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FIELD OF A CHARGED PARTICLE IN A MOVING MEDIUM

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The field produced by the motion of a charge through a moving medium is considered. The energy losses of the charge due to Cerenkov radiation and excitation of plasma waves are determined.

1. The problem of determining the field of a charged particle in a moving medium has a long history. The "inverse" Cerenkov effect, i.e., the Cerenkov effect as seen in the rest system of the particle, was first considered by Tamm.[1] This question has also been considered in a number of later papers.[2] The interest in the inverse Cerenkov effect arises in connection with a suggestion made by Veksler[3] concerning the application of this effect in the acceleration of charged particles; Veksler's proposal is based on the use of a fast electron stream as the moving medium.

Below we consider certain properties of the field of a charge in a moving medium which have not been considered in the earlier analyses. The results which are obtained may be useful, for example, in experimental investigations of relativistic plasma beams and in the investigation of the interaction of beams of charged particles with plasma oscillations.

Following Ryazanov,[5] we use the phenomenological equations of classical electrodynamics for a moving medium which have been developed by Tamm.[4]

2. Let $\epsilon(\omega)$ and $\mu(\epsilon)$ be the dielectric constant and the permeability of the medium in the system in which it is at rest. We introduce the quantity

$$\varkappa = \varepsilon\mu - 1. \tag{1}$$

Now suppose that the medium moves with the 4-velocity u_i:

$$u_{1,2,3} = u_{x,y,z}/\sqrt{1-u^2/c^2} \qquad u_4 = c/\sqrt{1-u^2/c^2},$$

where $\mathbf{u}$ is the three-dimensional velocity of the medium. The dielectric-magnetic permeability tensor is then written in the form

$$\varepsilon_{ikst} = \mu^{-1}(\delta_{is} + \varkappa c^{-2} u_i u_s)(\delta_{kt} + \varkappa c^{-2} u_k u_t) \tag{2}$$

Maxwell's equations assume the form

$$\partial_l F_{ke} + \partial_k F_{le} + \partial_e F_{lk} = 0, \qquad \partial_k H_{ik} = -(4\pi/c)\, j_i,$$
$$H_{ik} = \varepsilon_{ikst} F_{st}, \tag{3}$$

where $\partial_i(-\nabla, \partial/c dt)$ is the four-dimensional gradient, F_{ik} is a field tensor (electric field and magnetic induction field) and H_{ik} is a field tensor (magnetic field and electric induction field). The product of two 4-vectors A_i and B_i is written in the form

$$(AB) = A_i B_i = A_4 B_4 - A_1 B_1 - A_2 B_2 - A_3 B_3.$$

Introducing the 4-potential A_i in accordance with the relation $F_{ik} = \partial_i A_k - \partial_k A_i$ and imposing the following supplementary condition on the components:

$$(\partial_k + \varkappa c^{-2} u_k u_i \partial_i) A_k = 0,$$

we find a simple equation for the potentials:

$$\mu^{-1}[\partial_k^2 + \varkappa c^{-2}(u_k \partial_k)^2](\delta_{st} + \varkappa c^{-2} u_s u_t) A_t = -(4\pi/c)\, j_s. \tag{4}$$

In a medium which is at rest $(\mathbf{u} = 0)$ and in vacuum $(\kappa = 0)$ Eq. (4) becomes the well-known relation.

We may note that if the quantity κ in Eq. (1) is a function of frequency in the medium at rest, in the moving medium κ will depend on the four-dimensional product (uk), where k $(\mathbf{k}, \omega/c)$ is the four-dimensional wave vector.

3. We now consider the solution of Eq. (4). We multiply both sides by u_s and sum over s. Introducing the notation

$$L = \partial_k^2 + \varkappa c^{-2}(u_k \partial_k)^2, \tag{5}$$

we have

$$L(Au)(1+\varkappa) = -\frac{4\pi}{c}(ju). \tag{6}$$

On the other hand, from Eq. (4) it follows that

$$LA_s = -(4\pi\mu/c)\, j_s - u_s c^{-2} L\varkappa (Au). \tag{7}$$

Substituting the value of (Au) from Eq. (6) in Eq. (7) we have

$$A_s = -\frac{4\pi}{c} L^{-1}\left[\delta_{st} - \frac{\varkappa}{c^2(1+\varkappa)} u_s u_t\right]\mu j_t. \tag{8}$$

Equation (8) determines the field in the moving medium produced by an arbitrary source. To this same expression we can add the electromagnetic field in a medium at rest, using the appropriate Lorentz transformation.

4. We now assume that a point charge q with a velocity **v**, moves through a medium which itself is moving with velocity **u**. The current density and charge density are written in the form:

$$\mathbf{j} = \mathbf{v} q \delta(\mathbf{x} - \mathbf{v}t), \qquad \rho = q\delta(\mathbf{x} - \mathbf{v}t). \tag{9}$$

We use the Fourier expansion of the δ function and obtain the following solutions for the vector and scalar potentials:

$$\mathbf{A} = \frac{q}{2\pi^2}\int\left[\frac{\mathbf{v}}{c} - \frac{\varkappa}{1+\varkappa}\frac{\mathbf{u}}{c}\frac{1-\mathbf{uv}/c^2}{1-u^2/c^2}\right] \times \frac{\mu \exp\{i\mathbf{k}(\mathbf{x}-\mathbf{v}t)\}\, d\mathbf{k}}{(\mathbf{kv})^2/c^2 - k^2 + (\varkappa/c^2)(\mathbf{ku}-\mathbf{kv})^2/(1-u^2/c^2)},$$

$$\varphi = \frac{q}{2\pi^2}\int\left[1 - \frac{\varkappa}{1+\varkappa}\frac{1-\mathbf{uv}/c^2}{1-u^2/c^2}\right] \times \frac{\mu \exp\{i\mathbf{k}(\mathbf{x}-\mathbf{v}t)\}\, d\mathbf{k}}{(\mathbf{kv})^2/c^2 - k^2 + (\varkappa/c^2)(\mathbf{ku}-\mathbf{kv})^2/(1-u^2/c^2)} \tag{10}$$

5. The energy loss of the charge in the moving medium is given by

$$\frac{dW}{dx} = q\frac{\mathbf{Ev}}{v}\Big|_{\mathbf{x}=\mathbf{v}t} = -\frac{q_i}{2\pi^2 v}\int\left[1 - \frac{v^2}{c^2} - \frac{\varkappa}{1+\varkappa}\frac{(1-\mathbf{uv}/c^2)^2}{1-u^2/c^2}\right] \times \frac{\mu(\mathbf{kv})\, d\mathbf{k}}{(\mathbf{kv})^2/c^2 - k^2 + (\varkappa/c^2)(\mathbf{ku}-\mathbf{kv})^2/(1-u^2/c^2)} \tag{11}$$

It should be kept in mind that the quantity $\kappa = \epsilon\mu - 1$ is a function of the scalar product

$$(uk) = (\mathbf{kv} - \mathbf{ku})/(1-u^2/c^2)^{1/2}.$$

When the sign of the wave vector **k** changes, the integrand in Eq. (11) becomes the complex conjugate expression. Hence the integration is limited to values of **k** for which $\mathbf{k}\cdot\mathbf{v} > 0$ and the real part which is obtained is multiplied by two.

If the velocity vector of the charge **v** does not coincide in direction with the velocity of the medium **u**, a deflecting force acts on the charge in addition to the decelerating force.

6. We consider the integrand in Eq. (11). It is pure imaginary if we consider a transparent medium (ϵ and μ real). A contribution to the real part of the integral can be obtained only from poles along the path of integration. As is apparent from Eq. (12), poles will exist if one of the following conditions is satisfied:

$$1 + \varkappa = 0, \quad \text{i.e.,} \quad \varepsilon(k_i u_i) = 0, \tag{12}$$

or

$$\frac{(\mathbf{kv})^2}{c^2} - k^2 + \frac{\varkappa}{c^2}\frac{(\mathbf{ku}-\mathbf{kv})^2}{1-u^2/c^2} = 0. \tag{13}$$

The first equation determines the losses which are the analog of the longitudinal losses in a medium at rest. When $\mathbf{u} = 0$, from Eq. (11) we obtain the usual condition, $\epsilon(\omega) = 0$. The second equation determines the Cerenkov loss. This equation will now be investigated in detail.

7. We take

$$\mathbf{kv} = \omega = ck/n, \qquad \mathbf{ku} = ku\cos\psi. \tag{14}$$

Then, from Eq. (13) we obtain an equation which determines the phase velocity c/n for waves which propagate at an angle ψ with respect to the direction of motion of the medium:

$$c^2/n^2 - c^2 + \eta(c/n - u\cos\psi)^2 = 0. \tag{15}$$

The solution of this equation is

$$\frac{c}{n_{1,2}} = \frac{\eta\, u\cos\psi \pm \sqrt{c^2 + \eta(c^2 - u^2\cos^2\psi)}}{1+\eta}, \tag{16}$$

where

$$\eta = \varkappa/(1-u^2/c^2) = (\varepsilon\mu - 1)/(1-u^2/c^2). \tag{17}$$

We now plot the "surface of normals": In any direction, at an angle ψ, with respect to the velocity of the medium **u**, we lay off the segment $c/n(\psi)$. The ends of these segments then form the surface of normals. In other words, if there is a radiation pulse at the origin, the surface of constant phase for waves of a given frequency (after unit time) is the surface of normals. It is apparent from Eq. (16) that the surface of normals is a surface of rotation with axis along **u**. Furthermore, Eq. (16) defines one surface rather than two because

$$c/n_1(\pi - \psi) = -c/n_2(\psi), \tag{18}$$

i.e., both solutions describe one surface (if 1/n is negative, we assume that the phase velocity of the wave is opposite to the direction of the wave vector).

It is apparent from Eq. (16) that the properties of the surface of normals are determined by the parameter $\sqrt{\epsilon\mu}\, u/c$. The entrainment of the radiation of the moving medium appears in the fact that when $\epsilon\mu > 1$ the surface of normals is displaced in the direction of the moving medium (with respect to the origin). If $\epsilon\mu < 1$ the surface is displaced in the opposite direction.

From the surface of normals c/n we can go over to a surface of refractive indices n. When $\epsilon\mu u^2/c^2 < 1$ this surface is an ellipsoid, when $\epsilon\mu u^2/c^2 = 1$ it is a parabaloid, and when $\epsilon\mu u^2/c^2 > 1$ it is a hyperboloid of rotation.

8. Equation (13) indicates a remarkable property of media for which

$$\varepsilon\mu = 1 + \mathrm{const}/\omega^2. \tag{19}$$

In media of this kind the surface of normals is a sphere of fixed radius in any system of coordinates, regardless of the velocity of the medium. An example of a medium for which (19) is satisfied is an isotropic electron plasma.

We may also note that if the velocity of the moving medium is large $(1 - u^2/c^2 \ll 1)$ the argument of ϵ and μ, $(\omega - \mathbf{k}\cdot\mathbf{u})/(1 - u^2/c^2)^{1/2}$, becomes large and the asymptotic expressions can be used for ϵ and μ:

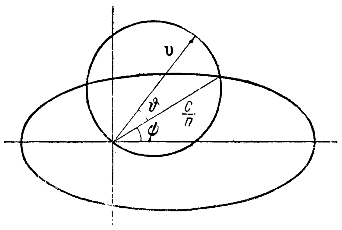
$$\mu = 1, \quad \varepsilon = 1 - 4\pi nc^2/m\omega^2. \tag{20}$$

In this case the propagation relation is the same as for an electron plasma [cf. Eq. (19)]. It should be noted, however, that when $\omega = \mathbf{k}\cdot\mathbf{u}$ this statement no longer applies. In the latter case the c/n surface is a sphere whose diameter is the vector $\mathbf{u}$, laid off from the origin.

9. From the definition of the refractive index (14) it follows that the vector $\mathbf{k}$, which satisfies Eq. (13), forms an angle ϑ with the velocity of the particle $\mathbf{v}$; this angle is given by the familiar relation

$$\cos\vartheta = c/vn(\psi). \tag{21}$$

Real waves can be emitted only when $c^2/v^2n^2(\psi) \leq 1$. This is the familiar Cerenkov radiation condition.



10. If the surface of normals is known the Cerenkov cone can be determined graphically. From the center of the surface of normals we lay off the vector corresponding to the particle velocity $\mathbf{v}$; on $\mathbf{v}$, we lay off a sphere with the vector as a diameter (cf. figure). In general this sphere intersects the surface of normals in some closed curve. A normal which connects the center of the surface with a point on the intersection curve gives the possible direction of the wave vector associated with the Cerenkov radiation while a line which connects the intersection point with the end of the vector $\mathbf{v}$ gives the wave front. If the intersection curve is traversed the normal describes the wave-vector cone while the line connecting the ends of the vector with the intersection curve describes the wave-front cone. In certain directions it may be impossible for radiation to be excited if the electric vector of the wave is perpendicular to the velocity of the charge.

11. We consider several simple cases of radiation of a charge in a moving medium. Let the velocity of a charge be parallel to the velocity of the medium. Then, from Eq. (11) we obtain the following expression for the Cerenkov radiation of the charge:

$$\frac{dW}{dx} = \frac{q^2}{v^2}\int\left[1 - \frac{v^2}{c^2} - \frac{\varkappa}{1+\varkappa}\,\frac{(1 - uv/c^2)^2}{1 - u^2/c^2}\right]\mu\omega d\omega, \tag{22}$$

where the integration extends over frequency regions which satisfy the inequality

$$1 - v^2/c^2 - \eta(v-u)^2/c^2 < 0, \qquad \eta = \varkappa/(1 - u^2/c^2). \tag{23}$$

The radiation cone is circular, with a uniform distribution of intensity over the geneatrices. The opening angle of the cone is given by the relation

$$\tan^2\vartheta = v^2/c^2 + \eta(u - v)^2/c^2 - 1. \tag{24}$$

If the velocity of the particle is parallel to the velocity of the medium the quantity κ is a function of the argument $\omega(1 - u/v)/(1 - u^2/c^2)^{1/2}$.

12. Let the velocity of the charge form an arbitrary angle with the velocity of the medium. Since the Cerenkov losses are given by Eq. (13), Eq. (11) can be written in the form

$$\frac{dW}{dx} = \frac{q^2}{\pi v^2}\int\limits_{\mathbf{kv}>0}\left[1 - \frac{v^2}{c^2} - \frac{\varkappa}{1+\varkappa}\,\frac{(1 - \mathbf{uv}/c^2)^2}{1 - u^2/c^2}\right]$$

$$\times\ \mu\mathbf{kv}\delta\left\{\frac{(\mathbf{kv})^2}{c^2} - k^2 + \frac{\varkappa}{c^2}\,\frac{(\mathbf{ku} - \mathbf{kv})^2}{(1 - u^2/c^2)}\right\}d\mathbf{k}. \tag{25}$$

In $\mathbf{k}$-space we introduce a cylindrical coordinate system with its k_z axis along the velocity of the charge. We take $\mathbf{k}\cdot\mathbf{v} = \omega$, $k^2 = \omega^2/v^2 + \sigma^2$, $\sigma = \mathbf{k} - \mathbf{v}(\mathbf{k}\cdot\mathbf{v})/v^2$. It will be apparent that

$$d\mathbf{k} = \sigma d\sigma d\varphi d\omega/v,$$

where φ is the angle in the plane perpendicular to $\mathbf{v}$. Carrying out the integration over σ, we have

$$\frac{dW}{dx} = \frac{q^2}{\pi v^2}\int\left[1 - \frac{v^2}{c^2} - \frac{\varkappa(\sigma_0)}{1+\varkappa(\sigma_0)}\,\frac{(1 - \mathbf{uv}/c^2)^2}{1 - u^2/c^2}\right]$$

$$\times\ \frac{\mu(\sigma_0)\,\sigma_0}{F'_\sigma(\omega, \varphi, \sigma_0)}\,\omega d\omega d\varphi, \tag{26}$$

where $F(\omega, \varphi, \sigma)$ is the argument of the δ function in Eq. (25) and σ_0 is the value of σ for which $F = 0$. Since σ_0 depends on ω and φ, Eq. (26) gives the distribution of Cerenkov radiation over angle and frequency. The radiated waves form an angle ϑ with the velocity of the charge where

$$\tan\vartheta = v\sigma_0/\omega. \tag{27}$$

It is apparent that the radiation cone is not circular, but has a complicated shape (since σ_0 depends on φ). It also follows from Eq. (26) that the radiation intensity is not uniform over the geneatrices.

13. We may also consider the polarization losses of the charge in the medium. Carrying out calculations similar to those in Section 12, we have

$$\frac{dW}{dx} = -\frac{q^2}{\pi v^2}\frac{(1-\mathbf{uv}/c^2)^2}{1-u^2/c^2}$$

$$\times\sum\int\frac{\sigma_0\mu(\sigma_0)}{|\varkappa'_\sigma(\omega,\varphi,\sigma_0)|}\frac{\omega d\omega d\varphi}{(\omega/v)^2(1-v^2/c^2)+\sigma_0^2+\omega_s^2/c^2}, \tag{28}$$

where σ_0 is the value of $\sigma(\omega, \varphi)$ for which $1 + \kappa = 0$ and ω_S is the root of the equation $\epsilon(\omega_S) = 0$. The summation is taken over all roots ω_S. The polarization losses are determined by the equation

$$(\mathbf{kv}-\mathbf{ku})/(1-u^2/c^2)^{1/2} = \omega_s. \tag{29}$$

It is interesting to note that the group velocity of the polarization waves is equal to the velocity of the medium **u**, i.e., the energy of these waves is "frozen" in the medium.

The relativistic Doppler shift in the frequency of the excited longitudinal oscillations of the medium can be used for measuring its velocity.

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