EXPANSION OF THE AMPLITUDE FOR A REACTION WITH PRODUCTION OF THREE LOW-ENERGY PARTICLES IN POWERS OF THE THRESHOLD MOMENTA

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It is shown that in the expansion near threshold of the amplitude for an arbitrary process in powers of the threshold momenta, the linear terms and the part of the quadratic terms that determines the angular distributions can be obtained from a consideration of the analytic properties of the amplitude.

1. As has been shown by Gribov,[1] the wave function of three low-energy particles whose interaction can be described by a potential that depends only on the distances between them differs from the wave function at zero energy by the factor

$$1+ik_{12}a_{12}+ik_{13}a_{13}+ik_{23}a_{23},$$

$$\mathbf{k}_{ik}=(\mathbf{p}_i m_k-\mathbf{p}_k m_i)/(m_i+m_k), \qquad (1)$$

where $\mathbf{p}_i$ and m_i are the momentum and mass of the i-th particle, and a_{ik} is the scattering amplitude of particles i and k at zero energy.

In the present paper this result is obtained for the amplitude of an arbitrary process in which the final state contains three low-energy particles with $|\mathbf{p}_i| \ll m$, where m is the minimum mass determining the radius of the interaction ($\hbar = c = 1$). The derivation is based on the analytic properties of the amplitude, which is represented as the combination of all possible Feynman diagrams. In addition it is shown that this method makes it possible to find the terms of second order in the threshold momenta $|\mathbf{p}_i|$ that determine the angular correlations of the particles produced in the reaction,[1] and to find the energy dependence of the amplitude in reactions of the type $a + A \rightarrow b + B$ near the threshold for production of particles of a third type $c + C$; this energy dependence has been considered by Baz'.[2]

The extension of the results to the case of reactions with more than three particles in the final state is obvious.

2. Let us consider the amplitude of a reaction in which the collision of two particles with the four-dimensional momenta p_1' and p_2' leads to the production of three particles with momenta p_1, p_2, p_3. For simplicity we shall suppose that the particles have no spin, charge, etc., and are characterized by their momenta and masses. None of them have resonance interactions with each other in the possible states ($a_{ik} \sim 1/m$).

The amplitude of such a process is a function of five invariants ($p^2 = \mathbf{p}^2 - p_0^2$)

$$T = T\{(p_1+p_2)^2,\ (p_1+p_3)^2,\ (p_2+p_3)^2,\ (p_1'-p_1)^2,\ (p_1'-p_2)^2\}$$

$$\left(S = 1 + i(2\pi)^4\,\delta\left(\sum p_f - \sum p_i'\right)T\right). \qquad (2)$$

Near the threshold of the reaction, in the center-of-mass system $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = 0$, $|\mathbf{p}_i| \ll m$, so that the invariants in Eq. (2) can be expanded in powers of the threshold momenta:

$$-(p_i+p_k)^2 = (m_i+m_k)^2 + \mathbf{k}_{ik}^2\, m_i m_k/\mu_{ik}, \qquad (3)$$

$$-(p_1'-p_i)^2 = M_1'^2 + m_i^2 - 2\varepsilon_1 m_i + 2\mathbf{p}_1'\mathbf{p}_i + \varepsilon_1'\mathbf{p}_i^2/m_i. \qquad (4)$$

Here $\mu_{ik} = m_i m_k/(m_i + m_k)$, and if T were an analytic function of the invariants, its expansion in powers of the threshold momenta would contain only terms of the types $\mathbf{p}_i\mathbf{p}_k/m^2$ and $\mathbf{p}_i\mathbf{p}_1'/m\epsilon_1'$, that is, terms quadratic in the threshold momenta or terms that are linear but depend on the angle between a particle in the initial state and one in the final state.* For brevity we shall hereafter call both types quadratic terms, since a term $\mathbf{p}_i\mathbf{p}'$ makes a contribution to the cross-section integrated over the angle in question that is quadratic in the threshold momenta.

The amplitude T near threshold is not, however, an analytic function of its arguments, since the threshold point

*For example, the amplitude for the decay $\tau \to 3\pi$[1] does not depend on the invariants of the type $(p_i' - p_i)^2$, and no terms $\mathbf{p}_i\mathbf{p}_i'$ occur in the expansion.

$$(p_i + p_k)^2 = -(m_i + m_k)^2,$$
$$(p_1 + p_2 + p_3)^2 = -(m_1 + m_2 + m_3)^2 \qquad (5)$$

is a branch point of T as a function of the invariants (5) [$(p_1+p_2+p_3)^2$, the square of the total energy in the center-of-mass system, is a combination of the invariants $(p_i+p_k)^2$]. The amplitude T depends on arguments of the type $I - I_0$, where I is some invariant or combination of invariants and I_0 is the value taken by this invariant or combination at some singular point. If the singularities (5) are the only ones close to the values I taken by the corresponding invariants (for the quantities (5) we have $I - I_0 \sim \mathbf{p}^2 \ll m^2$), then the appearance of terms other than quadratic in the expansion of T is due to the dependence of T on the quantities $(p_i + p_k)^2 + (m_i + m_k)^2$, $(p_1 + p_2 + p_3)^2 + (m_1 + m_2 + m_3)^2$, since (with $I' - I_0 \sim m^2$) the other arguments of T can lead only to terms of the types $\mathbf{p}_i \cdot \mathbf{p}_k/m^2$ and $\mathbf{p}_i \cdot \mathbf{p}'/m\epsilon'$ in the expansion.

The absence of any other nearby singularities in addition to (5) means, first, that the threshold for production of the particles with masses m_1, m_2, and m_3 is far from the thresholds for the production of any other groups of particles that are possible in the given reaction, that is*

$$(m_1 + m_2 + m_3)^2 - (m_1' + m_2' + m_3')^2 \approx m^2. \qquad (6)$$

Second, it is necessary that the singularities with respect to the invariants must lie far from the values taken by these invariants. These are invariants of the type of the momentum transferred, and for them the fact that this condition is satisfied follows, for example, from the analysis given by Landau[3] (for an amplitude represented as a sum of Feynman diagrams).

3. Thus, the terms other than quadratic ones in the expansion of T can be given only by the contribution from the diagrams that lead to singularities of the type (5). According to Landau a singularity with respect to $(p_i + p_k)^2$ at the value $-(m_i + m_k)^2$ arises in those diagrams that contain at least one pair of lines corresponding to virtual particles with the masses m_i and m_k and connecting the part of the diagram to which the momenta p_i and p_k are attached with the rest of the diagram (Fig. 1b and 1c). Similarly, for a singularity with respect to $(p_1 + p_2 + p_3)^2$ at the point $-(m_1 + m_2 + m_3)^2$ the diagram must be divisible into two parts connected only by three lines of virtual particles with the masses m_1, m_2,

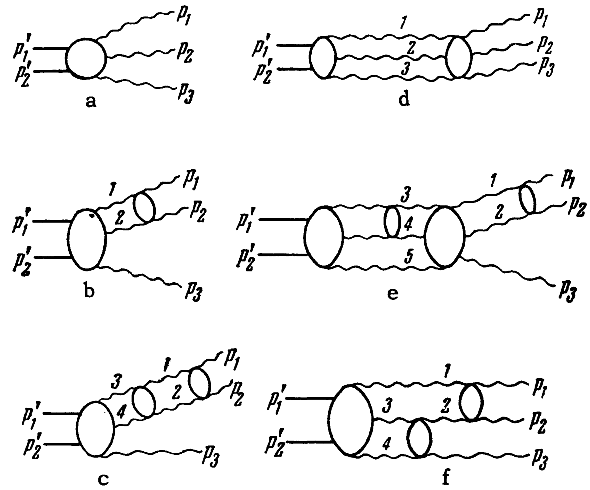

FIG. 2

m_3, with the initial momenta attached to one part and the final momenta to the other (Fig. 1d and 1e).

We shall draw as a block the parts of the diagrams in which one cannot distinguish the lines of the specified types, the connecting lines. Then the diagrams of Fig. 1a have no singularities (5) and contribute only to the quadratic terms. In the other diagrams the treatment for finding the terms other than quadratic can be simplified to a remarkable extent. As is well known, the integral corresponding to an arbitrary diagram can be represented in the form (after integrating over all virtual momenta):

$$\int_0^1\int_0^1 \ldots \frac{d\alpha\, d\beta\, d\gamma\, d\delta \ldots \delta(\alpha + \beta + \gamma + \delta + \ldots - 1)}{(A\alpha + B\beta + C\gamma + D\delta + \ldots)^n}, \qquad (7)$$

where A, B, C, D,... are the denominators corresponding to all the lines of the diagram:

$$A = q_A^2 + m_A^2, \quad q_A \sim \sum_i \alpha_i P_i. \qquad (8)$$

Here P_i is the external momentum contained in the i-th line.

If A and B are the denominators corresponding to a pair of connecting lines, then at the point $(p_i + p_k)^2 = -(m_i + m_k)^2$ the denominator in Eq. (7) goes to zero for (cf. reference 3)*

$$A = B = 0, \quad \alpha + \beta = 1, \quad \gamma = \delta = \ldots = 0. \qquad (9)$$

At a point beyond the threshold, $(p_i + p_k)^2 = -(m_i + m_k)^2 + O(\mathbf{p}^2)$, the denominator in Eq. (7)

*The argument that follows will make it evident that this condition is not essential, and in each individual case one can figure out the changes that are introduced by the presence of another nearby threshold.

*We are considering the case in which there is only one pair of connecting lines in the diagram; an analogous treatment of more complicated diagrams leads to the same results.

can go to zero in a region $\gamma \sim \delta \sim \ldots \sim \mathbf{p}^2/m^2$, since $A \sim B \sim \mathbf{p}^2$ and C, D, and so on do not vanish at $\gamma = \delta = \ldots = 0$. But then in the region where the denominator $\sim \mathbf{p}^2$ and where we cannot expand in a series with respect to the threshold momenta (obviously this is the only region in which terms other than the quadratic terms can arise, since in regions where the denominator does not go to zero only quadratic terms appear), we can neglect the dependence on these momenta in all the lines except the connecting lines. In fact, in the region near the values (9) the denominators C, D, ... are multiplied by quantities $\gamma \sim \delta \sim \ldots \sim \mathbf{p}^2/m^2$, and in A and B the momenta of the other lines occur multiplied by the corresponding quantities $\gamma \sim \delta \sim \ldots \sim \mathbf{p}^2/m^2$.

Consequently, in expanding T in powers of the threshold momenta to obtain the terms different from $\mathbf{p}_i \cdot \mathbf{p}_k/m^2$ and $\mathbf{p}_i \cdot \mathbf{p}'/m\epsilon'$ we can neglect the threshold momenta everywhere in the diagrams that represent T, except in the denominators corresponding to the connecting lines.

4. To separate out the terms in which we are interested from an arbitrary diagram we must examine the dependence on the threshold momenta that arises only from the connecting lines; that is, we must examine the integrals over the virtual momenta belonging to the connecting lines. As will become evident from the further argument, each of these integrals can be considered independently of the presence of other connecting lines in the given diagram. For example, from the diagram of Fig. 1c we can separate out independently the contributions from the lines 1, 2 and 3, 4; from the diagram of Fig. 1e, those from lines 1, 2 and from 3, 4, 5; and so on. The only exception is the case in which the same line belongs to two different groups of connecting lines (line 5 in Fig 1e and line 1 in Fig. 1f). All such groups must be considered simultaneously.

The main types of connecting lines are shown in Fig. 2. The integral corresponding to the diagram of Fig. 2a is of the form

$$-(2\pi)^{-4}\int T_1(q)\frac{d^4q}{\left[(P_1+q)^2+m_1^2\right]\left[q^2+m_2^2\right]}T_2(q), \qquad (10)$$

where $P_1 = p_1 + p_2$, and $T_1(q)$ and $T_2(q)$ are the expressions corresponding to the blocks T_1 and T_2 in the diagram (they can also contain connecting lines). The factor

$$\frac{-1}{(2\pi)^4} = \frac{(-i)^2}{(2\pi)^8}(2\pi)^4$$

comes from the two connecting lines and the δ function from one of the blocks T_1 or T_2. The integration over dq_0 gives the sum of the integrals over d^3q of the various residues with respect to q_0. Of these only the one corresponding to $q_0 = -(m_2^2 + q^2)^{1/2}$ (we close the contour in the upper half of the q_0 plane) gives denominators that cannot be expanded in terms of the threshold momenta (the singular part). The residues of the integral over dq_0 that correspond to the denominators of the lines contained in $T_1(q)$ and $T_2(q)$ cannot contribute to the singular part, since a singularity of the type (5) arises only from the simultaneous vanishing of the denominators corresponding to a pair of connecting lines, i.e., the lines 1 and 2 in the case of Fig. 2a. Furthermore, contributions to the singular part come only from the free propagation functions, so that we have substituted in Eq. (10) only $\Delta_F^{(0)}$.

Then (with $\mathbf{p}_1 + \mathbf{p}_2 = -\mathbf{p}_3$) Eq. (10) can be written in the form

$$T'(0) + \frac{\pi i}{(2\pi)^4}\int\frac{T_1(\mathbf{q})\,d^3\mathbf{q}\,T_2(\mathbf{q})}{\sqrt{m_2^2+\mathbf{q}^2}\left[\left(P_{10}-\sqrt{m_2^2+\mathbf{q}^2}\right)^2-m_1^2-(\mathbf{p}_3+\mathbf{q})^2\right]}$$

In $T'(0)$ the dependence of the connecting lines 1, 2 on the threshold momenta can give only quadratic terms (if T_1 and T_2 contain no connecting lines, then $T'(0)$ contains only quadratic terms in the expansion in terms of the threshold momenta). In the singular term the denominator goes to zero for $\mathbf{q} \sim \mathbf{p}$.

If we break up the integral over $|\mathbf{q}| = q$ into two integrals,

$$0 \leqslant q \leqslant \lambda \text{ and } \lambda \leqslant q, \qquad p \ll \lambda \ll m,$$

then the integral from λ to ∞ can be transferred to $T'(0)$, since in this integral the lines 1 and 2 give only quadratic terms. In the range from 0 to λ we can expand in terms of $|\mathbf{p}|$ and $|\mathbf{q}| \ll m$. In this region $T_i(\mathbf{q}) \sim T_i(0)$. We then have for the expression (10):

$$T(0) - \frac{i\pi\mu_{12}}{(2\pi)^4 m_1 m_2}T_1(0)\int_0^\lambda\frac{d^3\mathbf{q}}{(\mathbf{q}+\mu_{12}\mathbf{p}_3/m_1)^2-\mathbf{k}_{12}^2-i\varepsilon}T_2(0), \qquad (11)$$

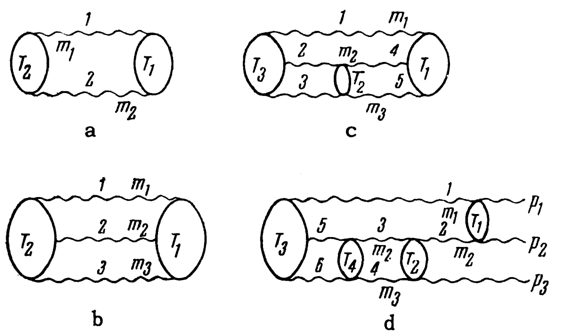

FIG. 2

or, putting the quadratic terms with T(0),

$$T(0) + \frac{\mu_{12}}{16\pi m_1 m_2} |\mathbf{k}_{12}| T_1(0) T_2(0) = T(0) + ik_{12} a_{12}^{(1)} \frac{T_2(0)}{2\sqrt{m_1 m_2}}, \tag{12}$$

$$T_1(0) = (4\pi i (m_1 + m_2)/\sqrt{m_1 m_2})\, a_{12}^{(1)}. \tag{13}$$

If we are interested only in the terms linear in the threshold momenta, then in the second term of Eq. (12) we can set $\mathbf{p}_i = 0$ in $T_1(0)$ and $T_2(0)$. Then $a_{12}^{(1)}$ is the contribution of the diagram T_1 to the scattering amplitude of particles 1 and 2 at zero energy, and $T_2(0)/2(m_1 m_2)^{1/2}$ is the contribution that the diagram T_2 makes to the amplitude of the process in question at the threshold energy of T(0).

In just the same way we must separate off from the T(0) in Eq. (12) the linear terms corresponding to the other connecting lines. It is then found that the connecting lines of the diagrams of Fig. 2b and c make the following contributions to the singular parts:

$$-\frac{\pi^2}{2m_1 m_2 m_3} T_2(0) \int_0^\lambda \frac{d^3\mathbf{q}\, d^3\mathbf{q}'}{\mathbf{q}'^2/2\mu_{23} + \mathbf{q}^2/2\mu_1 - E - i\varepsilon} T_1(0) \tag{14}$$

for diagram 1b (sic!), and

$$\frac{i\pi^3}{2m_1 (m_2 m_3)^2} T_1(0) \times \int_0^\lambda \frac{d^3\mathbf{q}\, d^3\mathbf{q}'\, d^3\mathbf{q}''\, T_2(0)\, T_3(0)}{(\mathbf{q}'^2/2\mu_{23} + \mathbf{q}^2/2\mu_1 - E - i\varepsilon)(\mathbf{q}''^2/2\mu_{23} + \mathbf{q}^2/2\mu_1 - E - i\varepsilon)} \tag{15}$$

for diagram 1c (sic!); here

$$1/\mu_1 = 1/m_1 + 1/(m_2 + m_3), \quad E = \mathbf{k}_{12}^2/2\mu_{12} + p_3^2/2\mu_3 \tag{16}$$

are the notations of Gribov's paper.[1]

It is easy to see that the expressions (14) and (15) do not contribute to the terms linear in the threshold momenta. In fact, for large q or q′ (q″) each integral has no singular denominators and contains only E. A singularity arises for $q \sim q' \sim |\mathbf{p}|$, but in this region the contribution to the integrals (14) and (15) is $\sim p^4$.

It can be shown that the contribution to the linear terms from a diagram of the type of Fig. 2d is given by the second term in Eq. (12), with appropriate values of $a_{12}^{(1)}$ and $T_2(0)$ (this contribution arises from lines 1 and 2 in Fig. 2d).

Thus terms linear in the threshold momenta are given only by diagrams of the type of Fig. 2a. The total contribution to the linear terms is obtained by summation of terms of the type (12) separated from all the connecting lines (of the type of Fig. 2a) of all possible diagrams. It can be shown that each product $a_{12}^{(1)} T_2(0)$ occurs in Eq. (12) only once. Therefore the sum of all the diagrams gives the expression

$$T = T(0)\,[1 + ik_{12}a_{12} + ik_{23}a_{23} + ik_{13}a_{13}], \tag{17}$$

where T(0) is the amplitude of the reaction in question when the threshold momenta are zero. The subsequent terms of the expansion in the threshold momenta contain either quadratic terms, or linear terms that depend on an angle between initial and final momenta, and thus make a quadratic contribution to the cross section integrated over this angle.

5. Neglect of the momenta in all the denominators except those of the connecting lines would mean neglect of terms of the types $\mathbf{p}_i \cdot \mathbf{p}_k/m^2$ and $\mathbf{p}_i \cdot \mathbf{p}'/m\epsilon'$.

There are also, however, quadratic terms of a different type: $|\mathbf{p}_l| k_{ik}$ or $E^{1/2} k_{ik}$ (for the case of three particles no terms $k_{ik} k_{lm}$ appear in the final state). These differ from the former terms in that they determine angular correlations of the particles produced in the reaction in all orbital angular-momentum states, whereas the former terms give correlations only in states with relative orbital angular momentum 1 (cf. reference 1). If no state with relative orbital angular momentum 1 is possible, as is the case for the $\tau \to 3\pi$ decay,[1] then the terms of first degree in the cosine of the angle between the momenta of the final particles must vanish in the final formulas, and to accuracy up to quadratic terms the angular correlations are completely determined by the terms $|\mathbf{p}_l| k_{ik}$ and $E^{1/2} k_{ik}$. Just as in Eq. (17), these terms are determined by the dependences on the momenta in the connecting lines. They arise from the integrals of the diagrams of Fig. 2d on the integrations over the virtual momenta of lines 1, 2 and 1, 3, 4 or 1, 5, 6, etc., and only from these integrations.

For the singular part of these integrals we have an expression of the form [in Eq. (18) $T_3(0)$ is the contribution from T_3 and T_4 in the diagram of Fig. 2d when integrated over the momenta of lines 1 and 2, 3 and 4]

$$a_{12}^{(1)} a_{23}^{(2)} \frac{T_3(0)}{\sqrt{8 m_1 m_2 m_3}} \frac{2\mu_{23}^{1/2}}{(2\pi)^4} \times \int_0^\lambda \frac{d^3\mathbf{q}'}{\mathbf{q}'^2/2\mu_{23} + \mathbf{q}^2/2\mu_1 - E - i\varepsilon} \frac{d^3\mathbf{q}}{(\mathbf{q} + \mu_{12}\mathbf{p}_3/m_2)^2 - \mathbf{k}_{12}^2 - i\varepsilon}. \tag{18}$$

From this we can separate out the quadratic terms of the type with which we are concerned,

$$\varkappa^2 a_{12}^{(1)} a_{23}^{(2)} T_3(0) J_{31}/\sqrt{8 m_1 m_2 m_3}, \tag{19}$$

where

$$J_{31} = \frac{1}{3\pi}\frac{\varkappa\gamma_1^{1/2}}{\mu_{12}p_3}\left\{\left[1 - \frac{(k_{12}+\mu_{12}p_3/m_2)^2}{\varkappa^2\gamma_1}\right]^{3/2}\cos^{-1}\frac{k_{12}+\mu_{12}p_3/m_2}{\varkappa\gamma_1^{1/2}}\right.$$

$$\left. - \left[1 - \frac{(k_{12}-\mu_{12}p_3/m_2)^2}{\varkappa^2\gamma_1}\right]^{3/2}\cos^{-1}\frac{k_{12}-\mu_{12}p_3/m_2}{\varkappa\gamma_1^{1/2}}\right\};$$

$$\varkappa = \sqrt{2E\left(\frac{m_1m_2m_3}{M}\right)^{1/2}}, \qquad \gamma_1 = \left(\frac{\mu_1}{\mu_{12}}\right)^{1/2},$$

$$M = m_1 + m_2 + m_3. \tag{20}$$

Summation of the contributions from all the diagrams gives the angular correlations of the particles produced, with accuracy up to terms quadratic in the threshold momenta (terms depending on the angle between $\mathbf{p}_3$ and $\mathbf{k}_{12}$). When we include linear terms and the quadratic terms that determine the correlation of $\mathbf{p}_3$, $\mathbf{k}_{12}$, the amplitude is

$$T = T(0)[1 + ik_{12}a_{12} + ik_{13}a_{13} + ik_{23}a_{23} + \varkappa^2a_{12}a_{23}J_{13} + \varkappa^2a_{12}a_{13}J_{23} + \varkappa^2a_{13}a_{23}J_{12} + \varkappa^2a_{23}a_{13}J_{21}] + O(p^2/m^2); \tag{21}$$

J_{13} and so on are obtained from Eq. (20) by interchange of indices. The remaining terms quadratic in the threshold momenta do not depend on the angle between $\mathbf{p}_3$ and $\mathbf{k}_{12}$.

By a similar argument one can determine the energy dependence of the amplitude for reactions of the type $a + A \to b + B$ near the threshold for production of a pair of particles $c + C$.[2] Then we can separate out from diagrams of the type of Fig. 2a, with connecting lines corresponding to particles c and C, the term linear in

$$k = \{-(p_a + P_A)^2 - (m_c + M_c)^2\}^{1/2}.$$

The amplitude will have the form

$$T_1(0) + ika_{aA\to cC}\,a_{cC\to bB}, \tag{22}$$

where $T_1(0)$ is the amplitude of the process $a + A \to b + B$ at energy equal to the threshold energy for the reaction $a + A \to c + C$. $a_{aA\to cC}$ and $a_{cC\to bB}$ are the amplitudes of the reactions in question at threshold. Below threshold $k \to -i|k|$.

Inclusion of spin and isotopic variables for the particles does not give rise to any difficulties, and means only that in the products $T(0)a_{ik}$ in Eq. (21) we must sum over all processes that are possible for the particles i and k at zero energy.

In conclusion the writer expresses his deep gratitude to V. N. Gribov for valuable discussions.

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265