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OSCILLATIONS OF AN INHOMOGENEOUS PLASMA IN A MAGNETIC FIELD

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Small oscillations of a hot plasma contained by a magnetic field are treated by means of the kinetic equation in the "drift" approximation without the collision integral.[1] Two waves can be excited when the propagation vector is in the plane perpendicular to the direction of the unperturbed magnetic field: a slow (drift) wave with a propagation velocity of the order of the mean drift velocity of the electrons (ions) in the unperturbed state, and a magneto-acoustic wave. The first is found in an inhomogeneous plasma only. If certain relations obtain between the zeroes of the magnetic field gradients, the plasma density, and the temperature, the drift current can cause amplification of these waves. Criteria for an instability of this kind are obtained.

1. INTRODUCTION. FORMULATION OF THE PROBLEM

In this paper, which is the first part of a work on oscillations in an inhomogeneous plasma, we consider the simplest geometric configuration for the unperturbed plasma: a straight pinch with an arbitrary distribution of longitudinal current.

We will consider oscillations at frequencies much higher than the collision frequency but smaller than the ion Larmor frequency. All characteristic dimensions are assumed to be large compared with the Larmor radius of the ions (electrons). Under these conditions it is convenient to describe the behavior of the plasma in terms of the drift approximation, using the kinetic equation.[1]

We introduce a cylindrical coordinate system for oscillations characterized by wave vectors which are transverse to the magnetic field (the magnetic field in the perturbed and unperturbed plasma is along φ).

The kinetic equation is

$$\frac{\partial f}{\partial t} + \nabla(\mathbf{v}_{\mathrm{dr}} f) + \frac{\partial}{\partial v_\varphi}\frac{dv_\varphi}{dt} f = 0. \qquad (1)$$

Here $f(v_\varphi, u, r, t)$ is the electron (ion) distribution function, μ is the magnetic moment of the electron (ion) and v_{dr} is the drift velocity, which is

$$\mathbf{v}_{\mathrm{dr}} = c\frac{[\mathbf{E}\times\mathbf{H}]}{H^2} + \frac{c}{e}\frac{\mu}{H^2}[\mathbf{H}\times\nabla H] + \frac{cm}{eH^2}\frac{v_\varphi^2}{r^2}[\mathbf{r}\times\mathbf{H}]. \qquad (2)$$

The first term in Eq. (2) corresponds to the electric drift, the second to the diamagnetic drift, and the third to the "centrifugal" drift.

The charge density and the current density are found from the distribution function

$$n = \int f d\mu dv_\varphi, \quad \mathbf{j} = c[\mathbf{H}\times\nabla p_\perp]/H^2 - \frac{c}{H}(p_\perp - p_\parallel)\operatorname{curl}\frac{\mathbf{H}}{H},$$

$$p = H\int \mu f d\mu dv_\varphi.$$

Maxwell's equations for the self-consistent field are

$$n_i = n_e \text{ (quasi-neutrality equation)} \qquad (3)$$

$$\operatorname{curl}\mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{4\pi nM}{H}c\frac{d}{dt}\frac{\mathbf{E}}{H}, \qquad \operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}. \qquad (4)$$

2. SMALL-OSCILLATION EQUATIONS

We investigate the time behavior of small departures from equilibrium.

The equilibrium state is given by the equation

$$\frac{1}{r}\frac{\partial}{\partial r}rH_0 = -\frac{4\pi}{H_0}\frac{d}{dr}\left(H_0\int\mu f_0 d\mu\right) \qquad (5)$$

The zeroth current term $\mathbf{j}_0$ is due to the motion of electrons with velocity

$$(c/en_0)[\mathbf{H}_0\times\nabla(p_i + p_e)]H_0^{-2}.$$

The ions are assumed to be fixed (cf., for example, reference 2). We seek small corrections to the equilibrium quantities in the form

$$A = A(r)\exp\{i(kz - \omega t)\}.$$

The equation for the perturbed distribution function is

$$-i(\omega - kv_{\mathrm{dr}}^0)f + \nabla(\mathbf{v}_{\mathrm{dr}} f_0) - \frac{\partial}{\partial v_\varphi}\frac{v_\varphi}{r}c\frac{E_z}{H_0}f = 0. \qquad (6)$$

Here

$$v_{\mathrm{dr}}^0 = -\frac{cm}{e}\frac{H_0'}{H_0} + \frac{cmv_\varphi^2}{eH_0 r} + c\frac{E_r^0}{H_0}, \qquad (7)$$

(primes denote differentiation with respect to r).

The correction to the distribution function is

$$f_1 = f_0 \frac{H}{H_0} + \left[\left(\frac{f_0}{H_0}\right)' - \frac{1}{r}\frac{v_\varphi}{H_0}\frac{\partial f_0}{\partial v_\varphi}\right]\frac{(kc\mu/e)H + icE_z}{\omega - kv^0_{\mathrm{dr}}}. \quad (8)$$

Substituting f_1 in Maxwell's equations we obtain three equations for the three quantities H_φ, E_r, and E_z. By means of simple transformations we eliminate E_r and H_φ, obtaining a single second-order differential equation for E_z, which is given here without the intermediate steps:

$$\frac{d}{dr}\frac{c^2}{V_A^2}H_0\frac{(1+4\pi a/H_0)E_z' - i(\omega 4\pi b/cH_0)E_z}{(1+4\pi a/H_0) - \omega^2/k^2V_A^2} - \frac{k^2c^2}{V_A^2}H_0E_z$$

$$+\frac{H_0}{r}\frac{(c^2/V_A^2)E_z' - i(k^2c4\pi b/\omega H_0)E_z}{(1+4\pi a/H_0) - \omega^2/k^2V_A^2} = 0, \quad (9)$$

where

$$a = \frac{2p_0}{H_0} + H_0\sum\int\left(\frac{f_0}{H_0}\right)'\frac{kc\mu^2/e}{\omega - kv^0_{\mathrm{dr}}}d\mu dv_\varphi,$$

$$b = icH_0\sum\int\left(\frac{f_0}{H_0}\right)'\frac{\mu}{\omega - kv^0_{\mathrm{dr}}}d\mu\, dv_\varphi,$$

$$V_A^2 = H_0^2/4\pi n_0 M. \quad (10)$$

The summation is taken over ions and electrons. The integrals over μ in Eq. (10) are taken along the real axis if $\mathrm{Im}(\omega) > 0$ and along a contour such that the pole of the integrand lies between the real axis and the contour if $\mathrm{Im}(\omega) < 0$, in agreement with the rules of analytic continuation.[3]

Thus, the problem of oscillations of an inhomogeneous plasma is reduced to that of finding the eigenvalues and eigenfunctions of a second-order differential equation with variable coefficients (9).

3. DRIFT WAVES

We consider oscillations whose frequencies satisfy the inequality $\omega \ll kH_0/\sqrt{4\pi n_0 M}$ (correspondingly, the phase velocity $\omega/k \ll V_A$).

Equation (9) can be simplified in this case:

$$\frac{c^2}{V_A^2}\frac{d^2E_z}{dr^2} - \frac{1}{r}\frac{4\pi k^2 cib}{\omega H_0}\frac{E_z}{1+4\pi a/H_0} = 0. \quad (11)$$

In Eq. (11) we have omitted the small parameter $\omega^2/(kV_A)^2$ in the coefficient of the second derivative.

Equation (11) is a generalized equation for the eigenvalues of a complex operator which is not self-adjoint.

We write this equation in the form

$$E'' - [U(x,\omega,k) + iV(x,\omega,k)]E = 0, \quad (12)$$

where U and V are real functions.

For solutions of Eq. (12) which converge in a bounded region of space we can write the integral conditions

$$\int |E'|^2 dx + \int U(x,\omega,k)|E^2|dx = 0,$$

$$\int V(x,\omega,k)|E|^2 dx = 0, \quad (13)$$

where the integration is taken over the entire region occupied by the plasma. The second of these conditions can be realized only if the quantity $V(x,\omega,k)$ passes through zero at some point in space.

We consider space localized solutions for which $U(x,\omega,k)$ passes through zero at the same point. The condition

$$U(x_0,\omega,k) + iV(x_0,\omega,k) = 0 \quad (14)$$

plays the role of a dispersion equation which relates ω and k.

Near this point Eq. (12) becomes an Airy equation with complex argument

$$E'' + (U_0' + iV_0')xE = 0.$$

We investigate the "high-gradient" case $(d/dr)_0 \gg 1/r$. In this case the "dispersion equation" is

$$b = icH_0\sum\int\left(\frac{f_0}{H_0}\right)'\frac{\mu d\mu dv_\varphi}{\omega - kcE_0/H_0 + kc\mu H_0'/eH_0} = 0. \quad (15)$$

It should be noted that although no curvature term appears in the dispersion equation, the derivation cannot be extended to the case $r \to \infty$ because terms of order ω/kV_A, which have been omitted in obtaining Eq. (11), become important at large values of r.

We now investigate Eq. (15). As an example we take $f_{0\mu}$ in the form $f_{0\mu} = n_0\delta(\mu - \mu_0)$. (All the particles at any point have the same magnetic moment.) Here we have already carried out the integration over the longitudinal velocity ($f_{0\mu} = \int f_0(\mu, v_\varphi)dv_\varphi$).

Integrating with respect to μ in Eq. (14) we have

$$\omega^2 = k^2 v_{\mathrm{dr}}^{02}\left(1 - 2\frac{\mu_0'}{\mu_0}\frac{n_0}{H_0}\Big/\left(\frac{n_0}{H_0}\right)'\right).$$

Whence, an instability develops if

$$\frac{\mu_0'}{\mu_0}\frac{n_0}{H_0}\Big/\left(\frac{n_0}{H_0}\right)' > \frac{1}{2}. \quad (16)$$

The case in which the Maxwellian distribution

$$f_0 = \frac{n_0H_0}{T_0}e^{-\mu H/T_0}$$

is used in Eq. (15) has been investigated in reference 4. The instability criterion is of the form

$$0 < -\frac{H_0^2/8\pi n_0T_0 + \partial\ln T_0/\partial\ln H_0}{1 - \partial\ln T_0/\partial\ln H_0} < 1. \quad (17)$$

4. MAGNETOACOUSTIC WAVES

We consider oscillations at frequencies $\omega \sim kV_A$. Omitting terms of order v_{dr}^0/V_A in Eq. (9) we obtain [for high gradients $(d/dr)_0 \gg 1/r$]:

$$\frac{d}{dr}\frac{c^2}{V_A^2}H_0\frac{(1+4\pi a/H_0)E_z' - i(\omega 4\pi b/cH_0)E_z}{(1+4\pi a/H_0) - \omega^2/k^2V_A^2} - \frac{k^2c^2}{V_A^2}H_0E_z = 0. \quad (18)$$

This is the equation for the magnetoacoustic waves. In the quasi-classical approximation ($E_z \sim \exp\{i\int k_r dr\}$, where k_r is a slowly varying function of r) we have

$$(k^2+k_r^2)(1+4\pi a/H_0) + \omega^2/V_A^2 = 0. \quad (19)$$

We consider oscillations close to the boundary of the stability region (small buildup). The real part of Eq. (19) determines the frequency of the oscillations and the imaginary part determines the increment factor ν $(\omega = \omega_1 + i\nu)$.

For a given value of H_0' there is always a pole in one of the integrands. For small increments $(\nu \ll \omega_1)$ we have

$$\int\frac{\mu^2(f_0/H_0)'\,d\mu}{\omega - k|v_{dr}^0|} = \fint\frac{\mu^2(f_0/H_0)'\,d\mu}{\omega - k|v_{dr}^0|} + \pi i\,\mathrm{Res}.$$

($\fint$ is the integral in the sense of the principal value).

Separating real and imaginary parts in Eq. (19) we have

$$\omega^2 = (k_r^2+k^2)\left(V_A^2 + 2\frac{p_0}{n_0}\right),$$
$$\nu = \frac{3}{2}\pi\omega_1\left\{\mu^2\left(\frac{f_0}{H_0}\right)'\right\}_{\omega_1 = kv_{dr}^0} \Big/ \left(\frac{p_0}{H_0^2}\right)' \quad (20)$$

In evaluating the integrals we have taken into account that for the magnetoacoustic branch $\omega \gg k\overline{v_{dr}^0}$; $\overline{v_{dr}^0}$ is the mean drift velocity. Thus the instability criterion $\nu > 0$ is of the form

$$H_0'(f_0/H_0)' < 0. \quad (21)$$

For a Maxwellian distribution function:

$$\partial\ln T_0/\partial\ln H_0 > 1. \quad (22)$$

The physical nature of this instability and the instability in the drift branch may be described as follows.

Electrons which move with the unperturbed velocity v_{dr}^0, close to the phase velocity of the wave, drift towards points of zero gradient, acquiring energy by virtue of the interaction with the wave. If the condition in Eq. (21) is satisfied [or, correspondingly, the condition in Eq. (17)] this interaction leads to an instability (the electrons lose energy to the wave). The time required for the development of this instability is large because in the case being considered ($\omega \gg k\overline{v}_{dr}$) the number of particles in resonance with the wave is an exponentially small quantity.

The increments can become large when the mean velocity of the unperturbed electron drift is of the order of the velocity of the magnetoacoustic wave. This situation can arise if the gradients are large at equilibrium: $H_0^{-1}\partial H_0/\partial r > R_H^{-1}$ (R_H is the ion Larmor radius). However, in this case the drift approximation no longer applies for the ions.

A simplified analysis of this case can be carried out on the basis of a "cold" ion model ($T_i \ll T_e$). In this case, Eq. (18) with $T_i = 0$ is valid up to characteristic dimensions of the order of the electron Larmor radius and the increments are

$$\nu = \pi\omega_1\left\{\mu^2\left(\frac{f_0}{H_0}\right)'\right\}_{\omega_1 = kv_{dr}^0} \Big/ \left(\frac{p_0}{H_0^2}\right)'. \quad (23)$$

(In this we assume that $\nu \ll \omega_1$ since we are interested in the stability boundaries.) The plasma is unstable if

$$\frac{\partial\ln T_e}{\partial\ln H_0}\left(\frac{\mu_1 H_0}{T_e} - 2\right) > \frac{H_0^2}{8\pi n_0 T_e} + \frac{\mu_1 H_0}{T_e},$$
$$\mu_1 = |eV_A H_0/cH_0'|. \quad (24)$$

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