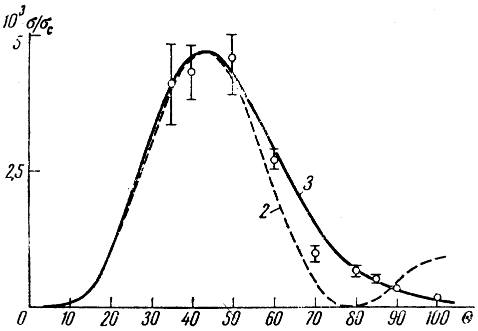

FIG. 3. Cross section of the inelastic scattering of electrons by the S^{32} nucleus with excitation of the 2.25-Mev level.

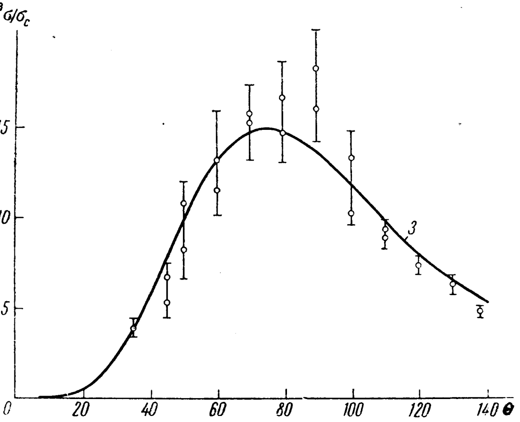

FIG. 4. Cross section of inelastic scattering of electrons by the C^{12} nucleus with excitation of the 4.43-Mev level.

C^{12} agrees well with the value $Q_0 = 0.3$ which follows from the data of Devons, Manning, and Towle,[12] if we take account of the low accuracy of their results.

The values of Q_0 obtained by us do not characterize directly the nonsphericity of the nuclei under consideration, because Q_0 depends not only on the nonsphericity but also on the size of the nucleus and on its charge. To obtain a criterion for the nonsphericity of the nucleus it is useful to introduce the concept of the equivalent ellipsoid: a uniformly charged nucleus with the shape of an ellipsoid of revolution with semi-axes a and b. This ellipsoid can be determined by the nonsphericity parameter $\eta = a/b$ and the mean square radius $\overline{r^2_{\text{nonsph}}} = \frac{3}{5} R^2$ (R is the radius of the equivalent sphere). It is easily shown, using relation (4.4), that this ellipsoid has the quadrupole moment

$$Q_0 = \frac{6}{5} \frac{\eta^2 - 1}{\eta^2 + 2} ZR^2. \qquad (5.5)$$

Using our values Q_0 and the values of R obtained from the experiments on the elastic scattering, we find the values of the nonsphericity parameter listed in the last column of Table II. We see that the nonsphericity for Mg^{24} is very large. Si^{28} has small nonsphericity. It is possible that in the case of a nucleus with such large nonsphericity the conditions for the validity of the strong coupling approximation in the unified model are violated to a considerable degree.

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