MAGNETOHYDRODYNAMICS OF WEAKLY CONDUCTING LIQUIDS

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The approximate form of the magnetohydrodynamic equations has been obtained for a fluid of low electrical conductivity (small magnetic Reynolds number) in an external magnetic field. Several characteristic problems which illustrate the physical behavior of such a liquid in a strong magnetic field are examined.

1. GALVANIC APPROXIMATION

As is well known (cf. for example, references 1 — 3), in magnetohydrodynamics there are additional characteristic dimensional quantities beyond those in ordinary hydrodynamics: the Alfvén velocity, $c_A = B(4\pi\mu\rho)^{-1/2}$, and the magnetic field diffusion coefficient, $D_m = c^2/4\pi\mu\sigma$. Here, B is the magnetic induction, ρ is the density, σ the electrical conductivity, and μ the magnetic permeability of the liquid, while c is the velocity of light. To these correspond two additional dimensionless similitude criteria, the magnetic analogs of the Mach number, v/c_A, and of the Reynolds number, $R_m = lv/D_m$ or $R_m = l^2\omega/D_m$ (l, v, and ω are respectively the characteristic linear dimension, the velocity, and the frequency). The magnetic Reynolds number characterizes the entrainment of the magnetic lines of force by the liquid in its motion. If $R_m \gg 1$, the lines of force behave as though frozen in the material — this is the case which is usually considered in magnetohydrodynamics, astrophysics, etc. We consider the other limiting case: $R_m \ll 1$. The equations used below can be obtained from the equations of magnetohydrodynamics by an expansion in powers of R_m, but it is more convenient to introduce them directly.

The volume density of the electromagnetic forces **f** is

$$\mathbf{f} = c^{-1}[\mathbf{j}\times\mathbf{B}], \tag{1.1}$$

where **j** is the current density. The currents which arise in the liquid are proportional to its electrical conductivity (small), but may result in the appearance of rather large forces if the liquid is located in an external magnetic field. This is the case which is considered here.

Because of its motion, an emf of order (v/c) B and a current of order σ(v/c) B are produced in the liquid. The magnetic field due to this current is of order $B_1 \sim R_m B$ and can be neglected when $R_m \ll 1$. The variation of B_1 produces an induced electric field $E_1 \sim R_m(vB/c)$ which is always unimportant when $R_m \ll 1$. Thus, in the approximation being used here it may be assumed that the magnetic field is given and equal to the external field. This field satisfies the equations div **B** = 0 and curl **B** = 0 (we assume that μ is independent of the coordinates or is approximately unity).

In determining the force **f** it is sufficient to consider only the currents which are produced in the liquid; the fields produced by these currents can be neglected. This approximation may be called the "galvanic" or non-induction approximation.

The current density in the moving liquid is

$$\mathbf{j} = (\sigma/c)\{[\mathbf{v}\times\mathbf{B}] - \nabla\varphi - \partial\mathbf{A}/\partial t\}, \tag{1.2}$$

where φ is the scalar potential (in magnetic units, i.e., $\varphi = c\varphi_{abs}$), **A** is the vector potential of the magnetic field (external) and curl **A** = **B**. We use the gauge condition div **A** = 0. As in "galvanostatics" the equation for φ is obtained from the condition that the charge is neutralized.

$$\operatorname{div}\mathbf{j} = 0. \tag{1.3}$$

For a uniform liquid this condition yields

$$\Delta\varphi = \mathbf{B}\operatorname{curl}\mathbf{v}. \tag{1.4}$$

If the conductivity is a function of coordinates, however, we have

$$\Delta\varphi = \mathbf{B}\operatorname{curl}\mathbf{v} - c(\mathbf{j}\nabla)\sigma^{-1}. \tag{1.4'}$$

Equation (1.1) for the volume forces assumes the form

$$\mathbf{f} = \frac{\sigma}{c^2}\left\{[[\mathbf{v}\times\mathbf{B}]\times\mathbf{B}] - [\nabla\varphi\times\mathbf{B}] - \left[\frac{\partial\mathbf{A}}{\partial t}\times\mathbf{B}\right]\right\} \tag{1.5}$$

This force appears in the equation of motion of the liquid

$$\rho(\partial\mathbf{v}/\partial t + (\mathbf{v}\nabla)\mathbf{v}) = -\nabla p + \rho\mathbf{g} + \eta\nabla^2\mathbf{v} + \mathbf{f}. \tag{1.6}$$

Here p is the pressure, **g** is the acceleration of gravity, the viscosity force is written for an incompressible liquid, and the viscosity is $\eta = \rho\nu$.

In the galvanic approximation, in addition to the equation of motion of the liquid it is necessary to have one scalar equation (1.4), which does not contain differentiation with respect to time; on the other hand, in true magnetohydrodynamics it is necessary to add a vector equation for $\partial \mathbf{B}/\partial t$.

We assume that the magnetic field is constant, $\partial \mathbf{A}/\partial t = 0$. If the currents produced by the induced emf $c^{-1}(\mathbf{v} \times \mathbf{B})$ can flow freely, the interaction of the liquid with the magnetic field results in an anisotropic "friction force" $\mathbf{f} = -(\sigma B^2/c^2)\mathbf{v}_\perp$, where $\mathbf{v}_\perp = \mathbf{v} - (\mathbf{v} \cdot \mathbf{B})\mathbf{B}/B^2$. If however $\operatorname{div}(\mathbf{v} \times \mathbf{B}) = \mathbf{B} \cdot \operatorname{curl} \mathbf{v} \neq 0$, in accordance with Eq. (14), charges will be produced and will create an electric field $-\nabla\varphi$. Charges and an electric field are also produced if the boundary conditions are such that the currents $(\sigma/c)\mathbf{v} \times \mathbf{B}$ cannot flow freely. In these cases the interaction of the liquid with the magnetic field does not produce magnetic friction and is not localized but, as before, the force $\mathbf{f}$ is a linear function of velocity. Since φ is linear in $\mathbf{v}$ (if the boundary conditions are linear), in the general case $\mathbf{f}$ is a linear function of the velocity taken at the same instant of time.

We may note that the magnetic field and the conductivity appear in $\mathbf{f}$ only in the combination σB^2. In the galvanic approximation this is the only additional dimensional parameter which appears beyond the usual hydrodynamic quantities.

In our analysis we shall consider a uniform incompressible liquid in a uniform magnetic field. In this case it is possible to obtain a convenient equation for the force $\mathbf{f}$ directly. Assuming that $\operatorname{div} \mathbf{v} = 0$, from Eq. (1.2) we find

$$\operatorname{curl} \mathbf{j} = (\sigma/c)(\mathbf{B}\nabla)\mathbf{v} - (\sigma/c)\partial\mathbf{B}/\partial t \tag{1.7}$$

and, using Eqs. (1.1) and (1.3)

$$\operatorname{curl}\operatorname{curl} \mathbf{f} = (\sigma B^2/c^2)(\mathbf{b}\nabla)^2\mathbf{v}, \tag{1.8}$$

where $\mathbf{b} = \mathbf{B}/B$ is a unit vector in the direction of the magnetic field. It is apparent that in the interaction of the liquid with the field there is a considerable change in the velocity component in the direction of the field.

Viscosity effects and the Joule heating effects cause dissipation of energy; this energy is given by

$$Q_\eta = (\eta/2)(\partial v_i/\partial x_k + \partial v_k/\partial x_i)^2,$$

$$Q_\sigma = j^2/\sigma = (\sigma/c^2)([\mathbf{v} \times \mathbf{B}] - \nabla\varphi)^2. \tag{1.9}$$

If $\nabla\varphi = 0$, the Joule dissipation can be regarded as the result of magnetic friction:

$$Q_\sigma = -\mathbf{f}\mathbf{v} = (\sigma B^2/c^2)\mathbf{v}_\perp^2.$$

In order-of-magnitude terms the ratio of the magnetic force to the viscosity force and the ratio of the corresponding dissipations are determined by the dimensionless Hartmann number, $M = \varkappa l$, where

$$\varkappa^2 = \sigma B^2/\eta c^2. \tag{1.10}$$

The ratio of the magnetic force to the inertia force is determined by the dimensionless number

$$N = \gamma_m/\omega = \gamma_m l/v = M^2 R^{-1},$$

where R is the Reynolds number,

$$\gamma_m = \sigma B^2/\rho c^2. \tag{1.11}$$

Which terms predominate in the equation of motion depends on the relative magnitudes of the quantity σB^2, the velocity, and the linear dimensions.

1. If σB^2 is very small, so that $M \ll 1$, the magnetic field has only a small effect on the motion of the liquid which, in the first approximation, is the same as in ordinary hydrodynamics. The chief new effects are the production of currents and potentials which, for a given velocity distribution, are determined by Eqs. (1.2) and (1.4). These effects can be used, for example, for measuring velocities. From a knowledge of the current one can find the force $\mathbf{f}$; then, substituting in Eq. (1.6) if necessary, it is possible to determine the small corrections to the velocity and pressure. This case is typical for electrolytes; for example for a 25% solution of NaCl in water $\sigma = 2 \times 10^{11}$ and for $B = 10^3$ gauss, $\kappa = 0.15\ \text{cm}^{-1}$. For liquid metals under laboratory conditions it is easy to make $M \gg 1$.

All the numerical examples below are given for mercury: the pertinent parameters are as follows: $\sigma = 0.95 \times 10^{16}\ \text{sec}^{-1}$, $\rho = 13.6\ \text{g-cm}^{-3}$, $\eta = 1.56 \times 10^{-2}\ \text{g-cm}^{-1}\ \text{sec}^{-1}$, $\nu = \eta/\rho = 1.15 \times 10^{-3}\ \text{cm}^2\ \text{sec}^{-1}$, $D_m = c^2/4\pi\sigma = 0.75 \times 10^4\ \text{cm}^2\ \text{sec}^{-1}$. For $B = 10^3$ gauss, $\kappa = 26\ \text{cm}^{-1}$, $\gamma_m = 0.78\ \text{sec}^{-1}$. The ratio $R/R_m = D_m/\nu = 1.5 \times 10^7$ so that the galvanic approximation applies for values of the Reynolds number up to $R \lesssim 10^7$.

2. At large values of the Reynolds number it is possible to have $M \gg 1$, but $N \ll 1$. In this case the magnetic forces are small compared with the inertial forces and the behavior of the liquid is approximately that of an ideal liquid. If the motion is laminar the magnetic forces play an important role in the dissipation of energy (damping of oscillations etc.). Furthermore, the magnetic forces change the nature of the boundary layers and this, in turn can change the flow as a whole and the stability conditions as, for example, in flow in tubes.

Flow between parallel planes has been shown theoretically[4] and experimentally[5] to be stable in a transverse magnetic field up to $R \sim 10^3 M$. If a turbulence develops, however, the effect of the magnetic field is small. For motion on the basic scale the magnetic forces are small compared with the inertial forces since $N \ll 1$; however, the frequencies of the turbulent motions increase with a reduction in scale and in all smaller scales it is certainly true that $N = \gamma_m/\omega \ll 1$. When $\lambda_0 \sim lR^{-3/4}$, however, where the basic dissipation occurs (cf. reference 2, §32), the viscosity dominates: $\kappa\lambda_0 \sim N^{1/2}R^{-1/4} \ll 1$ so that the magnetic forces are also unimportant in dissipation.

3. The effect of the magnetic field is most pronounced when $M \gg 1$, $N \gg 1$, in which case the force $\mathbf{f}$ dominates. The high Joule dissipation characteristic of this case has a stabilizing effect on the flow. Stuart[6] has shown, for example, that Poiseuille flow between plates is stabilized by a longitudinal magnetic field when $N \gtrsim 0.1$. However, the stabilizing effect of the Joule dissipation may be insufficient if the production of the instability is associated with small-scale high frequency motions, for which N is already small. This is the case, for example, in the experiment carried out by Lehnert.[7] Turbulent motion for $M \gg 1$ and $N \gg 1$ is also subject to a strong effect of the magnetic field, which plays a decisive role in motion of large-scale vortices and may change the entire motion pattern.

At small Reynolds numbers, if $\operatorname{curl}(\mathbf{v}\cdot\nabla)\mathbf{v} = 0$, because of symmetry, from Eqs. (1.6) and (1.8) we can obtain the following equation for stationary flows; this equation determines the "leakage" of the fluid through the magnetic field:

$$\Delta\Delta\mathbf{v} - \varkappa^2(\mathbf{b}\nabla)^2\mathbf{v} = 0. \qquad (1.12)$$

From Eqs. (1.4) and (1.6), $\operatorname{curl}\mathbf{f} = c^{-1}(\mathbf{B}\cdot\nabla)\mathbf{j}$ and Eq. (1.2) we obtain the same equation for φ:

$$\Delta\Delta\varphi - \varkappa^2(\mathbf{b}\nabla)^2\varphi = 0. \qquad (1.12')$$

Equation (1.12) for $\operatorname{curl}\mathbf{v}$ has already been obtained by Lehnert[8] for stationary small perturbations by linearization of the equations of magnetohydrodynamics with an arbitrary conductivity.

Suppose that a characteristic dimension of the system in the direction perpendicular to the field is l, and along the field is $l_{\|}$. When $M = \kappa l \gg 1$ both terms in Eq. (1.12) may be of the same order if $l_{\|} \sim \kappa^{-1} \ll l$ or $l_{\|} \sim \kappa l^2 \gg l$. An example of a solution of the first type, which describes a perturbation transverse to the field, gives the boundary layer formed in the Hartmann[9] analysis of flow between plane plates in a perpendicular magnetic field:

$$v = v_0[\cosh\varkappa h - \cosh\varkappa(h-y)]/\cosh\varkappa h. \qquad (1.13)$$

The magnetic field is along y, the velocity along z, and the distance between the plates is 2h.

A solution of the second type describes perturbations which are along the magnetic field. For example, they may act like tangential discontinuities. Equation (1.12) is satisfied by the expressions $\mathbf{v} = c_+\mathbf{v}_+ + c_-\mathbf{v}_-$, where $\mathbf{v}_\pm$ represents solutions of the equation

$$\{\Delta \mp \varkappa(\mathbf{b}\nabla)\}\mathbf{v}_\pm = \text{const}_\pm.$$

Let $B = B_y$, $v = v_z$ and $\partial/\partial z = 0$. If we neglect $\partial^2/\partial y^2$ compared with $\partial^2/\partial x^2$ in the Δ operator, the curly brackets will contain the same operators as those in the usual thermal-conductivity equation so that we can write the approximate solution immediately:

$$v_+ = \frac{v_0}{2}\Phi\left(\sqrt{\frac{\varkappa}{4(y_0+y)}}(x-x_0)\right), \quad \Phi(\xi) = \frac{2}{\sqrt{\pi}}\int_0^\xi e^{-t^2}dt. \qquad (1.14)$$

This solution describes the velocity jump from $-v_0/2$ to $+v_0/2$ at $x = x_0$. The effective half width of the discontinuity δ is

$$\delta = \sqrt{4(y_0+y)/\varkappa}. \qquad (1.15)$$

If $\kappa(y_0+y) \gg 1$, $\partial^2/\partial y^2$ can be neglected compared with $\partial^2/\partial x^2$.

We now compute the current. From the equilibrium equation $f_z + \eta\Delta v = 0$, we obtain $j_x = -(c/B)\times\eta\Delta v$; however, $\Delta v = \kappa\partial v/\partial y$ so that $j_x = -\sqrt{\sigma\eta}\times\partial v/\partial y$. Then, integrating Eq. (1.3) we have $j_y = \sqrt{\sigma\eta}\,\partial v/\partial x$. Thus, $\sqrt{\sigma\eta}\,v(x,y)$ is a function of the current for the vector $\mathbf{j}$. The "surface" current flowing along the discontinuity is $J_y = \int j_y dx = \sqrt{\sigma\eta}\,v_0$ or

$$\mathbf{J}_+ = -\sqrt{\sigma\eta}([\mathbf{n}\times\mathbf{v}]_1 + [\mathbf{n}\times\mathbf{v}]_2); \qquad (1.16)$$

the subscripts 1 and 2 refer to the left and right side of the discontinity, $\mathbf{n}_{1,2}$ corresponds to the corresponding inward normals. Knowing the current, it is easy to find the potential gradient $\partial\varphi/\partial y = -(B/\kappa)\partial v/\partial x$, $\partial\varphi/\partial x = -vB + (B/\kappa)\partial v/\partial y$. Whence it is apparent that there is a tangential discontinuity in the charge and the discontinuity in the electric field is

$$(\partial\varphi/\partial x)_2 - (\partial\varphi/\partial x)_1 = -v_0 B. \qquad (1.17)$$

These same results apply if there is a fixed pressure gradient $\partial p/\partial z = -\Gamma$; it is only necessary to add the constant terms $j_x(\infty) = -c\Gamma/B$ and $c^2\Gamma/\sigma B$, to j_x and $\partial\varphi/\partial x$.

The tangential discontinuity may also be regarded as a jet of electric current along the magnetic field. In the "galvanostatics" of solid conductors such jets are not possible because the current tends to flow along the line of least resistance and diverges to a large cross section; in a liquid, however, the force due to the j_x component produces a velocity v_z, which in turn results in an emf which supports the current in the jet.

Changing the sign in front of y in Eq. (1.14) we obtain the solution v_-, which corresponds to the current J_- and differs from Eq. (1.16) only in sign.

It will be shown below that in the motion of a liquid the current frequently appears only in thin layers and has a surface density $\sqrt{\sigma\eta}\, v$. In these cases the region of applicability of the galvanic approximation can be expanded since the magnetic field which is produced is of order $B_1 \sim (4\pi\mu/c) \times \sqrt{\sigma\eta}\, v \sim (R_m/M) B$ rather than $R_m B$.

2. SMALL OSCILLATIONS

In small oscillations we can neglect the quadratic term $(\mathbf{v}\cdot\nabla)\mathbf{v}$. The equation of motion is written in the form

$$\partial \mathbf{v} / \partial t = \nu \nabla^2 \mathbf{v} + \mathbf{F}, \quad \operatorname{div} \mathbf{v} = 0, \tag{2.1}$$

where $\mathbf{F} = -\nabla p'/\rho + \mathbf{f}/\rho$ and p' is the deviation of the pressure from the equilibrium value p_0. In accordance with Eq. (2.1), $\operatorname{div} \mathbf{F} = 0$, and from Eq. (1.8) we have

$$\Delta \mathbf{F} = -\gamma_m (\mathbf{b}\nabla)^2 \mathbf{v}. \tag{2.2}$$

Volume waves. We consider a plane wave in which all quantities are proportional to $\exp(i\mathbf{k}\cdot\mathbf{r}\ i\omega t)$. From Eqs. (2.1) and (2.2) we find that the possible motions for a plane wave are two slipping motions $\mathbf{v}\perp\mathbf{k}$, which are damped with the same decrement

$$\omega = -i(\nu k^2 + \gamma_m (\mathbf{bn})^2), \tag{2.3}$$

where $\mathbf{n} = \mathbf{k}/k$. The magnetic damping is due only to the field component parallel to the gradient. The perturbation of the motion is found from $\operatorname{div} \mathbf{F} = 0$; the quantities $\mathbf{j}$ and φ are found from Eq. (1.7) and Eq. (1.4):

$$\frac{p'}{\rho} = -\frac{i\gamma_m}{k^2}(\mathbf{bk})(\mathbf{bv}), \qquad \mathbf{j} = -\frac{\sigma B^2}{ck^2}(\mathbf{bk})[\mathbf{k}\times\mathbf{v}],$$

$$\varphi = -\frac{iB}{k^2}([\mathbf{b}\times\mathbf{k}]\,\mathbf{v}). \tag{2.4}$$

Thus, in the galvanic approximation magnetohydrodynamic waves are not possible. The small conductivity leads to the appearance of an additional mechanism but, in contrast with the high-conductivity case, there is no elastic restoring force. When $\omega \sim \gamma_m$, $R_m = \gamma_m / D_m k^2$; this result has been obtained by Lundquist;[10] the galvanic approximation applies if R_m is small. For mercury with $B = 10^4$ gauss, $k = 1\ \text{cm}^{-1}$ and $R_m = 10^{-2}$.

If the compressibility of the liquid is taken into account we find that in addition to the damped slipping motion it is also possible to have longitudinal sound waves with frequencies given by $\omega = c_s k$. The acoustic velocity is large (for mercury $c_s = 1.5\times 10^5$ cm/sec) so that it is not very sensitive to the magnetic field ($\gamma_m \ll \omega$); however, the damping of the acoustic wave may be a strong function of magnetic field. The current $(\sigma/c)\,\mathbf{v}\times\mathbf{B}$ flows freely along $\mathbf{k}\times\mathbf{B}$ so that the Joule dissipation can be regarded as the result of magnetic friction. The mean dissipation rate is $\sigma B^2/c^2\, \overline{v_\perp^2} = \gamma_m (\mathbf{b}\times\mathbf{n})^2 \rho \overline{v^2}$, where $\rho\overline{v^2}$ is the energy density of the wave and the magnetic damping is

$$\gamma_{\text{mag}} = [\mathbf{b}\times\mathbf{n}]^2 \gamma_m / 2. \tag{2.5}$$

We note that this quantity is anisotropic and independent of frequency. Equation (2.5) applies only when $R_m = c_s^2/\omega D_m \ll 1$, i.e., for high (and not low!) frequencies; for example, $\omega \gg 10^6$ for mercury. The ratio of the magnetic damping to viscosity is of order $\gamma_m/\nu k^2 = \kappa^2 k^2$; with $\omega \sim 10^6\ \text{sec}^{-1}$, $k \sim 10\ \text{cm}^{-1}$ and $B = 10^4$ gauss, this ratio is approximately 10^3.

Surface waves. Suppose that the z axis is vertical and the x axis is in the direction of propagation, so that all quantities vary as $\exp(ikx - i\omega t)$. The unperturbed surface is given by $z = 0$ and the perturbed surface by $z = \zeta = \zeta_0 \times \exp(ikx - i\omega t)$. The unperturbed pressure is $p_0 = -\rho g z$. We limit ourselves, for simplicity, to the case in which viscosity can be neglected $k^2 \ll \kappa^2$. In this case the boundary conditions at $z = 0$ (with surface tension taken into account) assume the form (cf. reference 2, §61)

$$v_z(z=0) = -i\omega\zeta, \qquad p'(z=0) = (\rho g + \alpha k^2)\zeta, \tag{2.6}$$

where α is the surface tension coefficient. An additional condition at the surface is $j_z = 0$.

It is apparent that it is sufficient to consider the velocity components v_x and v_z, taking $v_y = 0$. In this case the force $\mathbf{F}$, as follows from Eq. (2.2), has no y component; curl $\mathbf{v}$ and the current density $\mathbf{j}$ are along the y axis so that the condition $j_z(z=0) = 0$ is satisfied trivially. Thus, B_y of the magnetic field has no effect on the wave.

We seek a solution proportional to e^{mz} and introduce the complex vector $\mathbf{q}(k, 0, -im)$. From Eqs. (2.1) and (2.2) we find the condition which must be satisfied for a non-trivial solution

$$i\omega q^2 = \gamma_m (\mathbf{bq})^2, \tag{2.7}$$

and also $\mathbf{q}\cdot\mathbf{v} = 0$. The quadratic equation (2.7) determines m; we must take values which give damping inside the liquid, Re m > 0. The expression for p' can be obtained by replacing $\mathbf{k}$ by $\mathbf{q}$ in Eq. (2.4). Using $\mathbf{q}\cdot\mathbf{v} = kv_x - imv_z = 0$, we can express v_x and p' in terms of v_z:

$$p'/\rho = (i\omega/k)(\mathbf{bq})^{-1}(mb_x - ikb_z)v_z. \quad (2.8)$$

Here we have also made use of Eq. (2.7). Substituting Eq. (2.8) in Eq. (2.6), we obtain the relation for the frequency:

$$\omega^2(mb_x - ikb_z) = \omega_0^2(bq), \quad (2.9)$$

where ω_0 is the frequency in the absence of the magnetic field:

$$\omega_0^2 = gk + \alpha k^3/\rho. \quad (2.10)$$

After some elementary algebraic transformations, we obtain from Eqs. (2.7) and (2.9)

$$m/k = (\omega_0^2 b_x + \omega^2 i b_z)(\omega^2 b_x + \omega_0^2 i b_z)^{-1}, \quad (2.11)$$

$$\omega^4 - \omega_0^4 + \omega^3 i\gamma_m(b_x^2 + b_z^2) = 0. \quad (2.12)$$

If $\gamma_m \ll \omega$, damped oscillations are possible:

$$\omega = \omega_0 - (i\gamma_m/4)(b_x^2 + b_z^2). \quad (2.13)$$

The motion in this case is approximately potential motion, $m \approx k$, as is to be expected for $N \ll 1$. However, if $\gamma_m \gg \omega$ the perturbations of the surface are damped aperiodically:

$$\omega = -i\gamma_m(b_x^2 + b_z^2). \quad (2.14)$$

These results agree with the experimental data. It has been found in reference 8 that in a field of 10^4 gauss surface oscillations in mercury are completely suppressed. Under these conditions $\gamma_m = 78$ and the oscillations are damped rapidly; when $k < 4\ \mathrm{cm}^{-1}$, generally speaking it is impossible to have oscillations at all.

3. FLOW ALONG A TUBE

We consider stationary flow of a liquid in a long tube transverse to the magnetic field. We take the z axis along the axis of the tube: $v = v_z$ and $B = B_y$. All quantities are independent of z except for the pressure, which has a constant derivative in this direction.* This problem has been solved by Schercliff[11] for a non-conducting tube with cross section symmetric with respect to the x axis. The solution in reference 11 was obtained by substitution of variables; for the case of a conducting tube or an asymmetric tube this procedure is not quite as effective. Below a more general case is considered.

*In the liquid there are induced currents in the xy plane; the magnetic field in the z direction due to these currents has no effect on the force f_z since there are no time variations so that for an infinite tube length it is not necessary that R_m be small.

Projecting Eqs. (1.1), (1.2), and (1.6) on the z axis we have

$$\Gamma + \eta\Delta v + f_z = 0,$$

$$f_z = c^{-1}j_x B = -(\sigma B^2/c^2)(v + B^{-1}\partial\varphi/\partial x). \quad (3.1)$$

Here $\Gamma = -\partial p/\partial z$ and for an inclined tube $\Gamma = -\partial p/\partial z + \rho g \sin\alpha$, where α is the angle of inclination with respect to the horizontal.

When $M \ll 1$, viscosity predominates and the flow is approximately the same as ordinary Poiseuille flow. We consider the opposite limiting case, $M \gg 1$. In this case viscosity is important only in the thin layers close to the walls of the tube.

Boundary layer. We will approximate a small section of a wall as a plane and introduce the coordinates s, along the periphery of the cross section, and r, along the normal $\mathbf{n}$. Taking derivatives with respect to r only we obtain the solution (1.12) corresponding to the Hartmann layer $v = v_0 \times [1 - \exp(-\kappa_n r)]$, where $\kappa_n^2 = \kappa^2(\mathbf{b}\cdot\mathbf{n})^2$. By v_0 we mean the value of the velocity obtained from the solution for the inner region for a given point of the periphery.

Let the wall be insulating; then, close to the wall $j = j_s$. Using Eq. (1.7) we have

$$j_s = (\sigma B_n/c)\exp(-\varkappa_n r) + j_{s\infty}.$$

The "surface" current in the layer is

$$J_s = \int (j_s - j_{s\infty})\,dr = \sqrt{\sigma\eta}\,v_0$$

or

$$\mathbf{J} = \sqrt{\sigma\eta}\,[\mathbf{n}\times\mathbf{v}_0](\mathbf{bn})/|(\mathbf{bn})|. \quad (3.2)$$

If $j_{s\infty} \ll j_s$, using Eq. (1.9) it is possible to compute the energy dissipation in the Hartmann layer. The viscous loss and Joule dissipation are the same and their sum per unit surface is

$$Q_1 = \sqrt{\sigma\eta}\,c^{-1}|B_n|v_0^2. \quad (3.3)$$

If the conditions in the layer change slowly and a current $\mathbf{j}$ flows toward the insulating wall, it is possible to write an equation for conservation of charge for the layer

$$dJ_s/ds + (\mathbf{jn}) = 0. \quad (3.4)$$

Let the wall of the tube be highly conducting; in this case it assumes a fixed potential φ_0. If the electric field outside the layer is much smaller than the field inside we can write the potential drop in the layer $\varphi_\infty - \varphi_0$.

Using Eq. (1.4) we have

$$\varphi = \varphi_0 + B_s v_0 \varkappa_n^{-1}[1 - \exp(-\varkappa_n r)].$$

The potential discontinuity in the layer is

$$\varphi_\infty - \varphi_0 = (B/\varkappa_n)(\mathbf{b}\,[\mathbf{n}\times\mathbf{v}_0]). \quad (3.5)$$

The relations in Eqs. (3.3) – (3.5) serve as the effective boundary conditions for the equations inside the region.

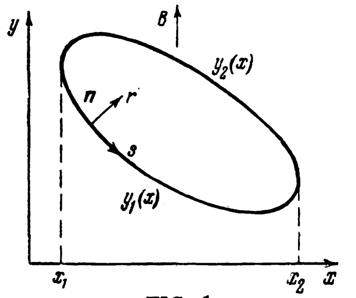

FIG. 1

It should be noted that this picture of the layer is valid only when $\kappa_n l \gg 1$; if, however, $B_n \to 0$, the relations which have been obtained no longer hold. This situation pertains, for example, to the regions near the points x_1 and x_2 (Fig. 1). If the wall of the tube is parallel to the magnetic field a longitudinal surface layer of the type considered by Shercliff[11] is formed; this is similar in structure to the tangential discontinuity (1.14). The layer thickness is of order $\sqrt{l/\kappa}$ [cf. Eq. (1.15)].

Viscosity can be neglected in the inner part of the tube and Eq. (1.12) yields $\partial^2 v/\partial y^2 = 0$, $\partial^2\varphi/\partial y^2 = 0$. Whence

$$v(x, y) = v_0(x) + v_1(x)\,y, \quad \varphi(x, y) = \varphi_0(x) + \varphi_1(x)\,y, \quad (3.6)$$

where φ_0 and φ_1 are related to v_0 and v_1 in accordance with Eq. (3.1). In the inner region, from Eq. (3.1) j_x is constant and since div $\mathbf{j} = 0$, j_y depends on x and is determined by Eq. (1.7)

$$j_x = -(c\Gamma/B), \quad dj_y/dx = (\sigma B/c)\,v_1(x). \quad (3.7)$$

Tube with insulating walls. Let the cross section of the tube be given by two single-valued curves $y_1(x)$ and $y_2(x)$ (cf. Fig. 1). The effective boundary conditions (3.2) at these boundaries can be written in the form

$$J_1 = \sqrt{\sigma\eta}\,v(x, y_1), \quad J_2 = -\sqrt{\sigma\eta}\,v(x, y_2). \quad (3.8)$$

In terms of the difference of these quantities, using Eq. (3.6) it is easy to express the total flow of liquid V through the cross section of the tube:

$$V = \int v\,dx\,dy = 2^{-1}(\sigma\eta)^{-1/2}\int_{x_1}^{x_2}(J_1 - J_2)(y_2 - y_1)\,dx.$$

Now, making use of Eq. (3.4) and taking account of the fact that $n_y\,ds = dx$ and $n_x/n_y = -dy/dx$, we obtain

$$d(J_1 - j_x y_1)/dx = -j_y, \quad d(J_2 - j_x y_2)/dx = -j_y. \quad (3.9)$$

Subtracting the second equation from the first and integrating, we have

$$J_1 - J_2 = -j_x(y_2 - y_1) = (c\Gamma/B)(y_2 - y_1).$$

Thus, the flow of liquid in a tube of arbitrary cross section with non-conducting walls is

$$V = (c\Gamma/B\sqrt{\sigma\eta})\int_{x_1}^{x_2}[y_2(x) - y_1(x)]^2\,dx. \quad (3.10)$$

Summing Eq. (3.9), taking account of Eqs. (3.7) and (3.8), and neglecting the term of order $1/\kappa l$, we have

$$j_y = -(c\Gamma/2B)\,d(y_1 + y_2)/dx. \quad (3.11)$$

If the tube is symmetric with respect to the x axis $j_y = 0$, $v_1 = 0$ and Eq. (3.10) coincides with the result which has been obtained by Shercliff. If the shape is not symmetric, then $j_y \sim j_x$ but the velocity shows almost no change along the field $v_1 y \ll v_0$. Actually, from Eq. (3.10) $v \sim c\Gamma l/B\sqrt{\sigma\eta}$ whereas from Eq. (3.7) $v_1 l \sim (c/\sigma B)\,j_y \sim v/\kappa l$.

Tube with highly conducting walls. A completely different picture obtains if the walls of the tube are highly conducting. In this case, the current can flow freely across the magnetic field, being closed in the walls of the tube, the potential of which may be assumed constant. In the zeroth approximation the interaction of the liquid with the field leads to magnetic "friction" and the term associated with the production of a potential difference is small, approximately M^{-1}. In this approximation, for the entire inner region we have

$$v = c^2\Gamma/\sigma B^2, \quad j = j_x = \sigma B v/c, \quad \varphi = 0. \quad (3.12)$$

The corrections associated with the next approximation can also be found easily. As before, in the inner region we can neglect viscosity since $\eta\Delta v \sim M^{-2}$ so that v and φ inside are of the form given in Eq. (3.6). The electric field inside $(\sim \varphi/l)$ is much smaller than in the layer $(\sim \kappa\varphi)$ so that we can use the effective boundary condition in (3.5). If $y = y_1$ and $y = y_2$, this condition becomes

$$\varphi(x, y_1) = (Bv/\varkappa)\,dy_1/dx, \quad \varphi(x, y_2) = -(Bv/\varkappa)\,dy_2/dx.$$

The quantities φ_0 and φ_1 are determined by the conditions:

$$\varphi_0(x) = (Bv/\varkappa)(y_2 - y_1)^{-1}\,d(y_1 y_2)/dx,$$
$$\varphi_1(x) = -(Bv/\varkappa)(y_2 - y_1)^{-1}\,d(y_2 - y_1)/dx. \quad (3.13)$$

Tubes with electrodes. Comparing the velocity in tubes with conducting walls and nonconducting walls, we see that for a given Γ, the velocity in the first case is approximately M times smaller

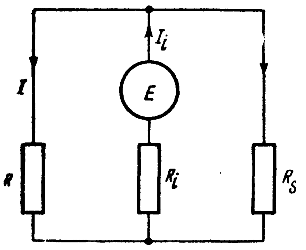

FIG. 2

than in the second. If the walls of the tube are insulating, but the upper and lower walls have electrodes which are connected through an external resistance R, it is possible to obtain a continuous transition between the two cases. The equivalent electric circuit of this tube is shown in Fig. 2. The current I_i produced by the induction emf $\mathscr{E}$ flows across the tube through the liquid (R_i) part of it returning through the resistance R and part through the liquid in the boundary layer (R_s).

We consider in greater detail a tube of rectangular cross section ABCD in which the sides AB = CD = l, perpendicular to the magnetic field, are insulating and the sides AC = BD = h, parallel to the field, are highly conducting and connected through an external resistance. The length of the tube is L. On sides AB and CD Hartmann layers are formed; on walls AC and BD longitudinal boundary layers similar to the Shercliff layer are formed. Since the thickness of the longitudinal layers [cf. Eq. (1.15)] is approximately $(h/\kappa)^{1/2} \sim M^{1/2}\kappa^{-1}$ and is much thicker than the Hartmann layers, the resistance of these layers is much smaller and the role they play in dissipation is not important. In general they need not be considered. From Eq. (3.6) and the symmetry conditions, over the entire inner region of the tube $\varphi = \varphi(x)$; consequently, $j_y = 0$. Whence it follows that the currents in the layers are constant and, in accordance with Eqs. (3.2) and (3.6), the velocity v is constant over the entire inner region. For this case of uniform flow it is easy to find the current which issues from the tube:

$$I = j_x hL + 2J_s L = -c\Gamma hL/B + 2\sqrt{\sigma\eta}vL,$$

and the difference of potentials between the conducting walls

$$l d\varphi/dx = -lvB + lc^2\Gamma/\sigma B.$$

On the other hand, $l\, d\varphi/dx = cIR$ so that

$$v = \frac{c^2\Gamma}{\sigma B^2}\left(1+\frac{R}{R_i}\right)\left(1+\frac{R}{R_s}\right)^{-1}, \qquad (3.14)$$

where we have used the notation

$$R_i = l/\sigma hL, \qquad R_s = \varkappa l/2\sigma L, \qquad \mathscr{E} = (v/c)Bl. \qquad (3.15)$$

The meaning of Eq. (3.14) is simple. When $R \gg R_s$, the electromagnetic force IBl/c vanishes since $I \approx 0$. The magnetic field is found to have only an indirect braking effect, producing a thin layer in which the friction force is $\eta\kappa v$; hence $v = \Gamma h/2\eta\kappa$ is approximately M times smaller than in Poiseuille flow. If $R \ll R_s$, the basic retardation effect is due to the force IBl/c, where $I = \mathscr{E}(R + R_i)^{-1}$ so that $v = (c^2\Gamma/\sigma B^2)(1 + R/R_i)$.

For other cross sections the circuit (Fig. 2) is still valid but Eq. (3.14) is accurate only to an order of magnitude. We may note that if the side walls are inclined rather than parallel to the field, Hartmann layers will also be formed on them. This means a certain increase in resistance; on the other hand there is an increase in the stability of flow at high velocities since the effective Reynolds number $R_\delta = v\delta/\nu$ is reduced (δ is the thickness of the layer).

Self-excited dynamo. If the current flowing out of the tube I is used for producing the magnetic field in the tube, under certain conditions it is possible to obtain a self-excited system. In this case the magnetic field is proportional to the current B = KI, where $K \sim 4\pi n/ch$ and n is the number of turns of the coil which produces the field. Assuming that $\mathscr{E} = (v/c)Bl$, we obtain the equation which determines the current:

$$LdI/dt + R_{\text{eff}} I = 0,$$

$$R_{\text{eff}} = R + R_i R_s (R_i + R_s)^{-1} - R_s (R_i + R_s)^{-1} K l v/c, \qquad (3.16)$$

where L is the self-inductance of the coil. The criterion for self-excitation is $R_{\text{eff}} < 0$ or $v > v_{\text{exc}}$ where $v_{\text{exc}} \sim (c^2 h/4\pi n l)[R_i + R(1 + R_i/R_s)]$. At small fields $R_s \sim R_i$. In order to increase K it is convenient to increase the number of turns of the coil; conversely R can be reduced by reducing the number of turns. In a system without iron the turns are wound along the tube $R \sim n^2 L/\sigma_c S$, where σ_c is the conductivity of the wire in the coil, S is the total cross section of the turns ($S \lesssim hl$). The most favorable conditions are obtained when $R \sim R_i$ or $n^2 \sim \sigma_c S/\sigma L^2$. When $\sigma_c/\sigma = 60$ (copper and mercury) this condition gives $n \sim 3-4$. For mercury, with L = 10 cm an estimate yields $v_{\text{exc}} \sim 10^3$ or several times 10^2 cm/sec. If a ferromagnetic material is used in the electromagnet, it is possible to reduce the dimensions of the turns and a greater number of turns can be used for the same R. With n = 10 and L = 10 cm, for example, for mercury we obtain $v_{\text{exc}} \sim 10^2$ cm/sec.

After excitation the field grows so long as its retarding effect does not reduce the velocity in such a way that $R_{\text{eff}} = 0$ or $4\pi n l v/c^2 h = R + R_i$.

Substituting this criterion in Eq. (3.14) we find the magnetic field $B^2 = 4\pi m\delta p$, where $\delta p = \Gamma L$ is the pressure difference in the gap with the magnetic field present. With $\delta p \sim 1$ atmos, values of several kilogauss are obtained. If iron is used, the field can be limited by saturation; this causes a reduction in K.

This scheme represents an example of the transformation of mechanical energy of a liquid into magnetic energy although, obviously, it is not a "true" hydromagnetic dynamo since solid conductors and insulators are used.

Flow along a channel (trough). If the tube is symmetric with respect to the x axis, in the median plane the boundary conditions for a free surface $j_y = 0$ and $\partial v/\partial y = 0$ are satisfied; hence the problem of a channel reduces to the problem of a tube with $\Gamma = -g\rho \sin\alpha$. For example, the flow of a liquid layer of thickness h along an inclined plane is the same as the flow (1.13) between plates with a spacing 2h. When $\kappa h \gg 1$ the flow velocity along an insulating inclined plane is κh times smaller than along a conducting plane.

The damping of surface waves has been considered in Section 2 for short wave lengths, i.e., $kh \gg 1$. On the other hand, if $kh \ll 1$ it is possible to have (cf. reference 2, §13) long gravitational waves with propagation velocities $\sqrt{gh}$. In these waves the velocity is essentially parallel to the bottom. We now find the damping of these waves (assuming that it is small and that $\kappa h \gg 1$) in the presence of a vertical magnetic field. If the trough is conducting, the currents flow freely and the damping is computed in exactly the same way as for the acoustic waves. The decrement is $\gamma = \gamma_m/2$ [cf. Eq. (2.5)]. If the trough is insulating, dissipation takes place chiefly in the boundary layer at the bottom. According to Eq. (3.3) the mean dissipation per unit surface is $\sqrt{\sigma\eta}\,(B/c)\overline{v^2}$; comparing this with the energy of the wave $h\rho v^2$, we find

$$\gamma = \sqrt{\sigma\eta}\,B/2h\rho c = \gamma_m/2\kappa h. \qquad (3.17)$$

For a channel of arbitrary cross section the damping can be determined by using Eq. (3.10) to compute the resistance of a corresponding symmetric tube.

4. ROTATION OF A LIQUID

We introduce the cylindrical coordinates r, φ and z. We assume that $\partial/\partial\varphi = 0$, that the magnetic field is vertical, $B = B_z$, and that the velocity of rotation $v_\varphi = v(r,z)$. We will also assume that $v_r, v_z \ll v$ and neglect $\operatorname{curl}(\mathbf{v}\cdot\nabla)\mathbf{v}$ so that Eq. (1.12) can be used. We will also neglect the meniscus at the free surface; at small rotation velocities this procedure is valid.

Rotation of a Liquid Caused by Rotation of the Bottom of the Container

A. Insulating Bottom. Let the annular region $R_1 \le r \le R$ at the bottom be rotated with a constant angular velocity Ω. The boundary conditions are of the following form: on the bottom $z = 0$, $v = v_d$ ($v_d = \Omega r$ for $R_1 \le r \le R$, $v_d = 0$ for $r < R_1$ and $r > R$), $j_z = 0$ or $\partial\varphi/\partial z = 0$; at the free surface $z = h$, $\partial v/\partial z = 0$, $j_z = 0$ or $\partial\varphi/\partial z = 0$. In a strong magnetic field, $M \gg 1$, only the layer of liquid above the rotating part of the bottom rotates. It is easy to construct the approximate solution (1.12) in this case. At $r = R_1$ and $r = R$ there are tangential discontinuities in the velocity $v = \Omega R_1(\frac{1}{2} + \overline{\Phi})$ and $v = \Omega R(\frac{1}{2} - \overline{\Phi})$, where [cf. Eq. (1.14), $x = r - R_1$ or $x = r - R$]

$$\overline{\Phi} = {}^1/_4[\Phi(\sqrt{\varkappa/4z}\,x) + \Phi(\sqrt{\varkappa/4(2h - z)}\,x)], \qquad (4.1)$$

and between the discontinuities the liquid rotates as a whole, together with the bottom, with velocity Ωr. In this case, Eq. (1.4) (with the boundary conditions in (1.17) at the charge discontinuities) gives $\partial\varphi/\partial r = B\Omega r = vB$, whence $j_r = 0$ so that no tangential force acts on the liquid. The centrifugal force $\rho v^2/r$ is equalized by the pressure gradient. Across the rotating liquid there is a potential difference $(B\Omega/2)(R^2 - R_1^2)$. When $R_1 = 0$ there is no inner discontinuity.

We consider the velocity discontinuity in greater detail, for example, at $r = R$. Equation (4.1) is a superposition of solutions for current jets which flow upward and downward; for $z = h$, one is easily convinced, this equation yields $j_z = 0$, a current jet which behaves as though reflected from the free surface. At the bottom the current is closed by the Hartmann layer, which is formed at the interface of the rotating and fixed portions of the bottom at $r = R$. In accordance with Eq. (4.1), the velocity distribution at the interface of these layers is given by the relation

$$v_0 = (\Omega R/4)[3 - \Phi(\sqrt{\varkappa/8h}\,x)], \quad x < 0;$$

$$v_0 = (\Omega R/4)[1 - \Phi(\sqrt{\varkappa/8h}\,x)], \quad x > 0. \qquad (4.2)$$

The current entering the layer is $j_z = \sqrt{\sigma\eta}\,\partial v_0/\partial x$, and the current in the layer is given by Eq. (3.2), where, in place of v_0 we substitute $v_0 - v_d$. It is apparent that the equation of continuity (3.4) is satisfied; at $x = 0$ from the right and left there flow currents whose sum is exactly equal to the current in the jet which diverges from this "point." The only place at which the indicated solution does not apply is in a region of dimensions given roughly by κ^{-1} at $x = 0$, $z = 0$, which is a singularity point.

Integrating the friction expression $\eta\kappa(v_0 - \Omega R)$, we can compute the moment of the friction force M_z which acts on the rotating bottom:

$$M_z = \Omega\eta\sqrt{2\pi\kappa h}\,(R^3 + R_1^3). \qquad (4.3)$$

B. Conducting Bottom (Lehnert experiment[7]). In place of the condition $\partial\varphi/\partial z = 0$ at the bottom, here we have $\partial\varphi/\partial r = vB$ or $j_r = 0$. As in the preceding case, in the annular region $R_1 \le r \le R$ the liquid rotates with the bottom; $v = \Omega r$, $\partial\varphi/\partial r = vB$ and at $r = R_1$ and $r = R$ there are velocity discontinuities. However, in this case no Hartmann layer can be formed; hence the solution for the tangential discontinuity, although qualitatively similar to the earlier solution, is not made up of self-similar solutions (4.1). We may assume, from Eq. (4.2), that the half-width of the discontinuity is given by

$$\delta = \sqrt{8h/\kappa}. \qquad (4.4)$$

Under the conditions of the Lehnert experiment ($h = 0.6$ cm., $R_1 + R = 7$ cm, $R - R_1 = 1$ cm., $\Omega = 0.2$ rev/sec) the rotating ring of liquid starts to form at approximately $B = 600$ gauss; this result is in qualitative agreement with Eq. (4.4) which, for this field, yields $\delta = 0.55$ cm.

The experiments carried out by Lehnert indicate that the tangential discontinuity is not stable and that a vortex path of cylindrical vortices, which behave like the rollers in a roller bearing, is formed under the rotating ring. As has been indicated by Lehnert, this effect is to be associated with the influence of the magnetic field on the form of flow. The velocity profile in the tangential "jump" has an inflection point and flow patterns of this kind, as is well known from the theory of stability in ordinary hydrodynamics,[12] become unstable at Reynolds numbers of the order of several tens. For the highest field which was used ($B = 8000$ gauss), however, a calculation made on the basis of Eq. (4.4) gives $R_\delta = v\delta/\nu = 570$. Correspondingly, $N_\delta = \gamma_m\delta/v = 1.7$; this value is insufficient for stabilization since an instability may develop completely in regions with dimensions appreciably smaller than δ. It is reasonable to expect that with a several-fold reduction of the rotation velocity (so that the ratio $R/N = v^2/\nu\gamma_m$ is of the order of ten) the flow would be stable.

The value of N is rather large for the rotating liquid ring as a whole; for $l = R - R_1 = 1$ cm we find $N = 14$, $R = 3.8 \times 10^3$. In the experiment carried out by Lehnert the stabilizing effect of the magnetic field is felt and instead of the development of turbulent motion in the jet there is a proper vortex row with a relatively small number of large vortices.

Rotation of a Liquid by a Transverse Current

Let a container with liquid, a cylindrical condenser with inner radius R_1, outer radius R, and height h, be placed in a strong magnetic field $B = B_z$. The faces of the container are insulators while the side walls are cylindrical coaxial electrodes. If a current flows between these walls the liquid is set into rotation. At the electrodes there are longitudinal boundary layers with thickness of the order of δ in Eq. (4.4); Hartmann layers are formed at the bases, In the inner region the viscosity can be neglected in accordance with Eq. (1.12) and, in this region. $\partial^2 v/\partial z^2 = 0$ and $\partial^2\varphi/\partial z^2 = 0$. Invoking the symmetry conditions we have $v = v(r)$ and $\varphi = \varphi(r)$ so that $j_z = 0$. Thus, the total current in each of the Hartmann layers $2\pi r J_s$ is independent of radius and, since $J_s \sim v$, we have $v \sim r^{-1}$. In the inner region there is a potential flow. When $v \sim r^{-1}$ the viscosity force is exactly equal to zero and consequently there is no radial current. The longitudinal electric field here is compensated by the induction emf $\partial\varphi/\partial r = vB$. The total current I flows in layers at the faces ($I = 2 \cdot 2\pi r\sqrt{\sigma\eta}\, v$), whence

$$v = I\,(4\pi\sqrt{\sigma\eta}\,r)^{-1} \qquad (4.5)$$

In ordinary hydrodynamics, flow between cylinders is stable when $v \sim r^{-1}$; since the magnetic field increases the stability, one may expect that the flow will also be stable in the case considered here.

It is interesting to note that the velocity is independent of magnetic field although the rotational moment is proportional to the field. The point is that friction at the faces is also proportional to the field since the reciprocal thickness of the layer $\kappa \sim B$. It can be shown that the resistance to rotation arises at the intersection of the magnetic lines of force at the faces of the container. This flow pattern applies if the longitudinal layers (thickness of order $(h/\kappa)^{1/2}$) are thin enough, i.e., if $h \ll \kappa R^2$. If, for example, $h \sim R \sim 10$ cm this means that $B \gg 4$ gauss. For $h \to \infty$[13] the friction force is of order $\eta v/R$ and the rotational force is approximately jB/c so that $v \sim I_1 BR/2\pi c\eta$, where I_1 is the current per unit height. In a real cylinder of height h this relation applies only when $h > \kappa R^2$. In rotation of a gas of low density the thickness of the layer κ^{-1} may be smaller than the mean free path, in which case the gas does not adhere to the faces and the velocity again increases with field.

As long as the mode of rotation is not established the current flows over the entire cross section and communicates to the liquid a velocity given approximately by $(I/2\pi Rh)\,B/c\rho$; hence, for a constant current the time for establishing the velocity (4.5) is of order $t_0 = h\,(\nu\gamma_m)^{-1/2}$. For example, with $h = 10$ cm and $B = 5 \times 10^3$ gauss, $t_0 = 1$ min.

The rotation of the liquid produces a radial pressure gradient $\partial p/\partial r = \rho v^2$. In the layers at the faces the liquid has a smaller velocity than in the remaining part of the container so that the motion is characterized by circulation in the meridian planes; the liquid in the layers moves towards the axis while the liquid in the center part moves away from the axis. The azimuthal currents can flow freely and there is a radial magnetic friction force $-(\sigma B^2/c^2)\,v_r$ which equilibrates the pressure gradient. In this case the radial velocity in the layers is of order $v_r \sim v^2c^2/\sigma B^2 r = vN^{-1}$ while the velocities v_r and v_z in the center part are smaller by a factor of κh or κR. Above we have neglected $(\mathbf{v}\cdot\nabla)\mathbf{v}$ terms in the azimuthal velocity equations as compared with the term $\nu\Delta\mathbf{v}$. The ratio of the neglected terms to those which have been considered is of order $(v/\gamma_m R)^2 = N^{-2}$; for example, with $v = 10^2$ cm sec^{-1} and $B = 10^4$ gauss, we have $N^{-2} = 2 \times 10^{-2}$.

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