SOVIET PHYSICS JETP VOLUME 37 (10), NUMBER 5 MAY, 1960

EQUILIBRIUM OF A PLASMA WITH HELICAL SYMMETRY

B. B. KADOMTSEV

Submitted to JETP editor May 26, 1959

J. Exptl. Theoret. Phys. (U.S.S.R.) **37**, 1352-1354 (November, 1959)

A simple example of plasma equilibrium in a magnetic field with helical symmetry is considered.

THE problem of equilibrium of a plasma in a magnetic field involves the solution of the system of equations:

$$\operatorname{div} \mathbf{H} = 0, \qquad \nabla p = [\operatorname{curl} \mathbf{H} \times \mathbf{H}]/4\pi. \tag{1}$$

In this note we consider the case of helical symmetry, in which the plasma pressure p and the magnetic field components H_r, H_φ, H_z in a cylindrical coordinate system, depend only on r and $\zeta = kz - m\varphi$ (m is an integer and $\lambda = 2\pi/k$ is the period of the variation in p and $\mathbf{H}$ along the z axis). As has been shown by Johnson et al.[1] in this case the system of equations in (1) can be reduced to a single nonlinear equation. From the equation

$$\operatorname{div} \mathbf{H} = \frac{1}{r}\frac{\partial}{\partial r}(rH_r) + \frac{\partial}{\partial \zeta}\left(-\frac{m}{r}H_\varphi + kH_z\right) = 0$$

it follows that*

$$H_r = \frac{1}{r}\frac{\partial \psi}{\partial \zeta}, \qquad \frac{m}{r}H_\varphi - kH_z = \frac{1}{r}\frac{\partial \psi}{\partial r},$$

where ψ is an arbitrary function of r and ζ. Furthermore, from the relation $\mathbf{H}\cdot\nabla p = 0$ it follows that p is a function of ψ only, while from the equation $\nabla p \cdot \operatorname{curl} \mathbf{H} = 0$ it follows that $mH_z + krH_\varphi = \mathbf{I}$, where $\mathbf{I}$ is an arbitrary function of ψ. By means of these relations the component of the equilibrium equation (1) along $\nabla\psi$ can be transformed as follows:

$$\frac{1}{r^2}\frac{\partial^2\psi}{\partial\zeta^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{k^2r^2+m^2}\frac{\partial\psi}{\partial r}\right) + 4\pi\frac{dp}{d\psi} + \frac{I}{k^2r^2+m^2}\frac{dI}{d\psi} + \frac{2kmI}{(k^2r^2+m^2)^2} = 0. \tag{2}$$

If the functional dependence of p and $\mathbf{I}$ on ψ is given and Eq. (2) is solved, in principle it is possible to obtain all possible equilibrium configurations characterized by helical symmetry. However, the problem of solving the nonlinear equation is extremely difficult. It is only in the simplest case, in which $\mathbf{I}$ and $dp/d\psi$ are linear functions of ψ, that Eq. (2) is linear and can be solved by separation of variables.

Here we consider a still simpler example of the equilibrium configuration: $\mathbf{I} = \text{const}$, $p = p_0 + (a/4\pi)\psi$. In this case Eq. (2) becomes

$$\frac{1}{r^2}\frac{\partial^2\psi}{\partial\zeta^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{k^2r^2+m^2}\frac{\partial\psi}{\partial r}\right) = -a - \frac{2kIm}{(k^2r^2+m^2)^2}. \tag{3}$$

Whence we find

$$\psi = -\frac{ar^2}{8}(k^2r^2+2m^2) - \frac{kr^2I}{2m} + A\left(\frac{k^2r^2}{2} + m^2\ln r\right) + \psi_1, \tag{4}$$

where $A = \text{const}$, and $\psi_1(r, \zeta)$ is an arbitrary solution of the homogeneous equation.

From the known function ψ we find the plasma pressure and the magnetic field:

$$p = p_0 + \frac{a}{4\pi}\left\{-\frac{ar^2}{8}(k^2r^2+2m^2) - \frac{kr^2I}{2m} + A\left(\frac{k^2r^2}{2} + m^2\ln r\right) + \psi_1\right\}, \tag{5}$$

$$H_r = \frac{1}{r}\frac{\partial\psi}{\partial\zeta} = H_{1r}, \tag{6}$$

$$H_\varphi = \frac{1}{m^2+k^2r^2}\left(m\frac{\partial\psi}{\partial r} + krI\right) = -\frac{mar}{2} + \frac{mA}{r} + H_{1\varphi}, \tag{7}$$

$$H_z = \frac{1}{m^2+k^2r^2}\left(-kr\frac{\partial\psi}{\partial r} + mI\right) = \frac{I}{m} - kA + \frac{akr^2}{2} + H_{1z}, \tag{8}$$

where $\mathbf{H}_1$ is the curl-free magnetic field, given by the function ψ_1.

It is apparent that the magnetic field is composed of the uniform magnetic field along the z axis, the field of the uniform longitudinal current ($j_z = \text{const}$) and the longitudinal current concentrated at the z-axis ($J_0 \sim A$), the field due to the azimuthal current j_φ, which increases linearly with r, and the arbitrary curl-free field $\mathbf{H}_1$ characterized by helical symmetry.

*If this relation is integrated over the volume bounded by the surface $\psi = \text{const}$ and the planes $z = 0$ and $z = 2\pi/k$, we obtain

$$\psi = \frac{k}{2\pi}(m\Phi_\varphi - \Phi_z),$$

where

$$\Phi_z(\psi) = \int_0^{2\pi}\int_0^{r(\psi)} H_z\, r\,dr\,d\varphi, \qquad \Phi_\varphi(\psi) = \int_0^{2\pi/k}\int_0^{r(\psi)} H_\varphi\, dr\,dz.$$

In this case ($I = \text{const}$) it can be shown that $j_\varphi = (kr/2\pi m)\, j_z$, i.e., the current flows along the helical lines $\zeta = \text{const}$ (this feature explains the possibility of applying an arbitrary field $\mathbf{H}_1$ with the same symmetry without disturbing the condition $\mathbf{j} \cdot \nabla p = 0$).

For simplicity we limit our analysis to the particular case in which

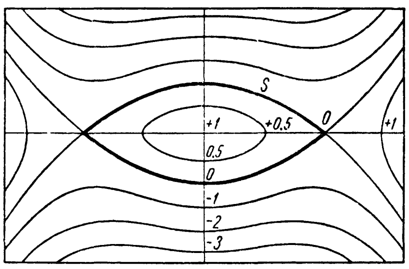
$$\psi_1 = \sin\zeta \cdot kr\,(BI'_m(kr) + CK'_m(kr)), \tag{9}$$

where $A = \text{const}$, $B = \text{const}$, I_m is a Bessel function of imaginary argument and K_m is the Macdonald function. The general solution of the equation for ψ_1 can be given in the form of a series of functions such as (9).

If we require that the pressure p and the function ψ be regular as $r \to 0$, we must set $A = C = 0$ and ψ becomes

$$\psi = -(ar^2/8)(k^2r^2 + 2m^2) - kr^2 I/m + B\sin\zeta \cdot krI'_m(kr). \tag{10}$$

Let $a > 0$ and $I > 0$; then, for small values of B (or if $m > 2$, for any value of B, but small values of r) close to the z axis there is a region in which ψ and the pressure p fall off with radius. This solution can be interpreted as characterizing equilibrium of a plasma inside a chamber whose walls coincide with the surface $\psi = -4\pi p_0/a$, where the plasma pressure vanishes. When $m = 2$, for example, the pressure distribution assumes the form shown in the figure, in which the pattern of the lines of constant pressure in the planes $z = \text{const}$ ($p_0 = 1$) is shown.



It is apparent from Eq. (10) that if the coefficient B becomes larger, i.e., if the magnetic field $\mathbf{H}_1$ produced by the external helical winding is increased, the equilibrium situation is worse; in particular, there is a reduction in the limiting pressure p_0 for which a plasma with zero pressure at the walls of the chamber can still be in equilibrium. In addition, the transverse dimension of the region in which $p > 0$ becomes smaller. This phenomenon is explained by the fact that the increase $\mathbf{H}_1$ causes a large curvature in the lines of force, resulting in a reduction of the region in which the lines of force do not extend to infinity in a radial direction. (In the figure this region is enclosed by the boundary marked S).

We consider another particular case: that in which $a > 0$ and $I < -am^3/4k$. If B is small, when $B > 0$ the function ψ will have a maximum $\psi = \psi_0$ at some helical line $r = r_0$, $\sin\zeta = 1$. Since ψ falls off with distance from this line, at low pressures there is an equilibrium state of the plasma in which the pressure vanishes along some helical tube $\psi = \text{const} < \psi_0$ which surrounds the line $r = r_0$, $\sin\zeta = 1$. Thus, this solution represents the simplest representation of the equilibrium states of a plasma in a chamber in the form of a helical tube. We may note that in the analysis of equilibrium in chambers of this type the constants A and C may be taken different from zero, allowing some extension of the family of simple solutions of this kind.

---

[1] Johnson, Oberman, Kusrud, and Frieman, Proceedings of the Second International Conf. on the Peaceful Uses of Atomic Energy, Geneva 1958, P/1875.

Translated by H. Lashinsky
269