APPLICATION OF THE DISPERSION RELATIONS METHOD IN QUANTUM ELECTRODYNAMICS

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Submitted to JETP editor June 1, 1959

J. Exptl. Theoret. Phys. (U.S.S.R.) **37**, 1361-1371 (November, 1959)

An approximate set of dispersion equations for the Green's function of the photon and vertex part has been derived in quantum electrodynamics on the basis of the dispersion relations and unitarity conditions. The "nonsubtraction" procedure is employed in the asymptotic investigation of the solutions of the equation. Agreement with the renormalized perturbation theory when the fine structure constant tends to zero has been used as the boundary condition. It is shown that the vertex function decreases asymptotically with increase in the square of the photon 4-momentum $q^2 = (p_+ + p_-)^2$ for $p_+^2 = p_-^2 < m^2$ where p_-, p_+ are the electron and positron 4-momenta. This leads to finite renormalization of the charge in the approximation under consideration.

1. INTRODUCTION

THE method of dispersion relations, based on very general requirements of covariance, causality, positiveness of the energy spectrum (spectrality), and unitarity has been developed intensively in recent years in quantum field theory.

This method permits us to express the Green's function (for the amplitudes of processes) in terms of invariant spectral functions which can in turn be connected with the Green's function of other processes by means of the unitarity conditions. Thus a set of equations for the Green's function can be obtained in principle. This set of equations possesses important advantages over the usual approach. First, it contains only renormalized quantities and consequently does not lead to the appearance of expressions that diverge at high momenta, the latter effect being characteristic for perturbation theory and the Schwinger-Dyson equation; second, it interrelates the amplitudes of various processes on the energy surface.

Serious hopes of avoiding the well-known difficulties attached to the approximate solutions of the Schwinger-Dyson equation[1-3] are connected with the method of dispersion relations.

A whole series of papers has appeared in which the method of dispersion relations is used for the derivation of approximate equations for the Green's function, Thus, Mandel'shtam[4] obtained an approximate equation from the dispersion relations for the amplitude of scattering of a meson by a nucleon. In the work of Drell and Zachariasen,[5] an attempt was undertaken at the determination of the approximate equation for the vertex part in quantum electrodynamics following from the dispersion relations (form factor of the electron). However, the basic results obtained in this research have raised objections.*

In the present research an attempt has been made at a more systematic analysis of quantum electrodynamics from the point of view of the dispersion relations. Inasmuch as spectral representations have been well studied only for the simplest Green's function, and also because of the extraordinary difficulties which arise in the calculation of higher Green's functions, we have limited ourselves to a discussion of the lowest approximation in the dispersion equations which contains only the Green's function of the photon and vertex part. The principal unsolved question today in such an approach is thus the consideration and estimation of the contribution of higher approximations, i.e., two-particle and more complicated Green's functions.

The choice of quantum electrodynamics is not accidental. First, there enters only a single constant here — the renormalized charge of the electron and the well-known boundary conditions — agreement with perturbation theory at low energies. Second, in the first stage of the investigation we can consider the interaction of photons only with the electron-positron field and disregard other particles (the "pure" quantum electrodynamics).

In the setting up of the dispersion relations,

*For details see below, Sec. 3..

we start out from the "nonsubtraction" procedure.* A formulation is given in Sec. 2 of the dispersion relations and the boundary conditions in quantum electrodynamics. In Sec. 3, we have investigated the asymptotic solution of a set of equations for the photon Green's function and vertex part in the simplest approximation. A discussion of the results obtained is given in Sec. 4. Appendices are included at the end of the paper.

2. DERIVATION OF THE DISPERSION RELATIONS

Derivation of the dispersion relations can be divided conditionally into three steps: I — the fundamental dispersion relations; II — an expression of the anti-Hermitian part of the spectral functions in terms of other amplitudes; III — the finding of reasonable boundary conditions.

It is natural to begin this process with the simplest single particle Green's function. The dispersion relations for the Green's function of the photon (the Källen-Lehman theorem[6]) are well known:†

$$D(q^2) = -\frac{1}{q^2} + \int_0^\infty \frac{\rho(q'^2)\,dq'^2}{q'^2 - q^2 - i\varepsilon}. \quad (1)$$

The longitudinal part of $D_{\mu\nu}(q)$ is chosen equal to zero; in this case,

$$D_{\mu\nu}(q) = (\delta_{\mu\nu} - q_\mu q_\nu / q^2)\, D(q^2). \quad (2)$$

By making use of the equation for the Heisenberg operators

$$\Box A_\mu(x) = j_\mu(x)$$

and decomposition over the whole set‡ (unitarity) one can express the spectral function $\rho(q^2)$ in terms of the amplitude of other processes.

$$\rho(q^2) = \frac{(2\pi)^3}{3}\,\mathrm{Sp}\left(\frac{1}{q^2}\right)^2 \sum_n \langle 0 \,|\, j_\mu(0)\,|\,n\rangle\langle n\,|\,j_\nu(0)|\,0\rangle\,\delta(p_n - q). \quad (3)$$

Equations (1) and (3) give the desired equation for $D_{\mu\nu}$.

Similarly, one can write down the dispersion relation for the Green's function of the electron. However, we shall not do this, inasmuch as it is not needed in what follows.

In order to extend the chain of equations, it is necessary to write down the dispersion relation for the matrix elements $\langle 0\,|\,j_\mu(0)\,|\,n\rangle$, which are on the right side of (3). The matrix element $\langle\,|\,j_\mu(0)\,|\,p_+, p_-\rangle$, where $|\,p_+, p_-\rangle$ is the state electron + positron, can be represented in the form

$$\langle 0\,|\,j_\mu(0)|\,p_+, p_-\rangle = \bar{u}_+(\mathbf{p}_+)\,\Lambda_\mu(p_+, p_-)\,u_-(\mathbf{p}_-),$$
$$\Lambda_\mu(p_+, p_-) = \gamma_\mu F_1(q^2) + \sigma_{\mu\nu} q_\nu F_2(q^2), \qquad q = p_+ + p_-. \quad (4)$$

from consideration of relativistic and gauge invariance. Here, $u_\pm$ are solutions of the Dirac equation for the positron and electron, respectively;

$$(\hat{p} \pm m)\,u_\pm(\mathbf{p}) = 0, \quad \gamma = \beta\alpha, \quad \gamma_0 = \beta$$
$$p_0 = E(\mathbf{p}) = +(\mathbf{p}^2 + m^2)^{1/2}, \qquad \sigma_{\mu\nu} = (\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu);$$

$F_1(q^2)$ and $F_2(q^2)$ are invariant functions which characterize the charge distribution and the magnetic moment of the electron. The quantity Λ_μ is connected with the vertex function Γ_ν for $p_+^2 = p_-^2 = m^2$:

$$\bar{u}_+\Lambda_\mu(p_+, p_-)\,u = -\bar{u}_+\Gamma_\nu(p_+, p_-\; u_-\cdot D_{\mu\nu}(q)\,q^2. \quad (5)$$

We emphasize that there is no necessity of taking Γ_μ into consideration. First, $\rho(q^2)$ [see Eq. (3)] is easily expressed* directly in terms of Λ_μ; second, inasmuch as Γ_μ in x space is not directly connected with the T product of the Heisenberg operators, the anti-Hermitian part of the vertex function can only be expressed indirectly in terms of the amplitude of other processes.

Up to the present time the dispersion relations for $F_i(q^2)$ have not been rigorously established. In the general case they can be shown in terms of q^2 under the condition $p_\pm^2 < 0$.† In the physical region $p_\pm^2 \to m^2$ ($p_\pm^2 \leq m^2$) there is a proof of the dispersion relations in any approximation of perturbation theory.[7] We shall start out from the validity of the following dispersion relation (for $p_\pm^2 \leq m^2$):

$$F_i(q^2) = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\,F_i(q'^2)\,dq'^2}{q'^2 - q^2 - i\varepsilon}. \quad (6)$$

In the derivation of (6), in addition to a knowledge of the analytic properties of $F_i(q^2)$ with respect to q^2, it is assumed that $F_i(q^2) \to 0$ for $|q^2| \to \infty$.‡ In this assumption is included the basic idea of the nonsubtraction procedure of Chew in the theory of dispersion relations. We note that the vanishing of $F_i(q^2)$ is the necessary condition for the finiteness of the charge renormaliza-

*In connection with the "nonsubtraction" procedure see, for example, reference 5 (see also below).

†These dispersion relations are obtained by starting out from the "nonsubtraction" procedure, i.e., under the assumption that $D(q^2) \to 0$ for $|q^2| \to \infty$. In our research the metric $q^2 = q_0^2 - \mathbf{q}^2$ is used throughout.

‡By way of the entire system one can with equal right make use of the states $|n_{in}\rangle$ and $|n_{out}\rangle$, which correspond to incident and diverging waves.

*See also Sec. 4.

†The difficulty with analytic continuation in the "physical" region $p_\pm^2 \to m^2$ has not been overcome.[8]

‡In the opposite case (if $F_i(q^2) = \text{const}$, or increases more slowly than $|q^2|$ for $|q^2| \to \infty$) the dispersion relations will be:

$$F_i(q^2) - F_i(0) = (q^2/\pi)\int_0^\infty \mathrm{Im}\,F_i\,[q'^2(q'^2(q'^2 - q^2 - i\varepsilon))]^{-1}\,dq'^2.$$

tion (i.e., the convergence of the integral $\int \rho(q^2)\,dq^2$).

The expression for $\mathrm{Im}\,F_i(q^2)$, which plays the role of spectral functions, can be found in the following fashion. We write:

$$\langle 0 | j_\mu(0) | p_+, p_-\rangle = -\int \bar{u}_+(\mathbf{p}_+)(i\hat{\nabla}_x - m)$$
$$\times \langle 0 | T\psi(x)\bar{\psi}(x') j_\mu(0) | 0\rangle (i\hat{\nabla}_{x'} + m) u_-(\mathbf{p}_-)$$
$$\times \exp(-ip_+x - ip_-x')\, d^4x d^4x' \ldots \quad (7)$$

The corresponding expression for $\Lambda_\mu(p_+, p_-)$ is obtained if we omit the "coverings" $\bar{u}_+$ and u_- in (7). Noting that

$$\gamma_0 \Lambda_\mu^*(-p_-, -p_+)\gamma_0 = -\int (i\hat{\nabla}_x - m)\langle 0 | T\psi(x)\bar{\psi}(x')$$
$$\times j_\mu(0) | 0\rangle (i\hat{\nabla}_{x'} + m) \exp(-ip_+x - ip_-x')\, d^4x d^4x', \quad (8)$$

where $\tilde{T}$ denotes the anti-product, and $\psi(x)$ is the Heisenberg operator of the electric field, we find

$$(2i)^{-1}\bar{u}_+(\mathbf{p}_+)(\Lambda_\mu(p_+, p_-) - \gamma_0\Lambda_\mu^*(-p_-, -p_+)\gamma_0)$$
$$\times u_-(\mathbf{p}_-) = \bar{u}_+(\mathbf{p}_+)(\gamma_\mu \,\mathrm{Im}\, F_1(q^2)$$
$$+ \sigma_{\mu\nu}q_\nu \,\mathrm{Im}\, F_2(q^2))\, u_-(\mathbf{p}_-) = \tfrac{1}{2}\int \bar{u}_+(\mathbf{p}_+)(i\hat{\nabla}_x - m)$$
$$\times \langle 0 | [j_\mu(0), \psi(x)]_- | p_-\rangle \exp(-ip_+x) d^4x$$
$$= \tfrac{1}{2}(2\pi)^4 \sum_n \delta(p_n - p_+ - p_-)\langle 0 | j_\mu(0) | n\rangle$$
$$\times \langle n | \bar{u}_+(\mathbf{p}_+)\eta(0) | p_-\rangle,$$
$$\eta(x) \equiv (i\hat{\nabla} - m)\psi(x). \quad (9)^*$$

Equations (6) and (9) are the desired dispersion equations for $\langle 0 | j_\mu(0) | p_+, p_-\rangle$.

Equations (1), (3), (6) and (9) should be supplemented by dispersion equations for the more complicated matrix elements $\langle 0 | j_\mu | n\rangle$ and $\langle n | \bar{u}_+\eta(0) | p_-\rangle$, which appear on the right hand side of (3) and (9). In principle, an infinite set of equations is obtained for all possible amplitudes. The difficulties that arise in this course were already noted in the introduction. In practice one always deals with a "broken" system of equations, in which amplitudes with a number of particles greater than some given value are discarded.† In the solution of such a system, the problem arises as to reasonable boundary conditions. In quantum electrodynamics, it is natural to assume for the boundary conditions

$$F_1(0) = e, \qquad F_2(0) = -\frac{1}{4\pi}\left(\frac{e^2}{4\pi}\right)\frac{e}{2m} = -\frac{1}{2}\Delta\mu, \quad (10)$$

where e is the renormalized charge and $\Delta\mu$ is the anomalous magnetic moment of the electron. It is seen that in the lowest approximation, which was considered in the present work (see Sec. 3), the condition (10) is insufficient for a unique determination of the solution [because of the homogeneity of the equation for $F_i(q^2)$].

As an additional condition we shall require agreement with perturbation theory for $e^2 \to 0$.

3. INVESTIGATION OF THE SIMPLEST APPROXIMATION

We shall neglect on the right side of (3) and (9) all matrix elements with a number of particles in the intermediate state greater than two. Then

$$\rho(q^2) = \frac{1}{3(2\pi)^3}\mathrm{Sp}\int d^4q_+ d^4q_- \theta(q_+^0)\theta(q_-^0)\delta(q_+^2 - m^2)\, 4q_+^0 q_-^0$$
$$\times \delta(q_-^2 - m^2)\delta(q_+ + q_- - q)$$
$$\times \langle 0 | j_\mu(0) | q_+, q_-\rangle\langle q_-, q_+ | j_\nu(0) | 0\rangle \ldots, \quad (11)$$

$$\bar{u}_+(\mathbf{p}_+)\{\gamma_\mu \,\mathrm{Im}\, F_1(q^2) + \sigma_{\mu\nu}q_\nu F_2(q^2)\}\, u_-(\mathbf{p}_-) =$$
$$= \frac{1}{2(2\pi)^2}\int d^4q_+ d^4q_- \theta(q_+^0)\theta(q_-^0)\delta(q_+^2 - m^2)$$
$$\times \delta(q_-^2 - m^2) 4q_+^0 q_-^0\, \delta(q_+ + q_- - p_+ - p_-)$$
$$\times \langle 0 | j_\mu(0) | q_+, q_-\rangle\langle q_-, q_+ | \bar{u}_+\eta(0) | p_-\rangle. \quad (12)$$

Together with (1) and (6), these relations form the simplest set of dispersion relations in which, in addition to the Green's function of the photon $D_{\mu\nu}$ and the matrix element $\langle 0 | j_\mu | q_+, q_-\rangle$, there also enters into (12) the exact matrix element $\langle q_-, q_+ | \bar{u}_+(\mathbf{p}_+)\eta(0) p_-\rangle$ of the scattering of the electron by a positron. In order to obtain a closed system, it is necessary to attempt to express $\langle q_-, q_+ | \bar{u}_+\eta | p_-\rangle$ in terms of $D_{\mu\nu}$ and $\langle 0 | j_\mu | q_+, q_-\rangle$. This can be done approximately if we limit ourselves in the expression for $\langle q_-, q_+ | \bar{u}_+\eta | p_-\rangle$ to the first irreducible Feynman diagram, pictured in the drawing, where the corresponding exact vertex functions Γ_μ are inserted at the junction, while the wavy line corresponds to $D_{\mu\nu}$.

We note that the conditions (9) are violated, generally speaking, in the approximate equation (12). Satisfaction of these conditions in each given approximation can be guaranteed if in (12), in place of the right hand part, we take the half sum of the matrix elements corresponding to solutions with diverging and converging waves (see footnote ‡ on page 969).

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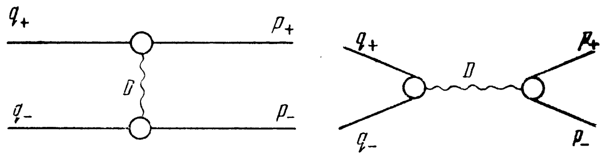
*The second term of the anticommutator does not make any contribution, since $p_-^0 = +(p_-^2 = m^2)^{1/2} > 0$.

†In the nonrelativistic region in mesodynamics, a similar device is used in the derivation of the Low equation.[9] Here there is also a formal analogy with the Tamm-Dancoff method,[10] where the cutoff is by the number of virtual particles.



Applying this rule to (12), and taking (5) into account, we obtain in the given approximation

$$\bar{u}_+(\mathbf{p}_+)\,(\gamma_\mu\,\mathrm{Im}\,F_1(q^2) + \sigma_{\mu\nu}q_\nu\,\mathrm{Im}\,F_2(q^2))\,u_-(\mathbf{p}_-) = \frac{1}{8\pi^2}$$
$$\times\int d^4q_+d^4q_-\theta(q_+^0)\,\theta(q_-^0)\,\delta(q_+^2 - m^2)\,\delta(q_-^2 - m^2)$$
$$\times\,\delta(q_+ + q_- - q)\cdot\bar{u}_+(\mathbf{p}_+)\{[\Lambda_\nu\mathrm{Sp}\,(\Lambda_\mu(-\hat{q}_+ + m)$$
$$\times\gamma_0\Lambda_\nu^*\gamma_0(\hat{q}_- + m)) + \Lambda_\nu^+\,\mathrm{Sp}\,(\Lambda_\mu^+(-\hat{q}_+ + m)\,\Lambda_\nu(-q)$$
$$\times(\hat{q}_- + m))]\,(q^2D(q^2)\,q^2)^{-1} - \tilde{\Lambda}_\nu(-\hat{q}_+ + m)$$
$$\times\,\mathrm{Re}\,\Lambda_\mu\cdot(\hat{q}_- + m)\gamma_0\tilde{\Lambda}_\nu^*\gamma_0\,(k^2D(k^2)\,k^2)^{-1}\}\,u_-(\mathbf{p}_-), \tag{13}$$

where the notation used is

$$\mathrm{Re}\,\Lambda_\mu \equiv \gamma_\mu\,\mathrm{Re}\,F_1(q^2) + \sigma_{\mu\nu}q_\nu\,\mathrm{Re}\,F_2(q^2),$$
$$\tilde{\Lambda}_\nu \equiv \gamma_\nu F_1(k^2) + \sigma_{\mu\nu}k_\nu F_2(k^2),$$
$$\Lambda_\mu^+ \equiv \gamma_\mu F_1^*(q^2) + \sigma_{\mu\nu}q_\nu F_2^*(q^2),$$
$$k = p_+ - q_+ = q_- - p_-. \tag{14}$$

In the work of Drell and Zachariasen,[5] the approximate equation for the spectral functions $F_i(q^2)$ was investigated. This corresponds to consideration only of the Born term in the matrix element $\langle q_-, q_+|\bar{u}_+\eta(0)|p_-\rangle$ of the scattering of the electron by a positron. However, it is not difficult to establish the fact that the expression obtained by them for $\mathrm{Im}\,F_i(q^2)$ is wrong. Actually, setting $\tilde{\Lambda}_\nu = e\gamma_\nu$ and $D(q^2)\,q^2 = -1$ in (14), and carrying out the necessary integrations, we find (omitting the "coverings")

$$\gamma_\mu\,\mathrm{Im}\,F_1(\lambda) + \sigma_{\mu\nu}q_\nu\,\mathrm{Im}\,F_2(\lambda) = \left(\frac{e^2}{4\pi}\right)\left(1 - \frac{2}{\lambda}\right)^{1/2}$$
$$\times\left\{\left(\mathrm{Re}\,F_1(\lambda) + \mathrm{Re}\,F_2(\lambda)\frac{q_\nu}{m}\right)\cdot\frac{1}{2}\left(\frac{\lambda-1}{\lambda-2}\right)\int_\xi^{\lambda-2}\frac{d\mu}{\mu}\right.$$
$$-((13/12)\mathrm{Re}\,F_1(\lambda)\,\gamma_\mu + \mathrm{Re}\,F_2(\lambda)\,\sigma_{\mu\nu}q_\nu/m)$$
$$-\gamma_\mu(3\lambda)^{-1}\,\mathrm{Re}\,F_1(\lambda) - {}^1/_2(\lambda-2)^{-1}(\mathrm{Re}\,F_1(\lambda)$$
$$\left.+\,8\,\mathrm{Re}\,F_2(\lambda))\,(\gamma_\mu + \sigma_{\mu\nu}q_\nu/4m)\right\}, \tag{15}$$

where for convenience the dimensionless quantity $\lambda = q^2/2m^2$ has been introduced; $F_1(q^2) \equiv F_1(\lambda)$; $F_2(q^2)\,m = F_2(\lambda)$; $\xi \approx (m^2 - p^2)/2m^2$; $p_+^2 = p_-^2 = p^2 \lesssim m^2$. Equation (15) differs from the similar equation (22) in reference 5 in that here the integral $\int_\xi^{\lambda-2} d\mu/\mu$ appears in the first term on the right hand side instead of $\int_{-1}^{+1}\frac{d\mu}{1-\mu}$ in the expression of Drell and Zachariasen.[5] The reason for this divergence is most simply understood from the example of perturbation theory. If we compute $\mathrm{Im}\,F_1$ in the first approximation of perturbation theory, then, because of the presence of the infrared catastrophe, we obtain different expressions depending on its capability for correction (see Appendix A); if we define $\mathrm{Im}\,F_1(\lambda)$ as the limit of $\mathrm{Im}\,F_1(\lambda, p^2)$ for $p^2 \to m^2$ in the region $p^2 < m^2$, then we get (15). As $p^2 \to m^2$ from the region $p^2 > m^2$, we obtain Eq. (22) of reference 5.* However, for $p^2 \gtrsim m^2$, $\mathrm{Im}\,F_1(\lambda)$ does not vanish everywhere in the spatially similar region $\lambda < 0$. Thus we can draw the important conclusion that the dispersion relations for $F_i(\lambda)$ in quantum electrodynamics in form (6) exist only for p_+^2 and $p_-^2 \lesssim m^2$. In the opposite case, integration in (6) must be carried out over all q'^2 from $-\infty$ to $+\infty$. This completes our proof, inasmuch as Drell and Zachariasen on the one hand use dispersion relations in the form (6) and on the other hand calculate $\mathrm{Im}\,F_1(q^2)$, essentially as the limit of the region $p^2 > m^2$.† The absence of a solution vanishing at infinity for $F_1(q^2)$ (i.e., in contradiction with the nonsubtraction procedure) and the negative value for the mean square of the radius of the distribution of charge of the electron in reference 5 were brought about in just this way. It is also necessary to emphasize that the approximation corresponding to replacement of $\langle q_-, q_+|\bar{u}_+\eta|p_-\rangle$ by the Born term [i.e., use of (15)] is scarcely valid. In this case, substitution of (15) in (6) leads to an integral equation for $F_1(q^2)$ which has a solution (corresponding to the perturbation theory for $e^2 \to 0$) which falls off for $q^2 \to \infty$ as $(q^2)^{-1/2}$ (see Appendix B). Such a behavior points up the essential role of $F_1(q^2)$ in the neglected terms in the matrix element $\langle q_-, q_+|\bar{u}_+\eta|p_-\rangle$ and the necessity of their calculation. Equation (13) again corresponds to an attempt to consider in first approximation the change in the Born term as $q^2 \to \infty$ because of $F_i(q^2)$ and $D_{\mu\nu}(q)$.

We now investigate the asymptotic solution of the set of integral nonlinear equations (1), (3), (6) and (9) as $q^2 \to \infty$. For this purpose we make the

*In order to find the first approximation of perturbation theory it is necessary in (15) and (22) of reference 5 to set $\mathrm{Re}\,F_1 = e$; $\mathrm{Re}\,F_2 = 0$.

†In fact, the infrared divergence of $\mathrm{Im}\,F_i(q^2)$ is removed in reference 5 by the elimination of the scattering of the electron by the positron at small angles under the assumption that the minimum obtainable angle in the center-of-mass system does not depend on q^2. However, such a situation is qualitatively equivalent to the calculation of $\mathrm{Im}\,F(q^2)$ in the region $p^2 \gtrsim m^2$.

assumption, first, that $|D(q^2)q^2|$ differs slightly from unity over the whole region of variation of q^2; second,* that $F_2(q^2)$ falls off as $|q^2| \to \infty$ not more slowly than $(q^2)^{-1}$. Then, neglecting terms in (11) and (13) which contain $F_2(q^2)$, we find†

$$\rho(\lambda) = \frac{\theta(\lambda-2)}{12\pi^2\lambda^2}\left(1-\frac{2}{\lambda}\right)^{1/2}(1+\lambda)\,|F_1(\lambda)|^2, \quad \rho(\lambda) \equiv 2m^2\rho(q^2), \tag{16}$$

$$\operatorname{Im} F_1(\lambda) = \operatorname{Re} F_1(\lambda)\left\{-\pi\lambda\rho(\lambda) + \frac{1}{4\pi}\left(1-\frac{2}{\lambda}\right)^{1/2}\left(\frac{1}{2}\left(\frac{\lambda-1}{\lambda-2}\right)\right.\right.$$
$$\left.\left.\times I_0(\lambda) - \left(\frac{\lambda}{\lambda-2}\right)I_1(\lambda) + \frac{\lambda+4}{4(\lambda-2)}I_2(\lambda)\right)\right\}, \; \lambda > 2 \tag{17}$$

where

$$I_0(\lambda) = \int_{\xi}^{\lambda-2} |F_1(-\mu)|^2\frac{d\mu}{\mu}, \quad I(\lambda) = \frac{1}{\lambda-2}\int_0^{\lambda-2}|F_1(-\mu)|^2 d\mu,$$
$$I_2(\lambda) = \frac{2}{(\lambda-2)^2}\int_0^{\lambda-2}|F_1(-\mu)|^2\mu\, d\mu. \tag{18}$$

Substituting (17) in (6), we obtain an integral equation for $F_1(\lambda)$:

$$F_1(\lambda) = \frac{1}{\pi}\int_2^{\infty}\frac{\operatorname{Re} F_1(\lambda')\, g(\lambda')\, d\lambda'}{\lambda'-\lambda-i\varepsilon}, \tag{19}$$

where $g(\lambda) = \operatorname{Im} F_1(\lambda)/\operatorname{Re} F_1(\lambda)$ is obtained from (17).

The formal, general solution of (19), which is finite at $\lambda = 0$, has the form[11]

$$F_1(\lambda) = P(\lambda)(\lambda-2)^{-n}\exp(\varphi(\lambda)), \tag{20}$$

$$\varphi(\lambda) = \frac{\lambda}{\pi}\int_2^{\infty}\frac{\tan^{-1} g(\lambda')\, d\lambda}{\lambda'(\lambda'-\lambda-i\varepsilon)}, \tag{21}$$

where $P(\lambda)$ is a polynomial and n is an integer. Agreement with the renormalized perturbation theory for $e^2 \to 0$ is achieved by the choice $n = 0$ and $P(\lambda) = e$.

It is seen from (21) that if $g(\lambda)$ is a nonvanishing function for $\lambda \to -\infty$, then

$$\lim_{\lambda\to-\infty}\varphi(\lambda) \to -\pi^{-1}\tan^{-1} g(|\lambda|)\ln|\lambda|. \tag{22}$$

Therefore, if $\tan^{-1} g(\lambda) \to \text{const} > 0$ for $|\lambda| \to \infty$, then, in accord with (20), we obtain a vanishing solution for $F_1(\lambda)$ with the asymptotic value

$$F_1(\lambda) \sim e\exp[-\pi^{-1}\tan^{-1} g(|\lambda|)\ln|\lambda|]. \tag{23}$$

On the other hand, it follows from (17) and (18) that if $F_1(\lambda) \to 0$ as $|\lambda| \to \infty$, then

$$g(\lambda) - \frac{1}{8\pi}\int_{\xi}^{\lambda}|F_1(-\mu)|^2\frac{d\mu}{\mu} \to \text{const} > 0, \tag{24}$$

since in this case the terms $\sim I_1$, I_2 and $\lambda\rho(\lambda)$ tend to zero with increase in λ.

Now it is easy to find the asymptotic value of $g(\infty)$. We assume that $g(\infty) \ll 1$, we replace $\tan^{-1} g(\infty)$ in (23) by $g(\infty)$, and then substitute (23) in (24). We obtain

$$g^2(\infty) = (e^2/16)(2/\xi)^{(2/\pi)g(\infty)}. \tag{25}$$

The divergence in (25) as $\xi \to 0$ (the infrared catastrophe) arises from the fact that the scattering amplitude $\langle q_-, q_+|\bar{u}_+\eta|p_-\rangle$ of the electron on the positron becomes infinitely great for forward scattering. Strict elimination of the infrared divergence in the method of dispersion relations lies outside the framework of our present article, and will be investigated separately.* Assuming that $(2/\pi)g(\infty)\ln\xi^{-1} \ll 1$, we have from (25)

$$g(\infty) \approx {}^1\!/_2(e^2/4)^{1/2} \ll 1. \tag{26}$$

The neglected terms have a maximum order of magnitude equal to $(2/\pi)g(\infty)\ln\xi^{-1}$ and are small for $(2/\pi)g(\infty)\ln\xi^{-1} \ll 1$ and $(2/\pi) \times g(\infty)\ln\lambda \gg 1$.

Finally, we obtain the following asymptotic solution of Eq. (19) for $(2/\pi)g(\infty)\ln\lambda \gg 1$.

$$F_1(\lambda) \approx e\lambda^{-\frac{1}{2}(e^2/4)^{1/2}}. \tag{27}$$

The most characteristic property of the vanishing asymptote is the nonanalytic dependence on e^2 for $\pi^{-1}(e^2/4)^{1/2}\ln\lambda \gg 1$. We note also that in the region $\lambda \gg 1$, and $(e^2/8\pi^2)(\ln\lambda)^2 \ll 1$, $F_1(\lambda)$ differs slightly from e and one can replace $\tan^{-1} g(\lambda)$ by $g(\lambda)$; in this case, $\varphi(\lambda)$ in (21) coincides with $F_1(\lambda)$, computed by perturbation theory (with accuracy to e^3) and, consequently, the dependence of $F_1(\lambda)$ on λ in (20) is obtained in the same way as in the research of Abrikosov.[12]

4. DISCUSSION OF RESULTS

We shall first investigate Eqs. (1) and (16) on the basis of (27) for $D(\lambda) \equiv 2m^2 D(q^2)$. Substitution of (27) in (16) gives the following behavior for $\rho(\lambda)$ in the region $\pi^{-1}(e^2/4)^{1/2}\ln\lambda \gg 1$:

$$\rho(\lambda) \approx (e^2/12\pi^2\lambda)\exp\{-(e^2/4)^{1/2}\ln\lambda\}. \tag{28}$$

The convergence of the integral $\int_0^{\infty}\rho(\lambda)\,d\lambda$ then follows, and consequently the finiteness of the con-

*For justification of the second assumption we can introduce the same arguments as in reference 5, which are still more forceful in our case because of the presence of a vanishing asymptotic value for F_1 (see below).

†Inasmuch as the set of equations with account of F_2 is rather cumbersome and is not analyzed in the present work, we shall not write it out.

*It is of interest to observe that in the approximation that we have considered, a solution also exists in the limiting case $\xi \to 0$.

stant of charge renormalization $Z_3^{-1} = (1 + \int_0^\infty \rho(\lambda)\,d\lambda)$ in the given approximation. Estimating Z_3 by means of the asymptotic expression (28) for ρ, we find

$$Z_3^{-1} \approx 1 + (3\pi)^{-1}(e^2/4)^{1/2}. \tag{29}$$

It is also not difficult to establish the fact that $\rho(\lambda)$ in this approximation does not have any resonance at $(e^2/12\pi^2)\ln\lambda \sim 1$, in contrast to the results of references 13 and 14 (see also below).

As is well known from the dispersion relations (1) for $D(\lambda)$, it follows that $|\lambda D(\lambda)|$ as $\lambda \to -\infty$ is smaller than Z_3^{-1}. We have obtained for Z_3^{-1} the finite value (29) which differs slightly from unity, justifying the initial assumption that $|\lambda D(\lambda)| \sim 1$ for $|\lambda| \to \infty$.*

It is interesting to compare our results with the results of other researchers. Lehman, Symanzik and Zimmermann,[15] starting out from the dispersion relations for $D(\lambda)$, found an important condition which must be satisfied by the quantity

$$M(\lambda) = \rho(\lambda)/|D(\lambda)|^2. \tag{30}$$

As is easy to prove, $M(\lambda)$ is connected with the imaginary part of the polarization operator $\Pi_{\mu\nu}(q)$. Thus

$$\pi M(\lambda) = \lambda\,\mathrm{Im}\,\pi(\lambda), \qquad \Pi_{\mu\nu}(\lambda) = (\delta_{\mu\nu}q^2 - q_\mu q_\nu)\,\pi(\lambda).$$

They have shown that if one considers (30) and (1) as the equation relative to $\rho(\lambda)$ for a given function $M(\lambda)$, then the solution of this equation exists only if

$$\int_0^\infty \lambda^{-2} M(\lambda)\,d\lambda \leqslant 1. \tag{31}$$

In the lowest approximation, $M(\lambda)$ in quantum electrodynamics is expressed in terms of the vertex parts

$$M(\lambda) = (e^2/12\pi^2)\,\theta(\lambda-2)(1-2/\lambda)^{1/2}\{(1+\lambda)|\Gamma_1(\lambda)|^2 + 4\lambda(\lambda+4)|\Gamma_2(\lambda)|^2 - 6\lambda(\Gamma_1(\lambda)\Gamma_2^*(\lambda) + \Gamma_1^*(\lambda)\Gamma_2(\lambda))\}, \tag{32}$$

where $\Gamma_i(\lambda)$ are determined from the relation

$$\bar{u}_+(p_+)\Gamma_\mu(p_+, p_-)u_-(p_-) = \bar{u}_+(p_+)(\gamma_\mu\Gamma_1(\lambda) + \sigma_{\mu\nu}q_\nu\Gamma_2(\lambda))u_-(p_-).$$

On the basis of (31) and (32), the authors of reference 15 drew a fundamental conclusion on the necessity of the vanishing of the vertex part with increase in λ for internal self-consistency of the theory. Inasmuch as $F_1(\lambda) = D(\lambda)\lambda\Gamma_1(\lambda)$ [see (5)] and for $\lambda \to -\infty$ $|\lambda D(\lambda)| \to Z_3^{-1}$, we conclude that, in the approximation that we have considered, $\Gamma_1(\lambda)$ falls off with increase of λ according to the same law as $F_1(\lambda)$. This guarantees satisfaction of the condition (31).

In the works of Redmond and Uretsky[13] and Bogolyubov, Logunov, and Shirkov,[14] a combination approach to the method of dispersion relations was employed, in which, besides the dispersion relations for obtaining a series of quantities in quantum field theory, series determined by perturbation theory were summed. The combination approach in the form in which it was formulated in these works, i.e., within the framework of single particle Green's functions only, possesses great ambiguity. The reasons for this ambiguity are discussed in reference 16; it is brought about principally by the fact that satisfaction of the dispersion relations for single particle Green's function is not a sufficient condition for fulfilling the requirements of causality and unitarity in the theory.

The concrete results of references 13 and 14 did not differ materially from ours. In the first place, although $D(\lambda)$ in references 13 and 14 does not possess non-physical poles, $\rho(\lambda)$ maintains a resonance character of behavior in the region of the former pole $[(e^2/12\pi^2)\ln\lambda \sim 1]$. In the second place, the specific non-analytic dependence of the superconducting type in $D(\lambda)$ in references 13 and 14 comes about from the use of the expression for $\rho(\lambda)$ outside the region of its applicability.

It is not without interest to note that if we set $\Gamma_1(\lambda) = 1$ in (32), and $\Gamma_2(\lambda) = 0$, then the formal solution of Eqs. (1) and (30) gives an expression for $\rho(\lambda)$ which coincides with that found in the work of Landau, Abrikosov, and Khalatnikov.[1]

In conclusion, we emphasize once more (see Sec. 1) that the most important problem in the investigation of "broken" dispersion equations (and in equal degree in the combination approach) is the calculation or even qualitative estimation of the role of higher approximations which include the more complicated matrix elements. In spite of the fact that corrections to the free Green's function of the photon in the simplest approximation turn out to be $\sim (e^2/4\pi)^{1/2}$ [see (29)], it is still impossible to say anything definite about the presence of a small parameter of expansion in such an approach. It appears to us that the solution of these questions permits us to shed additional light on the problem of the internal closed nature of quantum electrodynamics, and also to explain in what measure the method of dispersion relations is an escape from the framework of the Lagrangian formulation of quantum field theory. In equal measure this ap-

*This permits us, in place of the set of equations for $D(\lambda)$ and $F_1(\lambda)$, to limit ourselves to the initially asymptotic investigation of the equation only for $F_1(\lambda)$.

plies to all other interactions.

APPENDIX A

Let us consider $\bar{u}_+(p_+)\Gamma_\mu(p_+, p_-)u_-(p_-)$ according to perturbation theory. For simplicity, we set $p_+^2 = p_-^2 = p^2 \neq m^2$, $q = p_+ + p_-$;

$$\Gamma_\mu(p_+, p_-) = \frac{e^3}{i(2\pi)^4}\int \frac{\gamma_\nu(-\hat{p}_+ + \hat{k} + m)\gamma_\mu(\hat{p}_- + \hat{k} + m)\gamma_\nu d^4k}{[(p_+ - k)^2 - m^2 + i\varepsilon][(p_- + k)^2 - m^2 + i\varepsilon](k^2 + i\varepsilon)}. \quad \text{(A.1)}$$

Transforming the numerator with account of the "coverings" $\bar{u}_+$ and u_-, we get

$$\bar{u}_+\gamma_\nu(-\hat{p}_+ + \hat{k} + m)\gamma_\mu(\hat{p}_- + \hat{k} + m)\gamma_\nu u_- = \bar{u}_+[(-2q^2 + 4p^2)\gamma_\mu - 2\hat{k}\gamma_\mu\hat{k} - 2\gamma_\mu\hat{k}\hat{p}_+ + 2\hat{p}_-\hat{k}\gamma_\mu]u_-. \quad \text{(A.2)}$$

Infrared divergence occurs only in the first term on the right in (A.2). We limit ourselves to the consideration of the contribution of this term alone. Carrying out integration over d^4k in (A.1), we find

$$\Gamma_\mu(p_+, p_-) = e^3\gamma_\mu\frac{q^2 - 2p^2}{16\pi^2}\int_0^1 dx\int_0^1 dy \times [x(p^2 - q^2(1-y)y) - (p^2 - m^2) - i\varepsilon]^{-1}.$$

If we set $\Gamma_\mu(p_+, p_-) \equiv \gamma_\mu\Gamma_1(q^2, p^2)$, then for real q^2 and p^2,

$$\operatorname{Im}\Gamma_1(q^2, p^2) = e^3\frac{q^2 - 2p^2}{16\pi}\int_0^1 dx\int_0^1 dy\,\delta \times (x[p^2 - q^2(1-y)y] - (p^2 - m^2)) = \frac{e^3}{8\pi}\left(\frac{\lambda - 1}{\lambda - 2}\right)\left(1 - \frac{2}{\lambda}\right)^{1/2}\ln\frac{\lambda - 2}{\xi} \quad \text{for } p^2 \to m^2.$$

It is immediately seen from this expression that for $p^2 < m^2$, $\operatorname{Im}\Gamma_1(q^2, p^2)$ vanishes if $q^2 < 4m^2$, and for $p^2 > m^2$ and $q^2 < 0$,

$$\operatorname{Im}\Gamma_1(q^2, p^2) = \frac{e^3}{8\pi}\frac{q^2 - 2p^2}{16\pi|q^2|}\left(1 - \frac{4m^2}{q^2}\right)^{-1/2}\ln\frac{(1 - 4p^2/q^2)^{1/2} + 1}{(1 - 4p^2/q^2)^{1/2} - 1},$$

i.e., it is different from zero and finite for $p^2 \to m^2$ throughout the spatially similar region $q^2 < 0$. Consequently, the dispersion relations in the form (6) exist only for $p^2 < m^2$, which also supports our statement in Sec. 3.

APPENDIX B

Let us find the asymptote of $F_1(\lambda)$ from (15). Neglecting $F_2(\lambda)$ and substituting in (21) for $g(\lambda)$ the asymptotic value for $|\lambda| \to \infty$: $g(\lambda) \approx (e^2/8\pi^2)\ln(\lambda/\xi)$, we find

$$F_1(\lambda) \approx \frac{\lambda}{\pi}\int_2^\infty \frac{\tan^{-1}(e^2/8\pi^2)\ln(\lambda'/\xi)\,d\lambda'}{\lambda'(\lambda' - \lambda - i\varepsilon)}.$$

For $(e^2/8\pi^2)\ln(\lambda/\xi) \gg 1$ and $|\lambda| \to \infty$, the fundamental role in the integral (B.1) is played by $\lambda' \sim \lambda$. Therefore

$$F_1(\lambda) \to e\exp(-{}^1/_2\ln(\lambda/2)) = e(\lambda/2)^{-1/2},$$

Since

$$\tan^{-1}(e^2/8\pi^2)\ln(\lambda/\xi) \to \pi/2 \quad \text{for } \lambda \to \infty.$$

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Translated by R. T. Beyer
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