COMPARISON OF DIFFERENT COORDINATE CONDITIONS IN EINSTEIN'S GRAVITATION THEORY

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In this paper it is shown that the first-order coordinate conditions actually used by Einstein and Infeld in the problem of the motion of masses coincide with the harmonic ones and define a coordinate system in this order of approximation up to a Lorentz transformation. The difference between the Einstein-Infeld and the harmonic coordinate systems is characterized by the order of smallness of admissible non-Lorentz transformations [Eq. (10)]. These differences may be found explicitly. They are so small that they cannot affect the form of the equations of motion in the first post-Newtonian approximation. On the basis of the results obtained a criticism is given of the general attitude of Einstein and Infeld to the problem of coordinates.

1. INTRODUCTION

THE equations of motion for a system of masses in Einstein's gravitation theory are derived on the assumption that the system is of an insular nature and that at infinity space is Euclidean. Thus, in this problem the specific case is considered when there exists a coordinate system, known as harmonic, which is uniquely defined up to a Lorentz transformation (the existence of such a system follows from the uniqueness theorem for the wave equation). Because of this in the present problem the question of the coordinate system may be easily investigated to the end. Nevertheless this problem is discussed incorrectly in the papers of Einstein and Infeld. The aim of the present note is to establish in an explicit form the relationship between the coordinate systems which are actually used in the papers of Einstein and Infeld and the harmonic coordinate system. This will shed light also on the general principles of the problem under discussion which have been incorrectly dealt with by these authors.

2. COORDINATE CONDITIONS IN THE ZERO-ORDER AND THE FIRST-ORDER APPROXIMATIONS

In the present author's 1939 paper[1] and in all his later papers the harmonic coordinate system is adopted in which the quantities $\mathfrak{g}^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}$ satisfy the following equations

$$\frac{\partial \mathfrak{g}^{\mu\nu}}{\partial x_\mu} = 0. \qquad (1)$$

The time is adopted for the zero coordinate x_0: $x_0 = t$. The following general approximate expressions have been obtained for the quantities $\mathfrak{g}^{\mu\nu}$

$$\begin{aligned} \mathfrak{g}^{00} &= 1/c + 4U/c^3 + 4S/c^5, \\ \mathfrak{g}^{0i} &= 4U_i/c^3 + 4S_i/c^5, \\ \mathfrak{g}^{ik} &= -c\delta_{ik} + 4S_{ik}/c^3. \end{aligned} \qquad (2)$$

Expressions (2) lead to the following equations

$$\mathfrak{g}^{00} - 1/c = O(1/c^3), \qquad \mathfrak{g}^{0i} = O(1/c^3),$$
$$\mathfrak{g}^{ik} + c\delta_{ik} = O(1/c^3), \qquad (3)$$

which we shall call the zero-order coordinate conditions. In the zero-order approximation [when the right hand sides of (3) are neglected] relation (1) is satisfied because the main terms in the diagonal elements of $\mathfrak{g}^{\mu\nu}$ are constant and terms of order $1/c$ are absent in $\mathfrak{g}^{0i}$ and $\mathfrak{g}^{ik}$.

The same zero-order coordinate conditions have been adopted in the papers by Einstein and Infeld.[3-7] The notation in their papers differs somewhat from ours. They have adopted $x_0 = ct$, and the potentials $g_{\mu\nu}$ are expressed in terms of the auxiliary quantities $\gamma_{\mu\nu}$ by means of the formula

$$g_{\mu\nu} = e_\mu \delta_{\mu\nu}\left(1 - \tfrac{1}{2} e_\alpha \gamma_{\alpha\alpha}\right) + \gamma_{\mu\nu}, \qquad (4)$$

where $e_0 = 1$; $e_1 = e_2 = e_3 = -1$. For the quantities $\gamma_{\mu\nu}$ expansions in powers of the small parameter λ (for which one may take $1/c$) are used, viz:

$$\begin{aligned} \gamma_{00} &= \lambda^2 \underset{2}{\gamma}_{00} + \lambda^4 \underset{4}{\gamma}_{00} + \lambda^6 \underset{6}{\gamma}_{00} + \ldots, \\ \gamma_{0i} &= \lambda^3 \underset{3}{\gamma}_{0i} + \lambda^5 \underset{5}{\gamma}_{0i} + \ldots, \\ \gamma_{ik} &= \lambda^4 \underset{4}{\gamma}_{ik} + \lambda^6 \underset{6}{\gamma}_{ik} + \ldots. \end{aligned} \qquad (5)$$

The index below the letter γ denotes that this expression is the coefficient of the corresponding power of λ.

In connection with the radiation conditions on the potentials, and also in connection with the requirement of uniform convergence, doubts arise with respect to the justification of the use by Einstein and Infeld of infinite series of the form (5). However, we are interested only in the first terms of these series; therefore we shall here leave this question aside.

From a comparison of formulas (4) and (2) it follows after some calculation that

$$\gamma_{00} = -\frac{4U}{c^2} - \frac{4S - 6U^2}{c^4},$$
$$\gamma_{0i} = \frac{4U_i}{c^3} + \frac{4S_i - 8UU_i}{c^5},$$
$$\gamma_{ik} = \frac{2U^2}{c^4}\delta_{ik} - \frac{4S_{ik}}{c^4}. \qquad (6)$$

It may be easily seen that if $\lambda = 1/c$, then the conditions for the zero-order approximation (3) are algebraically equivalent to the requirement that the first terms of the series for $\gamma_{\mu\nu}$ should be of the form (5), i.e., that the series for γ_{00}, γ_{0i}, and γ_{ik} should begin with terms of order λ^2, λ^3, and λ^4 respectively. This may be written in the form

$$\underset{0}{\gamma_{00}} = 0, \qquad \underset{1}{\gamma_{0i}} = 0, \qquad \underset{2}{\gamma_{ik}} = 0. \qquad (7)$$

Thus, conditions (3) and (7) are equivalent. These conditions follow from an investigation of the approximate expressions for the components of the mass tensor in coordinates close to Cartesian ones. They were obtained in this way both in our paper,[1] and also in the papers by Einstein and his collaborators.[3,5]

It should be noted that although Einstein, Infeld and other authors admit that formulas (3) or (7) are essential for the derivation of the equations of motion, they deny that these formulas represent coordinate conditions. But it is possible to show (cf. Sec. 3) that conditions (3) define (in the approximation under consideration) the coordinate system up to a Lorentz transformation.

The coefficients U, U_i in expressions (2) for g^{00} and g^{0i} enter into the formulation of the coordinate conditions in the next approximation. The quantity U is the Newtonian potential, while the quantities U_i are the components of the vector potential, and the following equation holds

$$\frac{\partial U}{\partial t} + \frac{\partial U_i}{\partial x_i} = 0. \qquad (8)$$

In the given coordinate system the quantities U and U_i are uniquely determined by physical considerations (from a consideration of density and mass flux). Since the coordinate system is fixed in the zero-order approximation* we can likewise regard the quantities U and U_i as being fixed. But then we can make the coordinate conditions (3) more precise by writing them in the following form

$$g^{00} - \frac{1}{c} - \frac{4U}{c^3} = O\left(\frac{1}{c^5}\right); \quad g^{0i} - \frac{4U_i}{c^3} = O\left(\frac{1}{c^5}\right);$$
$$g^{ik} + c\delta_{ik} = O\left(\frac{1}{c^3}\right). \qquad (9)$$

We shall refer to these conditions [together with relation (8)] as the first-order coordinate conditions. We shall see later that these conditions fix the coordinate system (already in the next approximation) up to a Lorentz transformation, so that only a transformation of the following form

$$t' = t + a^0/c^6; \qquad x_i' = x_i + a^i/c^4. \qquad (10)$$

remains a non-Lorentz one.

Our first-order coordinate conditions are also equivalent to those of Einstein. Condition (8) is explicitly given by Einstein and Infeld[5] (although in somewhat different notation) in the following form

$$\left(\frac{1}{c}\frac{\partial \gamma_{00}}{\partial t} - \frac{\partial \gamma_{0i}}{\partial x_i}\right)_{\lambda^3} = 0, \qquad (11)$$

where the subscript λ^3 denotes that the coefficient of λ^3 in the corresponding expression should be taken.

As regards the method of fixing U and U_i (i.e., γ_{00} and γ_{0i} in the notation used by the two authors quoted earlier), this is explicitly dealt with on p. 227 of their article[3] [formulas (10.6) and (10.8)]. There it is directly stated that the values of the potentials U and U_i (in our notation) characterize the problem.

Thus, not only in the zero-order but also in the first-order approximation, the conditions of Einstein and Infeld are equivalent to our conditions which follow from the requirement that the coordinates be harmonic.

3. ADMISSIBLE TRANSFORMATIONS OF COORDINATES

We now examine the extent to which the zero-order and the first-order conditions restrict the coordinate system.

We carry out the following infinitesimal transformation of coordinates

*Without restricting generality we may take the remaining arbitrary Lorentz transformation to be equal to the identity transformation.

$$x'_\alpha = x_\alpha + \eta^\alpha(x_0 x_1 x_2 x_3). \tag{12}$$

By considering $g^{\mu\nu}$ as functions of their arguments we easily obtain*

$$\delta g^{\mu\nu} = g^{\mu\alpha}\frac{\partial\eta^\nu}{\partial x_\alpha} + g^{\nu\alpha}\frac{\partial\eta^\mu}{\partial x_\alpha} - \frac{\partial}{\partial x_\alpha}(g^{\mu\nu}\eta^\alpha), \tag{13}$$

and, if the initial system was harmonic, then

$$\delta g^{\mu\nu} = \frac{\partial}{\partial x_\alpha}(g^{\mu\alpha}\eta^\nu + g^{\nu\alpha}\eta^\mu - g^{\mu\nu}\eta^\alpha). \tag{14}$$

In the course of the following argument we shall ascribe to the quantities η^ν a definite order of magnitude with respect to $1/c$.

On taking for $g^{\mu\nu}$ approximate expressions which satisfy the zero-order condition we obtain

$$\delta g^{00} = \frac{1}{c}\left(\frac{\partial\eta^0}{\partial t} - \frac{\partial\eta^l}{\partial x_l}\right) + O\left(\frac{\eta}{c^3}\right),$$

$$\delta g^{0i} = -c\frac{\partial\eta^0}{\partial x_i} + \frac{1}{c}\frac{\partial\eta^i}{\partial t} + O\left(\frac{\eta}{c^3}\right),$$

$$\delta g^{ik} = -c\left(\frac{\partial\eta^k}{\partial x_i} + \frac{\partial\eta^i}{\partial x_k}\right) + c\left(\frac{\partial\eta^0}{\partial t} + \frac{\partial\eta^l}{\partial x_l}\right)\delta_{ik} + O\left(\frac{\eta}{c^3}\right). \tag{15}$$

We examine the consequences for $\delta g^{\mu\nu}$ arising from the zero-order coordinate conditions. It is evident that all $\delta g^{\mu\nu}$ must be of order not smaller than $1/c^3$. On the other hand, even if the quantities η^α should not contain powers of c in the denominator, the terms which we have denoted by $O(\eta/c^3)$ will be of order not smaller than $1/c^3$. By neglecting them and by setting $\delta g^{\mu\nu} = 0$, we obtain for the quantities η^α a system of equations which characterizes an infinitesimal Lorentz transformation.

In particular, if we do not consider translations and spatial rotations, we have

$$\eta^0 = -\frac{1}{c^2}(x_i V_i); \qquad \eta^i = -V_i t. \tag{16}$$

Thus, the zero-order conditions (3) fix the coordinate system up to a Lorentz transformation in which η^i are quantities of zero order† with respect to $1/c$.

Having established this, we may without restriction of generality assume that the Lorentz transformation given above reduces to the identity transformation, and we may proceed to investigate the case when the quantities η^i are of order $1/c^2$ or higher. By repeating the preceding arguments we may easily see that as a result of the first order conditions (9) the same system of equations is obtained for the quantities η^α, and we again obtain a Lorentz transformation in which the quantities V_i may now be of order q^3/c^2, where q is some velocity of zero order with respect to c. (Without a restriction of generality we may regard this Lorentz transformation as also being reduced to the identity transformation.)

We have thus proved that the first-order conditions adopted by Einstein and Infeld firstly coincide with ours and, secondly, define the coordinate system uniquely up to a Lorentz transformation.

The admissible non-Lorentz part of the transformation which does not violate the first-order conditions is equal to

$$\eta^0 = a^0/c^6, \qquad \eta^i = a^i/c^4. \tag{17}$$

where a^0 and a^i are of zero order with respect to $1/c$. It is quite evident that the difference between the two coordinate systems: the harmonic one (t, x_i) and the non-harmonic one (t', x'_i), where

$$t' = t + a^0/c^6, \qquad x'_i = x_i + a^i/c^4, \tag{18}$$

cannot yet affect the equations of motion in the first post-Newtonian approximation.

Thus the fact that the Einstein-Infeld equations coincide with the equations of motion in harmonic coordinates is to be explained not by saying that these equations allegedly do not depend on the coordinate conditions, but simply by the fact that the Einstein-Infeld coordinate system does not differ from the harmonic one in the approximation under discussion.

4. CONNECTION BETWEEN DIFFERENT COORDINATE SYSTEMS IN THE SECOND-ORDER APPROXIMATION

In order to show in the clearest possible way that the Einstein-Infeld coordinate system differs from the harmonic system only by small terms of the form (17), we shall obtain explicit expressions for these terms.

To do this it is necessary to examine the second-order conditions adopted in the papers of Einstein and Infeld and to compare them with the harmonic ones.

The second-order harmonic conditions will be obtained if in Eq. (1) we collect terms of order $1/c^5$ for $\mu = 0$ and terms of order $1/c^3$ for $\mu = 1$. We then have

$$\frac{\partial S}{\partial t} + \frac{\partial S_i}{\partial x_i} = 0, \qquad \frac{\partial U_i}{\partial t} + \frac{\partial S_{ik}}{\partial x_k} = 0. \tag{19}$$

As regards the Einstein-Infeld conditions, we shall write them in two different forms. However, in actual calculations only the second form is utilized.

*Cf., for example, our book,[2] formulas (48.20) and (48.21).

†From this it follows, in particular, that coordinate conditions (3) are sufficient to enable us to write Newton's equations of motion (cf. reference 6).

The first variant can be written in the following form

$$\left(\frac{1}{c}\frac{\partial\gamma_{00}}{\partial t}-\frac{\partial\gamma_{0i}}{\partial x_i}\right)_{\lambda^5}=0, \quad \left(\frac{1}{c}\frac{\partial\gamma_{i0}}{\partial t}-\frac{\partial\gamma_{ik}}{\partial x_k}\right)_{\lambda^4}=0, \tag{20}$$

and the second variant in the form

$$\left(\frac{1}{c}\frac{\partial\gamma_{00}}{\partial t}-\frac{\partial\gamma_{0i}}{\partial x_i}\right)_{\lambda^5}=0, \quad \left(\frac{\partial\gamma_{ik}}{\partial x_k}\right)_{\lambda^4}=0. \tag{21}$$

Here the subscripts λ^4 and λ^5 denote that the coefficients of λ^4 and λ^5 should be taken in the corresponding expressions.

If we denote the Einstein and Infeld coordinates by (t', x_i'), then in order to distinguish them from the harmonic coordinates (t, x_i), we ought to insert primes on the independent variables in formulas (20) and (21). However, in the present approximation the distinction between the two sets of coordinates is not significant and the primes may be omitted.

We assume that in the Einstein-Infeld coordinates the quantities S, S_i and S_{ik} in formulas (2) have the values S', S_i', S_{ik}' (while we retain the notation S, S_i and S_{ik} for the same quantities in the harmonic coordinates). As regards U and U_i, in virtue of the first-order coordinate conditions the values of these quantities in the two coordinate systems coincide. In accordance with (6) we now have

$$\gamma_{00}=-\frac{4U}{c^2}-\frac{4S'-6U^2}{c^4},$$
$$\gamma_{0i}=\frac{4U_i}{c^2}+\frac{4S_i'-8UU_i}{c^5},$$
$$\gamma_{ik}=\frac{2U^2}{c^4}\delta_{ik}-\frac{4S_{ik}'}{c^4}. \tag{22}$$

On substituting these expressions into (20), we obtain for the first variant*

$$\frac{\partial S'}{\partial t}+\frac{\partial S_i'}{\partial x_i}=\frac{3}{2}\frac{\partial(U^2)}{\partial t}+2\frac{\partial(UU_i)}{\partial x_i}, \qquad \frac{\partial U_i}{\partial t}+\frac{\partial S_{ik}'}{\partial x_k}=\frac{1}{2}\frac{\partial(U^2)}{\partial x_i} \tag{23}$$

In the second variant the term $\partial U_i/\partial t$ is absent in the left hand side of the second equation of (23). On setting

$$\delta S=S'-S; \qquad \delta S_i=S_i'-S_i; \qquad \delta S_{ik}=S_{ik}'-S_{ik} \tag{24}$$

and on utilizing the harmonic relations (19), we obtain

$$\frac{\partial\delta S}{\partial t}+\frac{\partial\delta S_i}{\partial x_i}=\frac{3}{2}\frac{\partial U^2}{\partial t}+2\frac{\partial(UU_i)}{\partial x_i} \tag{25}$$

in both variants and then

$$\frac{\partial\delta S_{ik}}{\partial x_k}=\frac{1}{2}\frac{\partial(U^2)}{\partial x_i} \quad \text{(1st variant)}, \tag{26}$$

$$\frac{\partial\delta S_{ik}}{\partial x_k}=\frac{1}{2}\frac{\partial(U^2)}{\partial x_i}+\frac{\partial U_i}{\partial t} \quad \text{(2nd variant)}. \tag{27}$$

Keeping in mind the fact that in formula (12) for the transformation of coordinates the quantities

$$\eta^0=a^0/c^6, \qquad \eta^i=a^i/c^4 \tag{28}$$

are so small that their squares may be neglected, we may treat this transformation as an infinitesimal one and calculate the differences (24) with the aid of the formulas

$$4\delta S=c^5\delta\mathfrak{g}^{00}, \qquad 4\delta S_i=c^5\delta\mathfrak{g}^{0i}, \qquad 4\delta S_{ik}=c^5\delta\mathfrak{g}^{ik}, \tag{29}$$

where $\delta\mathfrak{g}^{\mu\nu}$ have the values (15). On substituting into these equations expressions (17) for η^ν in terms of a^ν, we obtain

$$4\delta S=-\frac{\partial a^i}{\partial x_i}; \qquad 4\delta S_i=\frac{\partial a^i}{\partial t}-\frac{\partial a^0}{\partial x_i},$$
$$4\delta S_{ik}=\delta_{ik}\frac{\partial a^l}{\partial x_l}-\frac{\partial a^i}{\partial x_k}-\frac{\partial a^k}{\partial x_i}. \tag{30}$$

On introducing these expressions into the Einstein-Infeld coordinate conditions we obtain

$$-\Delta a^0=6\frac{\partial(U^2)}{\partial t}+8\frac{\partial(UU_l)}{\partial x_l}, \tag{31}$$

and then in the case of the first variant

$$-\Delta a^i=2\frac{\partial(U^2)}{\partial x_i} \tag{32}$$

and in the case of the second variant

$$-\Delta a^i=2\frac{\partial(U^2)}{\partial x_i}+4\frac{\partial U_i}{\partial t}. \tag{33}$$

Thus, in the transformation formulas (10) the quantities a^ν are determined from the Poisson equations so that if we wish, we can write explicit expressions for them. If we, moreover, require that at infinity the coordinates should go over into Galilean coordinates, then in the expression for a^ν the only undetermined terms will be linear terms corresponding to a Lorentz transformation.

5. CONCLUSION

The general aspect of the problem which we have just discussed of the relation between different coordinate conditions consists of the fact that we have here an obvious illustration of the danger associated with an incorrect application of the concept of relativity, and particularly of the term "general relativity," which does not have an exact meaning, but which often produces some sort of a hypnotic effect.

Einstein and Infeld, and later also some other scientists, have put forward the paradoxical (and incorrect) assertion, that allegedly the equations

*I. Fikhtengol'ts has assisted me in the derivation of some of these formulas.

of motion are not connected with the coordinate conditions or that they are not connected with harmonic coordinate conditions.

This incorrect assertion is repeated in practically every article. Thus, in the 1940 paper[4] it is stated: "We do not make any assumptions in advance with respect to the coordinate system beyond the fact that it is Galilean at infinity." In Sec. 13 of the 1949 paper[5] a similar assertion is repeated (in somewhat more careful form), while in fact zero-order and first-order coordinate conditions are used. In the 1954 paper[6] Infeld, in arguing against me, shows in fact that the form of the Newtonian equations of motion depends only on the zero-order coordinate conditions, but asserts that allegedly "the coordinate conditions have no relation whatsoever to the equations of motion not only in the Newtonian, but also in the next post-Newtonian approximation." This assertion is incorrect even if we say that the zero-order coordinate conditions are not "coordinate conditions," but a "method," as is done by Infeld. The same assertion is repeated in Infeld's 1957 paper.[7]

Starting with the first-order approximation [Eqs. (8) and (9)] Einstein and Infeld begin to use the term "coordinate conditions." But they do not notice that Eqs. (8) and (9), which they are in fact using, define the coordinate system up to transformations of the form (10), and that to this degree of accuracy the coordinate system is harmonic. Moreover, formulas (10) show that the second-order coordinate conditions certainly do not affect the form of the equations of motion in the first post-Newtonian approximation, so that the computations made by various authors in this connection are superfluous. Nevertheless, the whole attention of Einstein, Infeld and other authors is concentrated on the effect of the second-order coordinate conditions, and when after lengthy calculations it turns out that there is no such effect, the authors see in this a confirmation of the idea of "general relativity." But in actual fact the absence of such an effect is a trivial fact, which follows in an obvious way from the elementary formulas (10).

We must assert that in the papers of Einstein and Infeld quoted earlier a point of view predominates which compels them:

a) to deny the existence of coordinate conditions while in fact they have been introduced;

b) to ascribe particular significance to coordinate conditions which cannot affect the form of the equations of motion;

c) to deny that actually in the approximation needed for the formulation of the equations of motion the harmonic, and not any other, coordinate system is used, and, finally,

d) to deny even the fact that the harmonic coordinate system is defined uniquely up to a Lorentz transformation.

It seems to us that there is no doubt that this incorrect point of view is inspired by an incorrect concept of the idea of relativity. We hope that as a result of clarifying this problem our work will turn out to be useful not only because it will make unnecessary many complicated calculations, but also from the point of view of general principles.

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