RELATIVISTIC SPHERICAL FUNCTIONS. II.*

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Properties of infinite-dimensional representations of the Lorentz group are considered which are of interest for the solution of problems in relativistic elementary particle theory. Infinite-dimensional representations are applied to an analysis of the amplitude of the reaction $a + b \rightarrow c + d$.

1. INTRODUCTION

A study of the symmetry properties of a system of particles enables one to draw a number of important conclusions with respect to the behavior of the system even in the case when the nature of the interaction between the particles is unknown. The conditions of invariance of a problem under different groups of transformations impose strong limitations on possible types of solutions. These restrictions are widely used in the analysis of the processes of interaction between particles. Particularly wide use is made of the symmetry properties of a system with respect to the group of rotations and reflections of three-dimensional space. In contrast to the representations of the three-dimensional group the representations of the Lorentz group are utilized comparatively infrequently. This refers particularly to the infinite-dimensional representations of this group. But they may turn out to be useful both in the phenomenological analysis of processes involving relativistic particles, and also in the development of field theory.

One of the reasons for this insufficient utilization of the irreducible representations of the Lorentz group is the lack of mathematical apparatus similar to the one available in the case of the three-dimensional group. For finite-dimensional representations of the Lorentz group such a formalism was developed in an article by one of the authors.[1] The present paper is a continuation of the earlier one.[1] In the present paper a study is made of the infinite-dimensional representations of the Lorentz group in connection with their applications to different problems of the relativistic theory of elementary particles. A discussion is given of the possibility of utilizing the basis functions of this group for the solution of the equations of quantum field theory. The reaction $a + b \rightarrow c + d$ is discussed. The amplitude of this reaction is expanded in terms of the basis functions of the infinite-dimensional representation. The coefficients of such an expansion depend only on the nature of the interaction between the particles, and not on the kinematics of the process. For high-energy particles the method of analyzing the reaction amplitudes outlined here is simpler and more convenient than the usual phase analysis.

Infinite-dimensional representations of the Lorentz group have been utilized by a number of authors. Ginzburg and Tamm,[2] Gel'fand and Yaglom,[3] Dirac[4] and others have used them for the development of a theory of elementary particles with a spectrum of spins and masses, E. M. Lifshitz[5] has used them for the solution of the problem of the stability of an expanding universe, I. S. Shapiro[6] has used them for a relativistically-invariant classification of the states of elementary particles, etc. The results of the present paper may be applied to the problems enumerated above. One of the main methodological advantages of our method of investigating the infinite-dimensional representations as compared with the investigations of other authors is the possibility of utilizing the techniques of Clebsch-Gordan, Racah, and Fano which have become widespread in physics.

2. THE BASIS FUNCTIONS OF THE UNITARY INFINITE-DIMENSIONAL REPRESENTATIONS OF THE LORENTZ GROUP

The representations of the Lorentz group are determined by the eigenvalues of two invariants which can be constructed from the components of the operator for the infinitesimal rotation $M_{\alpha\beta}$ in four-dimensional space. If $\mathbf{H}$ is the space-like,

*This is a continuation of a paper by one of the authors,[1] published in JETP under the same title.

and $\mathbf{F}$ is the time-like part of $M_{\alpha\beta}$ then the invariants are given by $M^2 = H^2 - F^2$ and $(\mathbf{HF})$. If $M_{\alpha\beta}$ operates on functions which depend only on the coordinates and do not contain any spin variables, then the second invariant is equal to zero and the representation is determined by specifying the eigenvalue of M^2.

The basis functions of the irreducible finite-dimensional representations given in reference 1 for the time-like case ($t = \rho \cosh \alpha$, $r = |\rho| \times \sinh \alpha$, $0 \le \alpha \le \infty$, $-\infty \le \rho \le \infty$) are of the following form

$$\Psi_{nlm}(\alpha, \vartheta, \varphi) = \Pi_l(n, \alpha) Y_{lm}(\vartheta, \varphi), \tag{1}$$

$$\Pi_l(n, \alpha) = \frac{\sinh^l \alpha}{\sqrt{n^2(n^2-1^2)(n^2-2^2)\dots(n^2-l^2)}} \frac{d^{l+1}\cosh n\alpha}{d\cosh^{l+1}\alpha} \tag{2}$$

By replacing n in (2) by iN, where N is a real number which takes on all values in the range $0 \le N \le \infty$, we obtain the basis functions of the principal series of the unitary irreducible infinite-dimensional representations of the Lorentz group:

$$\Psi_{Nlm}(\alpha, \vartheta, \varphi) = \Pi_l(N, \alpha) Y_{lm}(\vartheta, \varphi)$$
$$= \frac{\sinh^l \alpha}{M_l} \frac{d^{l+1}\cos N\alpha}{d\cosh^{l+1}\alpha} Y_{lm}(\vartheta, \varphi),$$

$$M_l = \sqrt{N^2(N^2+1^2)\dots(N^2+l^2)}. \tag{3}$$

In the papers of Gel'fand and Naĭmark[7] it is shown that the basis functions of the Lorentz group must satisfy the following relations:

$$H_\mu \Psi_{Nlm} = \sqrt{l(l+1)}\, C^{lm+\mu}_{lm1\mu} \Psi_{Nlm+\mu},$$

$$F_\mu \Psi_{Nlm} = i \sum_{l'} C^{Nl}_{l'} C^{l0}_{l'010} C^{l'm+\mu}_{lm1\mu} \Psi_{Nl'm+\mu},$$

$$C^{Nl}_{l-1} = -\sqrt{N^2+l^2}, \quad C^{Nl}_{l+1} = \sqrt{N^2+(l+1)^2}; \tag{4}$$

$$\mathbf{H} = -i[\mathbf{n}\times\nabla^\omega], \quad \mathbf{F} = -i[\mathbf{n}\,\partial/\partial\alpha + \coth\alpha\nabla^\omega]. \tag{5}$$

Here $r\mathbf{n} = \mathbf{r}$, ∇^ω is the angular part of the operator ∇, and H_μ and F_μ are the cyclic components* of $\mathbf{H}$ and $\mathbf{F}$.

Formulas (4) define the basis functions of an infinite-dimensional irreducible unitary representation for which $k_0 = 0$, $c = iN$, where $c^2 - 1$ and k_0 are the eigenvalues of the two invariants of the group in the notation of reference 7. One can easily show by means of a direct substitution that (4) is valid for the function (3).

*The cyclic components of a vector $\mathbf{a}$ are related to its Cartesian components by the equations $a_{\pm 1} = \pm(a_x \pm i a_y)\sqrt{2}$, $a_0 = a_z$; our definition of the spherical harmonics $Y_{lm}(\vartheta, \varphi)$ is the same as the one used by Bethe,[8] and differs by the factor $(-1)^m$ from the functions listed by Condon and Shortley.[9]

From (4) we may obtain recurrence relations for $\Pi_l(N, \alpha)$:

$$d\Pi_l/d\alpha = -(l+1)\coth\alpha\Pi_l - \sqrt{N^2+l^2}\,\Pi_{l-1},$$
$$d\Pi_l/d\alpha = l\coth\alpha\Pi_l + \sqrt{N^2+(l+1)^2}\,\Pi_{l+1}, \tag{6}$$

which lead to the following second order equation

$$\frac{d^2\Pi_l}{d\alpha^2} + 2\coth\alpha\frac{d\Pi_l}{d\alpha} - \frac{l(l+1)}{\sinh^2\alpha}\Pi_l + (N^2+1)\Pi_l = 0. \tag{7}$$

The function $\Pi_l(N, \alpha)$ may also be expressed in terms of the following integral

$$\Pi_l(N, \alpha) = (-1)^{l+1}\frac{M_l}{\sinh^{l+1}\alpha}\int_0^\alpha \cos N\beta \frac{(\cosh\alpha - \cosh\beta)^l}{l!} d\beta. \tag{8}$$

It can be easily verified that (8) satisfies (6), and reduces to (3) when $l = 0$. This proves the identity of expressions (8) and (3). The normalization of $\Pi_l(N, \alpha)$ is such that

$$\int_0^\infty \Pi_l(N, \alpha)\Pi_l(N', \alpha)\sinh^2\alpha d\alpha = \frac{\pi}{2}\delta(N-N'). \tag{9}$$

Π_l has no singularities over the whole range of values of α and satisfies the condition $\Pi_l(N, 0) = -N\delta_{l0}$.

Relations analogous to (6) – (8) were obtained by Fock[10] for the four-dimensional spherical harmonics in Euclidean space.

To obtain the recurrence relations and the equation for the space-like case ($t = \rho \sinh \alpha$, $r = \rho\cosh\alpha$, $-\infty \le \alpha \le \infty$, $0 \le \rho \le \infty$), one should replace in (6) and (7) $\sinh\alpha$ by $\cosh\alpha$ and vice versa. As we have already noted in reference 1, the space-like function Ψ_{Nlm} can be obtained from (3) for the time-like case by means of replacing α by $\alpha \pm i\pi/2$. For our two linearly independent functions we may choose

$$\Psi^{\pm}_{Nlm}(\alpha, \vartheta, \varphi) = i\frac{\cosh\alpha}{M_l}\frac{d^{l+1}\cos N(\alpha \pm i\pi/2)}{d\sinh^{l+1}\alpha} Y_{lm}(\vartheta, \varphi). \tag{10}$$

These functions are orthogonal and are normalized by the condition

$$\int_{-\infty}^{\infty}\cosh^2\alpha d\alpha \int d\Omega\, \Psi^*_{N_1 l_1 m_1}(-\alpha, \vartheta, \varphi)\Psi_{N_2 l_2 m_2}(\alpha, \vartheta, \varphi)$$
$$= \pi\delta_{l_1 l_2}\delta_{m_1 m_2}\delta(N_1 - N_2). \tag{11}$$

3. MATRICES FOR THE ROTATION OPERATOR IN FOUR-DIMENSIONAL PSEUDO-EUCLIDEAN SPACE

Every proper Lorentz transformation may be represented in the form of three successive transformations: a) a spatial rotation, defined by the Eulerian angles $\varphi_1 = \Phi$, $\vartheta = \theta$, and $\varphi_2 = 0$; b) a transformation to a coordinate system which

has the velocity $v = \tanh\psi$ along the new z axis ($0 \le \psi \le \infty$, the velocity of light is $c = 1$), while the direction of the velocity is determined in the initial system by the polar angles θ and Φ;
c) a second spatial rotation, defined by the angles φ_1, ϑ, and φ_2.

To obtain the matrix elements of the rotation operator we have to construct the matrices for each of the three rotations a), b), c) and multiply them together. The Cayley-Klein coefficients for the spatial rotation have the following form

$$\alpha = \delta^* = \exp\left\{\frac{1}{2}i(\varphi_1 + \varphi_2)\right\}\cos\frac{\vartheta}{2},$$
$$\beta = -\gamma^* = -\exp\left\{-\frac{1}{2}i(\varphi_1 - \varphi_2)\right\}\sin\frac{\vartheta}{2}. \tag{12}$$

A rotation through the angle ψ in the (z, t) plane is specified by the quantities

$$\alpha = \exp(-\psi/2), \quad \delta = \exp(\psi/2), \quad \beta = \gamma = 0. \tag{13}$$

Let $U^{Jj}_{M\mu}$ be the basis function of an irreducible finite-dimensional representation of dimension $(2J+1)(2j+1)$, where J and j are defined by the eigenvalues of the two group invariants. In the notation of Gel'fand and Naĭmark[7] $k_0 = |J - j|$, $c = n = J + j + 1$. The transformation of $U^{Jj}_{M\mu}$ under four-dimensional rotations is defined by the following equation

$$U^{Jj}_{M\mu}(A) = \sum_{M'\mu'} D^{Jj}_{M\mu M'\mu'}(\Omega_2, \psi, \Omega_1)\, U^{Jj}_{M'\mu'}(A'), \tag{14}$$

where $U^{Jj}_{M\mu}(A)$ is taken in the original, and $U^{Jj}_{M'\mu'}(A')$ in the new coordinate system. The matrix elements of the four-dimensional rotation operator $D^{Jj}_{M\mu M'\mu'}(\Omega_2, \psi, \Omega_1)$ were given in reference 1 for the case $\Omega_2 = 0$. With the aid of (12) and (13) we can obtain, in a manner analogous to that used in reference 1, the general expression for the rotation matrix

$$D^{Jj}_{M\mu\, M'\mu'}(\Omega_2, \psi, \Omega_1) = \sum_{ll'\varkappa} (-1)^{\mu-\mu'} C^{lm}_{J-Mj\mu} C^{l'm'}_{J-M'j\mu'} D^{l}_{m\varkappa}$$
$$\times(\Phi, \theta, 0)\, D^{l'}_{\varkappa m'}(\varphi_1, \vartheta, \varphi_2)\, Q^{ll'}_{Jj\varkappa}(\psi). \tag{15}$$

$$Q^{ll'}_{Jj\varkappa}(\psi) = \sum_{\Lambda\lambda} C^{l\varkappa}_{J-\Lambda j\lambda} C^{l'\varkappa}_{J-\Lambda j\lambda} \exp\{(\Lambda + \lambda)\psi\}. \tag{16}$$

$D^{l}_{mm'}$ are the well-known matrix elements of the three-dimensional rotation operator:[11]

$$D^{l}_{mm'}(\varphi_1, \vartheta, \varphi_2) = \sum (-1)^k$$
$$\times \frac{[(l+m)!\,(l-m)!\,\,(l+m')!\,(l-m')!]^{1/2}}{(l+m'-k)!\,(l-m-k)!\,(m-m'+k)!\,k!}$$
$$\times\, e^{im\varphi_1}\left(\cos\frac{\vartheta}{2}\right)^{2l+m'-m-2k}\left(\sin\frac{\vartheta}{2}\right)^{2k-m'+m} e^{im'\varphi_2}. \tag{17}$$

In accordance with our definition of the angles of rotation the following equalities hold:

$$Y_{lm}(\theta, \Phi) = \sum_{m'} D^{l}_{mm'}(\varphi, \vartheta, \varphi_2)\, Y_{lm'}(\theta', \Phi'),$$
$$Y_{lm}(\vartheta, \varphi) = \sqrt{(2l+1)/4\pi}\, D^{l}_{m0}(\varphi, \vartheta, \varphi_2). \tag{18}$$

The function $D^{Jj}_{M\mu M'\mu'}$ transforms according to the $(2J+1)(2j+1)$-dimensional irreducible representation of the Lorentz group.

The rotation matrices for the functions Ψ_{nlm} can be easily obtained with the aid of (15) and formula (22) of reference 1. If

$$\Psi_{nlm}(\alpha, \vartheta, \varphi) = \sum_{l'm'} T^{n}_{lml'm'}(\Omega_2, \psi, \Omega_1)\, \Psi_{nl'm'}(\alpha', \vartheta', \varphi'), \tag{19}$$

then we have

$$T^{n}_{lml'm'}(\Omega_2, \psi, \Omega_1) = \sum_{\varkappa} D^{l}_{m\varkappa}(\Phi, \theta, 0)\, Q^{ll'}_{n\varkappa}(\psi)\, D^{l'}_{\varkappa m}(\varphi_1, \vartheta, \varphi_2), \tag{20}$$

where

$$Q^{ll'}_{n\varkappa} \equiv Q^{ll'}_{JJ\varkappa}, \quad n = 2J + 1.$$

On setting in (19) $\alpha' = \vartheta' = \varphi' = 0$, we obtain

$$\Psi_{nlm}(\alpha, 0, \Phi) = (n/\sqrt{4\pi})\, T^{n}_{lm00}(\Omega_2, \alpha, \Omega_1) \tag{21}$$

and, in particular,

$$\Pi_l(n, \alpha) = (n/\sqrt{2l+1})\, Q^{l0}_{n0}(\alpha). \tag{22}$$

The matrices which transform the basis functions of the infinite-dimensional unitary representation will differ from (20) only by the form of the function $Q^{ll'}_{N\kappa}(\psi)$, since the spatial parts of Ψ_{nlm} and Ψ_{Nlm} are the same. In order to determine $Q^{ll'}_{N\kappa}(\psi)$, we must find the explicit form of the operator corresponding to the infinitesimal rotation $\mathbf{F}$. This operator will depend on six variables θ, Φ, ψ, φ_1, ϑ, and φ_2. The result of its operation on $T^{N}_{lml'm'}$ is defined by a formula analogous to (4),

$$F_\mu T^{N}_{LMlm} = i\sum_{L'} C^{NL}_{L'} C^{L0}_{L'010} C^{L'M+\mu}_{LM1\mu} T^{N}_{L'M+\mu lm}. \tag{23}$$

We need not consider the operator $\mathbf{H}$, since it does not operate on ψ. The explicit form of F_μ can be found by means of the well-known method of constructing the operators for an infinitesimal rotation.[7] We obtain

$$iF_\mu = n_\mu\frac{\partial}{\partial\psi} + \coth\psi\nabla^{\Omega}_{\mu} + \frac{1}{\sinh\psi}$$
$$\times\{D^{1}_{\mu-1}(\Phi, 0, 0)\, L_{-1} - D^{1}_{\mu 1}(\Phi, 0, 0)\, L_1\}. \tag{24}$$

Here $n_\mu(\theta, \Phi)$ is a unit vector; ∇^{Ω}_{μ} is the operator which under rotations transforms in the same way as the gradient

$$\nabla_{\pm 1}^{\Omega} = \pm \frac{e^{\pm i\varphi_1}}{\sqrt{2}}\left[\cos\theta\frac{\partial}{\partial\theta} \pm \frac{i}{\sin\theta}\frac{\partial}{\partial\Phi} \mp \cot\vartheta\frac{\partial}{\partial\varphi_1}\right],$$

$$\nabla_0^{\Omega} = -\sin\theta\frac{\partial}{\partial\theta}; \tag{25}$$

L_μ is the operator of an infinitesimal rotation operating on the angles φ_1, ϑ, and φ_2:

$$L_{\pm 1} = \pm\frac{e^{\pm i\varphi_1}}{\sqrt{2}}\left[\pm\frac{\partial}{\partial\vartheta} + i\cot\vartheta\frac{\partial}{\partial\varphi_1} - \frac{i}{\sin\vartheta}\frac{\partial}{\partial\varphi_2}\right],$$

$$L_0 = -i\frac{\partial}{\partial\varphi_1}. \tag{26}$$

On substituting T^N_{LMlm} into (3), and on separating the factors that depend on the spatial angles, we obtain three recurrence relations which interrelate the functions $Q^{Ll}_{N\kappa}(\psi)$ corresponding to different values of L and κ:

$$C^{L'\varkappa}_{L\varkappa 10}(dQ^{Ll}_{N\varkappa}/d\psi + \gamma^L_{L}\coth\psi Q^{Ll}_{N\varkappa}) + C^{NL}_{L'}C^{L0}_{L'010}Q^{L'l}_{N\varkappa} = \frac{\sqrt{l(l+1)}}{\sinh\psi}$$
$$\times(C^{l\varkappa}_{l\varkappa-111}C^{L'\varkappa}_{L\varkappa-111}Q^{Ll}_{N\varkappa-1} - C^{l\varkappa}_{l\varkappa+11-1}C^{L'\varkappa}_{L\varkappa+11-1}Q^{Ll}_{N\varkappa+1}); \tag{27}$$

$$L' = L,\ L\pm 1, \quad \gamma^{L-1}_{L} = L+1,$$
$$\gamma^{L}_{L} = 1, \quad \gamma^{L+1}_{L} = -L,$$

$C^{NL}_{L'}$ is defined by formula (4).

The recurrence relations (27) completely define the function $Q^{Ll}_{N\kappa}(\psi)$. For $l = 0$ they reduce to (6). At the same time $\Pi_L(N, \alpha) = -(N/\sqrt{2L+1})\times Q^{L0}_{N0}(\psi)$. If we construct the invariant operator $M^2 = H^2 - F^2$, which depends on the six angles, then from the equation

$$M^2 T^N_{LMlm} = -(N^2+1)T^N_{LMlm} \tag{28}$$

we can obtain a system of second-order differential equations for $Q^{Ll}_{N\kappa}$:

$$\frac{d^2Q^{Ll}_{N\varkappa}}{d\psi^2} + 2\coth\psi\frac{dQ^{Ll}_{N\varkappa}}{d\psi} - \frac{L(L+1)+l(l+1)}{\sinh^2\psi}Q^{Ll}_{N\varkappa}$$
$$+\varkappa^2\frac{(\cosh\psi-1)^2}{\sinh^2\psi}Q^{Ll}_{N\varkappa} + (N^2+1)Q^{Ll}_{N\varkappa}$$
$$+2\sqrt{L(L+1)l(l+1)}\frac{\cosh\psi}{\sinh^2\psi}\sum_\mu C^{l\varkappa-\mu}_{l\varkappa 1-\mu}C^{L\varkappa-\mu}_{L\varkappa 1-\mu}Q^{Ll}_{N\varkappa-\mu} = 0. \tag{29}$$

We note that for $N = -in$, where n is an integer, (27) and (29) will define the rotation matrix for the basis functions ψ_{nlm} of the finite-dimensional representations; for complex values of N we shall obtain the rotation matrix for the basis functions of the infinite-dimensional nonunitary representations.

We can verify directly that the recurrence relations (27) and the equations (29) will be satisfied by the function $Q^{Ll}_{N\kappa}(\psi)$ which is obtained by means of analytic continuation of (16) into the region of purely imaginary values of $n = 2J+1 = iN$, $J = j$.

On taking into account the fact that $\Pi_l(n, \alpha)$ may be represented in the following form[1]

$$\Pi_l(n, \alpha) = \sum_\mu C^{J\mu}_{J\mu l0}e^{2\mu\alpha}, \quad n = 2J+1, \tag{30}$$

and on utilizing the explicit form (16) of $Q^{ll'}_{n\kappa}$, we obtain

$$Q^{ll'}_{n\varkappa}(\psi) = (-1)^\varkappa e^{\varkappa\psi}\sqrt{(2l+1)(2l'+1)/n}\times$$
$$\times\sum_s\sqrt{2s+1}\,C^{S0}_{l\varkappa l'-\varkappa}W(lJl'J;Js)\Pi_s(n,\psi). \tag{31}$$

W(abcd; ef) is the Racah function.[12] It may be expressed in terms of the Γ function and the generalized hypergeometric function ${}_4F_3$ (see reference 13), which are defined both for real and for complex values of their arguments. This enables us to go over from (31) to the case of an infinite-dimensional representation. To achieve this, we must replace n in expression (31) by a complex number. In particular, for the case of an infinite-dimensional unitary representation we must replace n by iN. We then obtain

$$Q^{ll'}_{N\varkappa}(\psi) = e^{\varkappa\psi}\sum_s(2s+1)C^{l'\varkappa}_{l\varkappa s0}F(Nll's)\Pi_s(N,\psi), \tag{32}$$

$$F(Nll's) = -(-i)^{l+l'+s}\sqrt{i(2l+1)/N}\,W$$
$$\times\left(\frac{iN-1}{2}l\frac{iN-1}{2}l';\frac{iN-1}{2}s\right) = \frac{(-1)^{l+l'+s}Nl!\,l'!\,s!}{M_lM_{l'}M_s}$$
$$\times\sqrt{\frac{(2l+1)(l+l'-s)!\,(l+s-l')!\,(l'+s-l)!}{(l+l'+s+1)!}}\,w(Nll's), \tag{33}$$

$w(Nll's)$

$$= i\sum_k\frac{(-1)^k\Gamma(l+l'+1-k+iN)}{\Gamma(-s-k+iN)(l-k)!(l'-k)!(l+l'-s-k)!\,k!(s-l+k)!(s-l'+k)!}.$$

We note several symmetry properties of the function $Q^{ll'}_{N\kappa}$. It follows from (3), (27), and (32) that

$$Q^{ll'}_{N\varkappa}(\psi) = (-1)^{l+l'}Q^{l'l}_{N\varkappa}(\psi) = Q^{ll'}_{N,-\varkappa}(\psi) = (-1)^{l+l'}Q^{ll'}_{N\varkappa}(-\psi). \tag{34}$$

The normalizing constant for T^N_{LMlm} can be calculated by utilizing (20) and the explicit form $Q^{ll'}_{N\kappa}$. On introducing the notation $d\Omega_1 = (1/4\pi)\sin\theta\,d\theta\,d\Phi$, $d\Omega_2 = (1/8\pi^2)\sin\vartheta\,d\vartheta\,d\varphi_1\,d\varphi_2$, we obtain

$$\int_0^\infty\sinh^2\psi d\psi\int T^{N'*}_{L'M'l'm'}T^N_{LMlm}\,d\Omega_1 d\Omega_2$$
$$= \frac{\pi}{2N^2}\delta(N-N')\delta_{LL'}\delta_{ll'}\delta_{MM'}\delta_{mm'}. \tag{35}$$

The addition theorem for Π_l in the finite-dimensional case follows from formula (19):

$$\sum_l(2l+1)\Pi_l(n,\alpha)\Pi_l(n,\beta) = n\Pi_0(n,\alpha+\beta). \tag{36}$$

The analogous formula for the infinite-dimensional representation has the following form

$$\sum_l(-1)^{l+1}(2l+1)\Pi_l(N,\alpha)\Pi_l(N,\beta) = N\Pi_0(N,\alpha+\beta). \tag{37}$$

4. THE CLEBSCH-GORDAN EXPANSION FOR THE INFINITE-DIMENSIONAL REPRESENTATIONS OF THE LORENTZ GROUP

In reference 1 we have obtained the Clebsch-Gordan expansion for the finite-dimension representations:

$$\Psi_{n_1l_1m_1}\Psi_{n_2l_2m_2}=\sum_{n,l}\sqrt{n_1n_2/4\pi n}\,A(n_1l_1n_2l_2nl)\,C^{lm}_{l_1m_1l_2m_2}\Psi_{nlm},$$

$$A(n_1l_1n_2l_2nl)=n\sqrt{(2l_1+1)(2l_2+1)}\,X(J_1J_1l_1,J_2J_2l_2,JJl), \tag{38}$$

X(abc, def, ghs) are the Fano functions whose explicit form together with tables of particular values are given by Matsunobo and Takebe.[14] The coefficients A satisfy the following orthogonality relations:

$$\sum_{l_1l_2}A(n_1l_1n_2l_2nl)\,A(n_1l_1n_2l_2n'l)=\delta_{nn'},$$

$$\sum_{n}A(n_1l_1n_2l_2nl)\,A(n_1l_1'n_2l_2'nl)=\delta_{l_1l_1'}\delta_{l_2l_2'}. \tag{39}$$

With the aid of (38) and (39) we may obtain the Clebsch-Gordan expansion

$$T^{n_1}_{l_1m_1l_1'm_1'}T^{n_2}_{l_2m_2l_2'm_2'}=\sum_{nll'}A(n_1l_1n_2l_2nl)$$
$$\times A(n_1l_1'n_2l_2'nl')\,C^{lm}_{l_1m_1l_2m_2}C^{l'm'}_{l_1'm_1'l_2'm_2'}T^{n}_{lml'm'} \tag{40}$$

and the inverse expansion

$$T^{n}_{lml'm'}=\sum A(n_1l_1n_2l_2nl)\ A(n_1l_1'n_2l_2'nl')$$
$$\times C^{lm}_{l_1m_1l_2m_2}C^{l'm'}_{l_1'm_1'l_2'm_2'}T^{n_1}_{l_1m_1l_1'm_1'}T^{n_2}_{l_2m_2l_2'm_2'} \tag{41}$$

In (41) the summation is carried out over all the allowable values of l_1, l_2, l_1', l_2', m_1, m_2, m_1', m_2'.

In accordance with reference 15 we shall seek the Clebsch-Gordan expansion of the product of basis functions Ψ_{Nlm} of the infinite-dimensional representation in the following form

$$\Psi_{N_1l_1m_1}\Psi_{N_2l_2m_2}=(1/\sqrt{4\pi})\sum_{l}$$
$$\times\int_0^\infty dN\,B(N_1N_2N)\,C(N_1l_1N_2l_2Nl)\,C^{lm}_{l_1m_1l_2m_2}\Psi_{Nlm}, \tag{42}$$

and we shall require that $C(N_10N_20N0)=-1$. By making use of the orthogonality of the functions Ψ_{Nlm} we obtain

$$B(N_1N_2N)=\pm\tfrac{1}{4}\sinh\pi N_1\sinh\pi N_2\sinh\pi N$$
$$\times\Big[\cosh\tfrac{\pi}{2}(N_1+N_2+N)\cosh\tfrac{\pi}{2}(N_1+N_2-N)$$
$$\times\cosh\tfrac{\pi}{2}(N+N_2-N_1)\cosh\tfrac{\pi}{2}(N+N_1-N_2)\Big]^{-1}. \tag{43}$$

The plus sign corresponds to the time-like, and the minus sign to the space-like case. To determine the expansion coefficient $C(N_1l_1N_2l_2Nl)$, we apply to both sides of (42) the operator F_μ (24), and then expand products of the type $\Psi_{N_1l_1m_1}\times\Psi_{N_2l_2'm_2+\mu}$, which will appear in the left hand side of the equation, again by utilizing (42). By equating coefficients of Ψ_{Nlm} on both sides of the equation, and by utilizing the orthogonality properties of the coefficients $C^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}$, we obtain the following recurrence relations

$$C(N_1l_1N_2l_2Nl')\,C^{Nl}_{l'}C^{l'0}_{l010}$$
$$=\sum_{l_1'}C(N_1l_1'N_2l_2Nl)\,C^{N_1l_1}_{l_1'}C^{l_10}_{l_1'010}U(l_1l_21l;\,l'l_1')$$
$$+\sum_{l_2'}C(N_1l_1N_2l_2'Nl)\,C^{N_2l_2}_{l_2'}C^{l_20}_{l_2'010}U(l_2l_11l;\,l'l_2'),\quad l'=l,l\pm1,$$
$$U(abcd;\,ef)=\sqrt{(2e+1)(2f+1)}\,W(abcd;\,ef). \tag{44}$$

The relations (44) enable us to define the function $C(N_1l_1N_2l_2Nl)$ for arbitrary l_1, l_2 and l. It has the following symmetry properties:

$$C(N_1l_1N_2l_2Nl)=C(N_2l_2N_1l_1Nl)$$
$$=(-1)^{l_2}\sqrt{\frac{2l+1}{2l_1+1}}\,C(N_2l_2NlN_1l_1), \tag{45}$$

$C(N_1l_1N_2l_2Nl)=0$ if l_1+l_2+l is an odd integer. These properties become obvious if we express $C(N_1l_1N_2l_2Nl)$ in terms of an integral of the product of three $\Pi_l(N,\alpha)$, by utilizing (42).

We shall seek the expansion which is the inverse of (42) in the form

$$\Psi_{Nlm}=(N\sqrt{4\pi}/N_1N_2)\sum_{l_1m_1l_2m_2}\tilde{C}(N_1l_1N_2l_2Nl)$$
$$\times C^{lm}_{l_1m_1l_2m_2}\Psi_{N_1l_1m_1}\Psi_{N_2l_2m_2}. \tag{46}$$

On setting $\alpha=0$ in both sides of this equation we obtain $\tilde{C}(N_10N_20N0)=-1$. The recurrence relations between $\tilde{C}$ and different values of l_1, l_2, and l can be obtained in the same way as for C. They are identical with (44), from which it follows that $\tilde{C}(N_1l_1N_2l_2Nl)$ coincides with $C(N_1l_1N_2l_2Nl)$.

From (42) and (46) we obtain the orthogonality relation*

$$\int_0^\infty NdN\,B(N_1N_2N)\,C(N_1l_1N_2l_2Nl)\,C(N_1l_1'N_2l_2'Nl)$$
$$=N_1N_2\delta_{l_1l_1'}\delta_{l_2l_2'},$$

$$NB(N_1N_2N)\sum_{l_1l_2}C(N_1l_1N_2l_2Nl)\,C(N_1l_1N_2l_2N'l)$$
$$=N_1N_2\delta(N-N'). \tag{47}$$

The coefficients $C(N_1l_1N_2l_2Nl)$ defined by the recurrence relations (44) are identical, up to a constant factor, with the Fano function of complex arguments:

*We note that in reference 15 the factor $B(n_1n_2n)$ has been omitted in the right hand sides of (7) and (8); the factor $-1/4$ is lacking in the expression for $B(n_1n_2n)$ [formula (6)].

$$C(N_1l_1N_2l_2Nl) = i^{l-1-l_1-l_2}\sqrt{iNN_1N_2(2l_1+1)(2l_2+1)}$$
$$\times X(J_1J_1l_1,\ J_2J_2l_2,\ JJl).$$
$$J_k = {}^1/_2(iN_k - 1). \tag{48}$$

To calculate the particular values of the Fano function in (48), we can use the usual formula

$$X(J_1J_1l_1,\ J_2J_2l_2,\ JJl) = \sum_\lambda (2J_2 + 2\lambda + 1)\, W(JlJ_1J_2;\ JJ_2+\lambda)$$
$$\times W(l_1lJ_2J_2;\ l_2J_2+\lambda)\, W(JJ_2J_1l_1,\ J_1J_2+\lambda), \tag{49}$$

in which λ must take on all integral values from $-l_0$ to $+l_0$, where l_0 is the smallest of the three numbers l_1, l_2, and l.

The explicit form of the functions W and others, which was obtained by Racah for real values of J_k, is also preserved in the case of complex $J_k = \frac{1}{2}(iN_k - 1)$ if all the factorials are replaced by the corresponding Γ functions of complex argument. In order not to lose a phase factor in this process, we must in the process of squaring and of extracting the square root retain all the factors i, and only in the very last stage can we put -1 for the i^2 which are outside the radical.

From (42) and (46) we obtain the Clebsch-Gordan expansion

$$T^{N_1}_{l_1m_1l_1'm_1'}T^{N_2}_{l_2m_2l_2'm_2'} = \frac{1}{N_1N_2}\sum_{ll'}\int_0^\infty NdNB(N_1N_2N)$$
$$\times C(N_1l_1N_2l_2Nl)\, C(N_1l_1'N_2l_2'Nl')\, C^{lm}_{l_1m_1l_2m_2}C^{l'm'}_{l_1'm_1'l_2'm_2'}T^N_{lml'm'}. \tag{50}$$

The inverse expansion may be obtained by utilizing the orthogonality properties of the coefficients $C(N_1l_1N_2l_2Nl)$. It has the following form

$$(N/N_1N_2)\, B(N_1N_2N)\sum C(N_1l_1N_2l_2Nl)\, C(N_1l_1'N_2l_2'N'l')$$
$$\times C^{lm}_{l_1m_1l_2m_2}C^{l'm'}_{l_1'm_1'l_2'm_2'}T^{N_1}_{l_1m_1l_1'm_1'}T^{N_2}_{l_2m_2l_2'm_2'} = T^N_{lml'm'}\delta(N - N'). \tag{51}$$

All the formulas of this section, with the exception of (47), hold for both the time-like and the space-like case. In (47) a minus sign will appear in the right hand sides if $B(N_1N_2N)$ is negative (space-like case).

5. APPLICATION OF THE INFINITE-DIMENSIONAL REPRESENTATIONS TO THE STUDY OF RELATIVISTIC PROCESSES

1. The infinite-dimensional representations of the Lorentz group can be utilized for the spectral representation of the reaction amplitude. As an example we consider the reaction $a + b \rightarrow c + d$, and assume that the particles are spinless. Then the reaction amplitude will depend on the masses of the particles, and on three independent momenta, p_a, p_b, and p_c. To describe the angular and the energy distributions of the particles, the reaction amplitude is usually expanded in terms of spherical harmonics of the angles that specify p_i, and the energy dependence of the coefficients of these functions is then obtained. Such an expansion is inconvenient for relativistic processes since the convergence of the expansion becomes worse as the energy increases. The reaction amplitude, regarded as a function of the variables p_a, p_b, p_c, depends not only on the nature of the interaction, but also on the kinematics of the process. Since the amplitude is a scalar, it contains only invariant combinations of the momenta. Along with $p_i^2 = m_i^2$, we can choose as such combinations, for example, the two scalar products (p_ap_b) and (p_cp_d).

In order to characterize the process independently of the kinematics, and to describe the angular distribution without expanding in terms of spherical harmonics, we can utilize the expansion of the amplitude in terms of the infinite-dimensional representations of the Lorentz group. We denote the reaction amplitude by $A(m_i^2, p_i)$. The expansion of the amplitude in terms of the basis functions of the infinite-dimensional representation has the following form

$$A(m_i^2,\ p_i) = \int_0^\infty\int_0^\infty dN_1dN_2A(m_i^2, N_1, N_2)\,\Pi_0(N_1, \gamma_1)\,\Pi_0(N_2, \gamma_2). \tag{52}$$

Here we have

$$\Pi_0(N, \gamma) = -\sin N\gamma/\sinh\gamma,\ \cosh\gamma_1 = \cosh\alpha_a\cosh\alpha_b - \sinh\alpha_a\sinh\alpha_b\cos\theta_{ab},$$
$$(p_ap_b) = m_am_b\cosh\gamma_1 = E_aE_b - \mathbf{p}_a\mathbf{p}_b,\ (p_cp_d) = m_cm_d\cosh\gamma_2.$$

The interaction process will be completely described by the quantity

$$A(m_i^2, N_1, N_2) = (4/\pi^2)\int d\omega_1d\omega_2A(m_i^2, p_i)\,\Pi_0(N_1, \gamma_1)\,\Pi_0(N_2, \gamma). \tag{53}$$

where $d\omega = \sinh^2\gamma\, d\gamma\, d\Omega$.

The whole angular and energy dependence of the amplitude is contained in the known functions $\Pi_0(N, \gamma)$. The expansions (52) and (53) hold for functions which are quadratically integrable over $d\omega$. If $A(m_i^2, p_i)$ is not such a function, then one can always separate the invariant factor in such a way that the remaining part will have the required properties.

If the particles that participate in the reaction have spin, then the amplitude will contain spin operators with factors that depend on the momenta and transform according to a finite-dimensional representation. The product of these factors and $\Pi_0(N, \gamma)$ can be expanded in terms of the infinite-dimensional unitary representations and, thus, an expansion for the reaction amplitude can be obtained in the general case.

2. The infinite-dimensional representations can be utilized for the solution of equations of quantum field theory. As an example let us consider the simplest case — the D'Alembert equation. It can be easily seen that in this case the basis functions $\Psi_{Nlm}(\alpha, \vartheta, \varphi)$ enable us to separate variables. Indeed, in accordance with reference 15, we have

$$\partial_{\alpha\beta} G_N(\rho)\Psi_{Nlm}(\alpha,\vartheta,\varphi) = \pm\sum_{L\varkappa f} i^{l-L}\left[\frac{\partial}{\partial\rho} - \varkappa\frac{iN-\varkappa}{\rho}\right] \times G_N(\rho)\sqrt{\frac{N(2f+1)}{2\nu}}\, C^{1/2\alpha}_{1/2\beta f-\sigma} C^{L\Lambda}_{lmf\sigma} A(Nl2f\nu L)\Psi_{\nu L\Lambda}(\alpha,\vartheta,\varphi), \tag{54}$$

$i\nu = iN + \kappa$, $\kappa = \pm 1$, $G_N(\rho)$ depends only on ρ, while

$$\partial_{\pm 1/2 \pm 1/2} = \partial/\partial t \mp \partial/\partial z, \qquad \partial_{\pm 1/2 \mp 1/2} = \pm(\partial/\partial x \mp i\partial/\partial y).$$

The $\pm$ signs in (54) refer respectively to the cases $t^2 - r^2 > 0$ and $t^2 - r^2 < 0$. By applying the operator $\partial_{\alpha\beta}$ twice we obtain

$$\Box G_N(\rho)\Psi_{Nlm}(\alpha,\vartheta,\varphi) = -{}^1\!/_2\sum_{\alpha\beta}\partial^{\alpha\beta}\partial_{\alpha\beta}G_N(\rho)\Psi_{Nlm}(\alpha,\vartheta,\varphi) = \left\{\frac{\partial^2}{\partial\rho^2} + \frac{3}{\rho}\frac{\partial}{\partial\rho} + \frac{N^2+1}{\rho^2}\right\}G_N(\rho)\Psi_{Nlm}(\alpha,\vartheta,\varphi). \tag{55}$$

For equations of a more complicated type, for example, the Bethe-Salpeter equation, a chain of equations in the variable ρ is obtained.

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