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EXCITATION OF VIBRATIONAL LEVELS AND COULOMB EXCITATION IN ALPHA DECAY

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The relative probability of excitation of vibrational levels in the α decay of even-even nuclei is calculated. An expression for the intensity of excitation of the daughter nucleus by α particles of the main (allowed) group is derived in the quasi-classical perturbation theory approximation. The results obtained are applied to an analysis of the experimental data on the fine structure of α decay.

CONSIDERABLE progress has been made recently in the interpretation of the fine structure of the α decay spectra of deformed nuclei.[1-7] These papers are based on the physical fact, first noted by Hill and Wheeler,*[8] that the spatial anisotropy of the potential barrier in nuclei with a nonspherical surface leads to an anisotropy in the angular distribution of the α particles: the intensity of the current in the direction a (see Fig. 1) will be greater than in the direction b owing to the greater penetrability of the Coulomb barrier. Indeed, the particles emitted in the direction a traverse a smaller width of barrier than the particles emitted in the direction b (r^* in Fig. 1 is the radius corresponding to the turning point). In other words, the wave function describing the α decay will contain not only the s wave but also waves with higher angular momentum. We therefore have a mixture of excited states of the daughter nucleus with angular momenta different from that of the initial nucleus. We shall assume that the wave function at the nuclear surface is constant and identical for all fine structure lines belonging to the same rotational band. This is justified qualitatively by the small absorption length of the α particles in the nucleus. The emission of the α particles by the nucleus may be thought of as a local surface process which is not affected by the deformation of the nucleus.

Besides this effect connected with the nonsphericity of the nuclear shape (i.e., with the anisotropy of the nuclear potential), there will always be some interaction between the α particles and the anisotropic component of the Coulomb field of the nucleus. This interaction leads to an additional possibility of energy and momentum exchange between the α particles and the nucleus even after the α particle has passed beyond the range of the nuclear interaction. This mechanism was first noted by Preston,[10] who also calculated its effect.

A close analogy exists between the Coulomb excitation in α decay and the Coulomb excitation of nuclei by passing charged particles. The difference between the two is that the excitation in α decay occurs mainly while the α particle is still beneath the barrier, whereas in the usual excitation the region of classical motion, $r > r^*$, is the more important. Coulomb excitation also occurs in undeformed nuclei; its probability gives the lower limit for the intensity of excitation of the weak lines of the spectrum in an intensive allowed decay. The first of the above-mentioned mechanisms comes into play not only for nuclei with rigid deformation, but also for spherical nuclei in the discussion of the levels connected with deformations of the surface, as, for example, the vibrational levels of the surface oscillations.

The surface effect for spherical nuclei can be calculated rather simply by taking account of the fact that in this case the spatial anisotropy of the Coulomb field of the daughter nucleus can be neglected in first approximation. The problem then consists in the determination of the wave function describing the decay on the sphere S (Fig. 1). In the region outside the sphere S the radial and angular coordinates separate, and the amplitudes of the partial waves at infinity are easily determined in the usual way. To find the wave function on the sphere S we may use the so-called adiabatic approximation. This consists in assuming that the nuclear surface is rigid during the time the α particle passes through the region of greatest interaction, i.e., in our case, through the region between the surface of the nucleus and the sphere S. A precise choice of the radius of this sphere is not important as long as it is sufficiently

*A similar remark was made earlier by Migdal.[9]

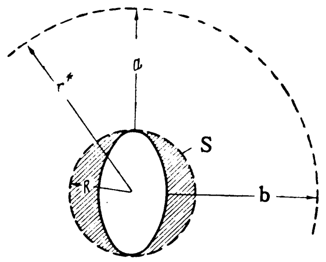

FIG. 1

close to the mean radius of the nucleus so that the change in velocity and the bending of the α particle trajectory can be neglected in the dashed region of Fig. 1.

For the applicability of the adiabatic approximation in this sense it is necessary that the time it takes for the α particle to traverse a distance of the order of the deformation of the nucleus be small compared with the period of oscillation of the surface, i.e., we must require

$$\alpha R\omega / v(R) \ll 1, \tag{1}$$

where $v(r)$ is the velocity of the α particles and αR and ω are the amplitude and the frequency of the surface oscillations. Substituting in (1) the value

$$\alpha \sim \sqrt{\hbar/2B_\lambda\omega} = \sqrt{\hbar\omega_\lambda/2C_\lambda},$$

where B_λ and C_λ are the mass parameter and the deformability for surface oscillations of the type λ, we find the following relation, which is equivalent to the inequality (1):

$$(\hbar\omega KR/2\,|\mathscr{E}(R)|)\sqrt{\hbar\omega/2C_\lambda} \lesssim 0{,}1 \ll 1,$$

where $\mathscr{E}(r)$ is the kinetic energy of the α particle, $K = \sqrt{2m\,|\mathscr{E}(R)|}/\hbar$ ($\hbar\omega \approx 0.5$ Mev, $|\mathscr{E}(R)| \approx 15-20$ Mev, $KR \approx 20$).

Fixing the shape of the nuclear surface for the moment, we find for the wave function of the α particle on the surface S the following expression in the quasi-classical approximation:

$$\psi(\mathbf{n})\Big|_S = \text{const}\cdot\exp\left\{-\int_{R(\mathbf{n})}^{r} K(r)\,dr\right\}\Big|_S \approx \text{const}\cdot\exp\left\{KR\sum_{\lambda\mu}\alpha_{\lambda\mu}Y_{\lambda\mu}(\mathbf{n})\right\}, \tag{2}$$

where $\mathbf{n}$ is the direction of the α particle. The surface of the nucleus is given by the equation

$$R(\Omega) = R\left[1 + \sum_{\lambda\mu}\alpha_{\lambda\mu}Y_{\lambda\mu}(\Omega)\right],$$

where $Y_{\lambda\mu}$ is a spherical harmonic.

Formula (2) does not take account of the bending of the trajectory of the α particle, since this effect leads at worst to corrections of the next highest order in α. We now introduce the phonon creation and annihilation operators in expression (2) by replacing α_μ by

$$\sqrt{\hbar/2B_\lambda\omega}\,(b_{\lambda\mu} + (-1)^\mu b^+_{\overline{\lambda\mu}}),$$

where $b_{\lambda\mu}$ and $b^+_{\lambda\mu}$ are the operators of annihilation and creation of phonons, respectively:

$$b_{\lambda\mu}b^+_{\lambda\mu} = n_{\lambda\mu} + 1, \qquad b^+_{\lambda\mu}b_{\lambda\mu} = n_{\lambda\mu},$$

where $n_{\lambda\mu}$ is the occupation number. The function (2) has now become an operator acting on functions of the occupation numbers $n_{\lambda\mu}$:

$$\psi(\mathbf{n})\Big|_S = \text{const}\cdot\exp\Big\{KR\sqrt{\hbar/2B_\lambda\omega} \times\sum_\mu [b_{\lambda\mu}Y_{\lambda\mu}(\mathbf{n}) + b^+_{\lambda\mu}Y^*_{\lambda\mu}(\mathbf{n})]\Big\}. \tag{3}$$

Since the commutator of the operators $\sum_\mu b_{\lambda\mu}Y_{\lambda\mu}$ and $\sum_\mu b^+_{\lambda\mu}Y^*_{\lambda\mu}$ is a c-number (it is equal to $\sqrt{(2\lambda+1)/4\pi}$), we can write the function (3) in the form

$$\psi(\mathbf{n})\Big|_S = \text{const}\cdot\exp(c/2)\exp\Big\{KR\sqrt{\hbar/2B_\lambda\omega}\sum_\mu b^+_{\lambda\mu}Y^*_{\lambda\mu}(\mathbf{n})\Big\} \times\exp\Big\{KR\sqrt{\hbar/2B_\lambda\omega}\sum_\mu b_{\lambda\mu}Y_{\lambda\mu}(\mathbf{n})\Big\}, \tag{4}$$

where

$$c = \sqrt{(2\lambda+1)/4\pi}\,(KR\sqrt{\hbar/2B_\lambda\omega})^2$$

is the commutator of the operator $KR\sqrt{\hbar/2B_\lambda\omega}\times\sum_\mu b_\mu Y_{\lambda\mu}$ with its conjugate operator.

The wave function (4) must be expanded in terms of the normalized (to unity) states of the daughter nucleus $\chi^{(n)}_{j,\mu}$ with a given number of phonons n and total angular momentum j, μ. In the case of an even-even nucleus the amplitude of such a state of the daughter nucleus is given by the matrix element of the operator (4) between the state $\chi^{(n)}_{j,\mu}$ and the vacuum state:

$$\langle n; j,\mu\,|\,\psi(\mathbf{n})\,|\,0; 0,0\rangle = \text{const}\cdot\frac{1}{n!}(KR\sqrt{\hbar/2B_\lambda\omega})^n\langle n; j,\mu\Big| \times\Big(\sum_\nu b^+_\nu Y_{\lambda\nu}(\mathbf{n})\Big)^n\Big|\,0; 0,0\rangle = \text{const}\cdot(KR\sqrt{\hbar/2B_\lambda\omega})^n \times\sum_{\nu_1\ldots\nu_n} Q^{j\mu;\,n}_{\lambda\nu_1\ldots\lambda\nu_n}Y^*_{\lambda\nu_1}(\mathbf{n})\ldots Y^*_{\lambda\nu_n}(\mathbf{n}), \tag{5}$$

where we used the following expression for the function $\chi^{(n)}_{j,\mu}$:

$$\chi^{(n)}_{j,\mu} = \sum_{\nu_1\ldots\nu_n} Q^{j\mu;\,n}_{\lambda\nu_1\ldots\lambda\nu_n}b^+_{\lambda\nu_1}\ldots b^+_{\lambda\nu_n}\,|\,0; 0,0\rangle. \tag{6}$$

The $Q^{\cdot\,\cdot}_{\cdot\,\cdot}$ are numerical coefficients,

$$Q^{j\mu;0} = \delta_{j0}, \quad Q^{j\mu;1}_{\lambda\nu} = \delta_{j\lambda}\,\delta_{\mu\nu}, \quad Q^{j\mu;2}_{\lambda\nu_1\lambda\nu_2} = \frac{1}{\sqrt{2}} C^{j\mu}_{\lambda\nu_1\lambda\nu_2}, \ldots,$$

and the $C^{c\gamma}_{a\alpha b\beta}$ are Clebsch-Gordan coefficients. In (5) we have

$$\sum_{\nu_1\ldots\nu_n} C^{j\mu;n}_{\lambda\nu_1\ldots\lambda\nu_n} Y^*_{\lambda\nu_1}(\mathbf{n})\ldots Y^*_{\lambda\nu_n}(\mathbf{n}) = A^{j,n}_{\lambda} Y^*_{j\mu}(\mathbf{n}).$$

The coefficients $A^{j,n}_{\lambda}$, just like the coefficients $C^{j\mu;n}_{\lambda\nu_1\ldots\lambda\nu_n}$, can be written as linear combinations of Clebsch-Gordan coefficients. For the important values $n = 0, 1, 2$ the coefficients $A^{j,n}_{\lambda}$ are equal to

$$A^{j,0}_{\lambda} = \delta_{j,0}, \quad A^{j,1}_{\lambda} = \delta_{\lambda j}, \quad A^{j,2}_{\lambda} = (2\lambda+1)\,[8\pi(2j+1)]^{-1/2} C^{j0}_{\lambda 0\lambda 0}. \tag{7}$$

Taking this into account, we find for the wave function describing the decay of a nucleus with spin zero

$$\psi = \text{const}\cdot \sum_{n=0,1,2} A_{j,n}\,(KR\sqrt{\hbar/2B_\lambda\omega})^n G_j(r)(k_j r)^{-1} \sum_{\mu} \chi^{(n)}_{j\mu} Y^*_{j\mu}(\mathbf{n}), \tag{8}$$

where $G_j(r)$ are radial wave functions describing the motion of an α particle with angular momentum $l = j$ and corresponding energy in a spherically symmetric Coulomb field.

According to (8) the amplitudes of the partial waves at infinity are equal to the amplitudes at the surface S multiplied by the corresponding penetration factors for the Coulomb and centrifugal barriers. The probability for α decay with the excitation of a vibrational state with the number of phonons n and angular momentum j, μ is proportional to

$$(2j+1)\,|\,A^{j,n}_{\lambda}(KR\sqrt{\hbar/2B_\lambda\omega})^n G_j(r\to\infty)\,|^2. \tag{9}$$

The ratio of the intensity of α decay with excitation of one phonon $(j = \lambda)$ over the intensity of α decay to the ground state is equal to

$$\xi^{(1)}_{\lambda=j} = (2j+1)(KR\sqrt{\hbar/2B_\lambda\omega})^2 \frac{P(\mathscr{E}', l=\lambda)}{P(\mathscr{E}_0, l=0)}, \tag{10}$$

and the relative intensity of the excitation of the two-phonon vibrational state is

$$\xi^{(2)}_{\lambda,j} = \frac{(2\lambda+1)^2}{4\pi}(KR\sqrt{\hbar/2B_\lambda\omega})^4 C^{j0}_{\lambda 0\lambda 0} \frac{P(\mathscr{E}'', l=j)}{P(\mathscr{E}_0, l=0)}, \tag{11}$$

$P(\mathscr{E}, l)$ denotes the penetrability of the Coulomb barrier for particles with energy $\mathscr{E}$ and orbital angular momentum l:

$$P(\mathscr{E}, l) = \exp\left\{-2\int_R^{r^*} |k_l(r)|\,dr\right\}$$
$$\approx \exp\{-2l(l+1)/\varkappa b\}\exp\{-2\varkappa\sqrt{b}\gamma\,\mathscr{E}R/2Ze^2\}, \tag{12}$$

where $b = 2Ze^2/R\mathscr{E}_0$, $\kappa = \sqrt{2m\mathscr{E}_0}\,R/\hbar$, and $\gamma(x)$ is a tabulated function (see, for example, the review article[11]). The quantity $\hbar/2B_\lambda\omega$ entering in (10) and (11) can be expressed in terms of the reduced probability of the radiative transition for a one-phonon excitation:[12]

$$B(E\lambda;\, j=0\to j=\lambda) = (2\lambda+1)\left(\frac{3}{4\pi} ZR_0^\lambda\right)^2 \hbar/2B_\lambda\omega. \tag{13}$$

We note that this formula, which expresses the relation between the radiative transition probability and the amplitude of the nuclear surface oscillations, contains the model assumption of irrotational flow of nuclear matter. The degeneracy of the two-phonon state with respect to the angular momentum number existing in the pure harmonic model can be removed by the residual interaction. If, however, the splitting of this level is small in comparison with the resolution of the apparatus, it is meaningful to speak only of the summed intensity of the α decay to the second excited vibrational state. This quantity is obtained by summing (12) over j, where we use the normalization of the Clebsch-Gordan coefficients $\sum_j (C^{j0}_{\lambda_0\lambda_0})^2 = 1$:

$$\xi^{(2)}_{\lambda} = \sum_j \xi^{(2)}_{\lambda,j} \approx \frac{(2\lambda+1)^2}{4\pi}(KR\sqrt{\hbar/2B_\lambda\omega})^4 \frac{P(\mathscr{E}'', l)}{P(\mathscr{E}_0, l=0)}, \tag{14}$$

where $P(\mathscr{E}'', l)$ is the average penetrability of the Coulomb barrier for the α particles belonging to the given group of states.

The lowest lying excited states with even parity of the Pb^{206} nucleus ($j = 2$, $E \approx 0.8$ Mev) and of the even isotopes of Po and Rn are possible examples of vibrational levels in the region of α active nuclei. In the table we list the results of the calculations for these nuclei, using formulas (10) to (14). The table also gives the relevant experimental data and the values of the effective radii $R = (r_0 A^{1/3} + 2.5)\times 10^{-13}$ cm used in the calculations (these values of R are taken from the review article[11]). R_0 in (13) was taken to be equal to $r_0 A^{1/3}\times 10^{-13}$ cm. The values of $\xi^{(2)}_{\lambda=2}$ in the table were calculated with the help of (14), since it is not excluded that the second excited state of the nuclei is in reality a vibrational state which is threefold degenerate with respect to the spin or a group of close lying levels. This possibility is not in disagreement with the experimental data.

As is seen from the data of the table, there is rather close agreement between the calculated and the experimental values of the quantities $B(E2; 0\to 2)$ or $\xi^{(2)}$, although the calculated value of $B(E2)$ lies somewhat below that obtained from Coulomb excitation. This discrepancy is particularly great in the case of Pb^{206} [$B(E2)$ α decay

Daughter nucleus	Pb^{206}	Po^{214}	Rn^{219}	Rn^{222}
$\mathcal{E}_0$, Mev	5.3	7.13	6.5	4.78
E_2, Mev	0.80	0.61	0.32	0.187
E_1, Mev			0.65	0.447
$10^{-13}R$, cm	$1.42A^{1/3}+2.5$	$1.58A^{1/3}+2.5$	$1.58A^{1/3}+2.5$	$1.54A^{1/3}+2.5$
$\xi^{(1)}$ experiment	$1,2\cdot 10^{-5}$	$2\cdot 10^{-3}$	$4\cdot 10^{-2}$	$5.7\cdot 10^{-2}$
$\xi^{(2)}$ experiment			$8\cdot 10^{-5}$	10^{-4}
$10^{-48}B\,(E2)$, cm^4 experiment	0.14	0.60	unknown	unknown
$KR\sqrt{\hbar/2B_2\omega}$ calculated from $\xi^{(1)}_{\exp}$	0.29	0.33	0.50	0.63
$10^{-48}B\,(E2)$, cm^4 calculated from $\xi^{(1)}_{\exp}$	0.022	0.44	1.00	1.20
$\xi^{(2)}$ calculated from $\xi^{(1)}_{\exp}$			$5\cdot 10^{-5}$	$5.5\cdot 10^{-5}$

$\approx {}^1\!/_7 B\,(E2)_{\mathrm{Coul}}$]. The disagreement even becomes slightly worse if the Coulomb excitation in α decay is taken into account (see below). A possible explanation of this disagreement is that in α decay only the part of the quadrupole moment connected with the deformation of the nuclear surface comes into play. In radiative transitions and Coulomb excitation, on the other hand, the total quadrupole moment of the nucleus enters into the calculation, including also the quadrupole moment of the nucleons in the unfilled shell. Formula (13) involves only the quadrupole moment of the core.

We note that the intensity of the α decay of the even isotopes of Ra to the ground, first, and second excited states of Rn with even parity also agrees with the assumption that these levels have rotational character. The parameter of quadrupole deformation, α, is here assumed to be equal to $0.07 - 0.09$,[4] which is in qualitative agreement with the magnitude of the moments of inertia of these nuclei. The smallness of the ratio of the energy of the second excited level over the energy of the first excited level $(2.0 - 2.4)$ may in this case be explained by the circumstance that the rotation of the nucleus is not adiabatic for a small moment of inertia of the nucleus.*

Let us now turn to the discussion of the Coulomb excitation in α decay. For simplicity we consider spherical nuclei, in which case the nonspherical Coulomb field outside the nucleus can be neglected (with the exception of the transition field). In the case of a spherical nucleus the wave function describing the α decay of the nucleus into states with spin I, M = 0 (the result is, of course, independent of the choice of the projection of the angular momentum) can be written in the form

$$\psi_{I,0} = \sum_{lj} r^{-1} a_{lj}(r) \sum_{m\mu} C^{I0}_{lmj\mu} \chi_{j\mu} Y_{lm}(\mathbf{n}) + \sum_{lj} r^{-1} b_{lj}(r) \sum_{m\mu} C^{I0}_{lmj\mu} \bar{\chi}_{j\mu} Y_{lm}(\mathbf{n}), \tag{15}$$

where $\chi_{j\mu}$ and $\bar{\chi}_{j\mu}$ are wave functions describing the internal state of the daughter nucleus in the intensive principal α decay and the state of the nucleus excited by the emitted α particle via the Coulomb interaction, respectively.

The Schrödinger equation satisfied by (15) has the form

$$\left(H(X) - \frac{\hbar^2}{2m}\Delta_r + V_0(\mathbf{r}) + V_1(X, \mathbf{r}) - E\right)\psi_{I,0}(X, \mathbf{r}) = 0, \tag{16}$$

where H(X) is the Hamiltonian of the daughter nucleus, and X denotes the internal coordinates of the nucleus. Here

$$V_0 = \frac{2Ze^2}{r},$$

$$V_1 = \frac{2Ze^2}{r} \sum_{\lambda=1,\ldots} \frac{4\pi}{2\lambda+1} \sum_{i,\nu} (r_i/r)^\lambda Y^*_{\lambda\nu}(\mathbf{r}_i/r_i)\, Y_{\lambda\nu}(\mathbf{n}),$$

the summation goes over all protons in the nucleus. We can obtain an equation for the functions b_{lj} by multiplying (16) by $\bar{\chi}^*_{j\mu} Y^*_{lm}(\mathbf{n})$ and integrating the resulting expression over the nuclear variables and the angular coordinates of the α particle. Since we assume that the effect of the transition field is weak, we can neglect the term containing the product of V_1 and b_{lj}. The resulting equation for b_{lj} has the form

$$\left(-\frac{\hbar^2}{2m}\Delta_r^{(l)} + V_0 - \mathcal{E}\right)\frac{b_{lj}}{r} C^{I0}_{l-\mu j\mu} = \sum_{l'j'} (a_{l'j'}/r) \times \sum_{m'\mu'} C^{I0}_{l'm'j'\mu'} \int d\mathbf{n} Y^*_{lm}(\mathbf{n}) < j\mu\,|\,V'\,|\,j'\mu' > Y_{l'm'}(\mathbf{n}),$$

$$\Delta_r^{(l)} = \frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr} - \frac{l(l+1)}{r^2}, \tag{17}$$

where the primes on the indices refer to the principal α decay.

An analogous equation can be written down for the functions $a_{l'j'}$. It is clear, however, that it is

*A different possible explanation of the smallness of the ratio of the second excited level to the energy of the first excited level 2^+ for these nuclei has to do with the asymmetry of their surface shapes (private communication by A. S. Davydov). This possibility is not considered here.

sufficient for the calculation of the b_{lj} in first approximation to know the $a_{l'j'}$ only in the zeroth approximation. This means that we can neglect the inverse transitions from the secondary group to the principal group. In this approximation the functions $a_{l'j'}$ coincide with the functions G introduced earlier.

We write the transition field $\langle j\mu \mid V_1(X, r) \mid j'\mu'\rangle$ in the form

$$\langle j\mu \mid V_1(X, r) \mid j'\mu'\rangle = \frac{2e^2}{r} \sum_{\lambda=1,\ldots} \frac{4\pi}{2\lambda+1} \sum_{\nu} r^{-\lambda} \langle j\mu \mid \mathfrak{M}^*(\lambda\nu) \mid j'\mu'\rangle Y_{\lambda\nu}, \tag{18}$$

where $\mathfrak{M}(\lambda\nu)$ is the operator for the electric multipole transition (λ, ν):

$$\mathfrak{M}(\lambda\nu) = \sum_i r_i^{\lambda} Y_{\lambda\nu}(\mathbf{r}_i / r_i).$$

The matrix element of the operator $\mathfrak{M}(\lambda\nu)$ can be conveniently expressed in terms of the reduced matrix element $\langle j \parallel \mathfrak{M}(\lambda) \parallel j'\rangle$, which is defined by

$$\langle j\mu \mid \mathfrak{M}(\lambda\nu) \mid j'\mu'\rangle = (-1)^{\lambda+j-j'} C^{j\mu}_{\lambda\nu j'\mu'} (2j+1)^{-1/2} \langle j \parallel \mathfrak{M}(\lambda) \parallel j'\rangle. \tag{19}$$

The reduced matrix element is related to the reduced transition probability $B(E\lambda;\, j' \to j)$ in the following way:

$$B(E\lambda;\, j' \to j) = \sum_{\mu'\nu} |\langle j\mu \mid \mathfrak{M}(\lambda\nu) \mid j'\mu'\rangle|^2 = |\langle j \parallel \mathfrak{M}(\lambda) \parallel j'\rangle|^2 \frac{1}{(2j'+1)}. \tag{20}$$

Substituting (19) in the right hand side of equation (17), we obtain, after summing over the spin projections, the following equation for the $b_{lj}(r)$:

$$\left(-\frac{\hbar^2}{2m}\Delta_r^{(l)} + V_0 - \mathscr{E}_l\right)\frac{b_{lj}}{r} = -\frac{2e^2}{r}\sum_{\lambda l'j'} (-1)^{j-j'} [4\pi(2l'+1)(2j'+1)/(2\lambda+1)]^{-1/2} \times C^{l0}_{l'0\lambda 0} B^{1/2}(\lambda;\, j' \to j)\, r^{-\lambda} W(jj'll'/\lambda I)\, [a_{l'j'}(r)/r], \tag{21}$$

where the $W(abcd \mid ef)$ are Racah coefficients.

The phase of $B^{1/2}(E\lambda)$ in (21) is taken to be that of the matrix element $\langle j \parallel \mathfrak{M}(\lambda) \parallel j'\rangle$. The angular momenta entering in equation (21) must satisfy the "triangular relations" $(jj'\lambda)$, $(ll'\lambda)$, (jlI), and $(j'l'I)$. These inequalities connecting the magnitudes of the angular momenta have the usual physical interpretation (see Fig. 2).

In place of the functions b_{lj} we introduce new functions $\zeta_{lj}(r)$, which are connected with the b_{lj} by the relation

$$b_{lj}(r) = \zeta_{lj}(r) \exp\{\hbar^{-1} S_{lj}(r)\}, \tag{22}$$

where

$$\hbar^{-1} S_{lj}(r) = \int_R^r |k_{lj}(r)|\, dr,$$

k_{lj} is the wave vector of the α particle [see Eq. (12)]. Substituting (22) in (21), we obtain for the function ζ_{lj} the equation

$$p_{lj}\, d\zeta_{lj}(r)/dr + \tfrac{1}{2}(dp_{lj}/dr)\,\zeta_{lj}(r) = \frac{2me^2}{\hbar R}\sum_{l'j'} A^{lj}_{l'j'}\left(\frac{R}{r}\right)^{\lambda+1} \times a_{l'j'}(r) \exp\{\hbar^{-1} S_{lj}(r)\}\, B^{1/2}(\lambda;\, j' \to j)\, R^{-\lambda},$$
$$p_{lj}(r) = |k_{lj}(r)|. \tag{23}$$

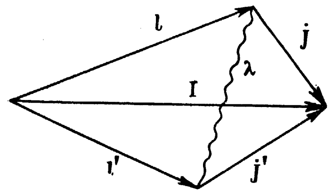

FIG. 2

In (23) we have omitted the term containing the second derivative of the function ζ_{lj}, which is smaller than the other terms by the factor KR, and

$$A^{lj}_{l'j'} = \{4\pi(2l'+1)/(2\lambda+1)\}^{1/2} C^{l0}_{l'0\lambda 0} W(jj'll'/\lambda I).$$

The solution of (23) can be written in the form

$$\zeta_{lj}(r) = \zeta^0_{l'j'} + (2me^2/\hbar\sqrt{p_{lj}(r)}) \sum_{l'j'} A^{lj}_{l'j'} B^{1/2}(\lambda;\, j' \to j)\, R^{-\lambda} \times \int_R^r (R/r')^{\lambda+1} d(r'/R)\, [p_{lj}(r')]^{-1/2} \exp\{h^{-1} S_{lj}(r')\}\, a_{l'j'}(r). \tag{24}$$

Here ζ^0_{lj} is the value of the amplitude on the nuclear surface. In our approximation this term corresponds to the nuclear excitation. In the particular case of vibrational levels it corresponds to the above-mentioned "surface" excitation. In this instance ζ^0_{lj} coincides with the amplitude of the "surface" excitation on the sphere S.

For the functions $a_{l'j'}$ we may use the quasi-classical approximation,

$$a_{l'j'}(r) = a^0_{l'j'} [p_{l'j'}(r)]^{-1/2} \exp\{-\hbar^{-1} S_{l'j'}(r)\}, \tag{25}$$

where $a^0_{l'j'}$ is the amplitude of the function a_{lj} on the surface of the nucleus. For $r > r^*$ the function b_{lj} has the form

$$b_{lj}(r) = (2me^2/\hbar\sqrt{p_{lj}(r)}) \times \sum_{l'j'} A^{lj}_{l'j'} B^{1/2} R^{-\lambda} a_{l'j'}(r^*) \exp\{i\hbar^{-1} S_{lj}(r)\} \times \left\{\int_R^{r^*} \left(\frac{R}{r'}\right)^{\lambda+1} d\left(\frac{r'}{R}\right) \exp\left\{\frac{1}{\hbar}[(S_{lj}(r') - S_{lj}(r^*)) - (S_{l'j'}(r') - S_{l'j'}(r^*))]\right\} [p_{lj}(r')\, p_{l'j'}(r')]^{-1/2} + \int_{r^*}^{r} \left(\frac{R}{r}\right)^{\lambda+1} d\left(\frac{r'}{R}\right) \times \exp\left\{\frac{1}{\hbar}[S_{lj}(r') - S_{l'j'}(r')]\right\} [p_{lj}(r')\, p_{l'j'}(r')]^{-1/2}\right\}. \tag{26}$$

To obtain the intensity of the line (l, j) we must multiply the square of (26) by $p_{lj}(r)$ and take the limit of the resulting expression as $r \to \infty$. The intensity of the line corresponding to a level with angular momentum j is obtained by summing over l. The ratio of the intensities is conveniently written in the form

$$\xi_{j' \to j;\, l \to j'} = (4me^2R / \hbar^2 b^\lambda \varkappa)\, B(\lambda;\, j' \to j)\, R^{-2\lambda}$$
$$\times \Big\{ \sum_l \Big| \sum_{l'} A_{l'j'}^{lj} a_{l'j'}(r^*)\, (J_1 + J_2) \Big|^2 \Big\} \Big/ \sum_{l'} |a_{l'j'}(r^*)|^2, \qquad (27)$$

where J_1 and J_2 are, respectively, integrals over the barrier region and over the region of classical motion.

The action function S_{lj} in (26) is now expanded into a sum over the angular momenta and energies of the excited nucleus. The integral J_1 can then be written in the form

$$J_1 = \int_0^{\sqrt{b-1}} dy\, (1 + y^2)^{\lambda - 1} \exp\{-\sigma_{lj,\, l'j'}(y)\},$$
$$\sigma_{lj,\, l'j'}(y) = c_{ll'}\, y + d_{jj'}\, (y / (1 + y^2) + \tan^{-1} y),$$
$$c_{ll'} = [l(l+1) - l'(l'+1)] / \varkappa b, \quad d_{jj'} = \varkappa b \Delta\mathscr{E} / 2\mathscr{E}.$$

A convenient form of the integral J_2 is

$$J_2 = \frac{1}{2} v_\infty b^\lambda R^\lambda \int_{r^*}^{\infty} r^{-(\lambda+1)}\, v^{-1}(r) \exp\{-i\omega_{jj'}\, t(r)\}\, dr,$$

where

$$t(r) = \int_{r^*}^{r} v^{-1}(r)\, dr, \quad \omega_{jj'} = (\mathscr{E}_j - \mathscr{E}_{j'}) / \hbar.$$

In the denominators of the integrands of the integrals J_1 and J_2 we neglected the dependence of p_{lj} on l and j. Integrals analogous to J_2 have been computed in the theory of the Coulomb excitation by charged particles.[13,12] The only difference is that the time integration in J_2 goes from 0 to ∞, and not from $-\infty$ to $+\infty$ as in the usual case, i.e., we integrate only over one half of the trajectory of the scattering particle. Furthermore, we neglected in J_2 the dependence of $t(r)$ on the orbital angular momentum. The integral J_2, therefore, corresponds to a head-on collision, i.e., to a classical orbit with eccentricity one.

In a head-on collision the incoming and outgoing branches of the trajectory give the same contribution to the amplitude of electric excitation. This allows us to express the integral J_2 directly in terms of the tabulated integral $I_{\lambda\mu}(\vartheta, \xi)$:[12,14]

$$J_2 = 2^{\lambda-2} I_{\lambda 0}(180°,\, d_{jj'}/2) = 2^{\lambda-2} \int_{-\infty}^{+\infty} \exp\{id_{jj'}(\sinh\omega$$
$$+\, \omega)/2\}\, (\cosh\omega + 1)^{-\lambda}\, d\omega,$$
$$d_{jj'} = (2Ze^2 / \hbar v_\infty)(\Delta\mathscr{E} / \mathscr{E}).$$

Formula (27) can be simplified considerably in the case of an even-even nucleus. Then $I = 0$, $l = j$, and $l' = j'$. If, moreover, the contribution from the state $l' = j' = 0$ is predominant, which is the case for spherical nuclei, we obtain from formula (27)

$$\xi_{0 \to j;\, 0 \to 0} = (4me^2R / \hbar b^\lambda \varkappa)\, [4\pi / (2\lambda + 1)]\, B(E\lambda;\, 0 \to j)$$
$$\times R^{-2\lambda} (J_1 + J_2)^2 \delta_{\lambda j}. \qquad (28)$$

In the decay of deformed nuclei several values of l usually have comparable intensity. Besides this, the Coulomb excitation in deformed nuclei is also different in that there exists an appreciable static quadrupole potential outside the nucleus. Nevertheless, even in the calculation of the Coulomb excitation of nonspherical nuclei one can in first approximation neglect the quadrupole field, unless, of course, the quadrupole potential is the transition potential itself.

This circumstance is due to the fact that even the electric quadrupole excitation contributes only little to the intensity of the principal lines of the even group $(j = 0, 2)$ (see below, and also reference 4). The electric excitation outside the sphere S is in this approximation also given by formula (27). It is convenient to write this formula in a somewhat different form for deformed nuclei. For this purpose we use a formula which expresses the reduced transition probability through the intrinsic multipole moment of the nucleus in the coordinate system fixed in the nucleus:[12]

$$B^{1/2}(E\lambda;\, j' \to j) = [(2\lambda + 1) / 4\pi]^{1/2} Q_\lambda^{(0)} C_{j'K\lambda 0}^{jK}.$$

We have

$$\xi_{\Sigma j' \to j;\, 0 \to 0} = (4me^2R / \hbar^2 \varkappa b^\lambda)^2 q_\lambda^2 \Big| \sum_{l'} \alpha_{l'} (C_{\lambda 0 j 0}^{l'0})^2 (J_1 + J_2) \Big|^2,$$
$$q_\lambda = Q_\lambda^{(0)} / R_0^\lambda, \quad \alpha_{l'} = a_{l'}(r^*) / a_0(r^*). \qquad (29)$$

For odd nonspherical nuclei we can use an expression for the transition amplitude which was obtained in the adiabatic approximation (see references 2 and 3):*

$$\alpha_{l'j'} = \alpha_{l'} C_{IKl'0}^{j'K}, \quad \alpha_{l'j'} = a_{l'j'}(r^*) / a_{0I}(r^*). \qquad (30)$$

Using (29) and (30), we find for an odd nucleus

*In the determination of the wave function in the adiabatic approximation the nucleus is considered at rest and the law of conservation of energy is not observed. The accuracy of the adiabatic approximation is improved considerably, if the conservation of energy is taken into account. The adiabatic approximation in this sense becomes invalid only to the same extent as the nonsphericity of the Coulomb field becomes important. It therefore has quite sufficient accuracy for our purposes.

$$\xi_{\Sigma j'\to j;\, I\to I} = (4me^2R / \hbar^2\varkappa b^\lambda)\, q_\lambda^2 \Big\{\sum_l \Big| \sum_{l'j'} C^{lj}_{l'j'} \alpha_{l'} \times (J_1 + J_2) \Big|^2\Big\} \Big/ \sum_{l'} |\alpha_{l'}|^2 (C^{II}_{II\,l'0})^2, \quad (31)$$

where

$$C^{lj}_{l'j'} = (-1)^{j-j'} [(2l'+1) \times (2j'+1)]^{1/2} C^{l0}_{l'0\lambda 0} C^{j'I}_{II l0} C^{jI}_{j'I\lambda 0} W(jj'll' | \lambda I).$$

In formula (31) the projection of the spin on the nuclear axis, K, is set equal to I, since we are considering a transition without change of K, which is equal to I in the initial nucleus.

In those cases where the reduced electromagnetic transition probability is known experimentally (for example, through the lifetime of the state or through the probability of electric Coulomb excitation) the probability of Coulomb excitation of a given state in α decay can be computed directly using formulas (28), (29), and (31). The values of B(E2; $0 \to 2$) are known for the nuclei Pb^{206} and Po^{210} (cf. the table). The integrals $J_1 + J_2$ for these nuclei are equal to 0.2 and 0.77, respectively (the integral J_1 is obtained by numerical integration, the values of the integral J_2 were taken from the above-mentioned tables[12,14]). According to formula (28) we obtain for the relative probability of electric quadrupole excitation for these nuclei the values 2.0×10^{-4}% and 0.03%, respectively. This means that pure Coulomb excitation could explain only approximately $1/6$ to $1/7$ of the observed intensity of the decay to the level 2^+. In order to explain, in the case of Ra^{222}, the observed intensity of the decay to the first excited state 2^+ of the daughter nucleus Rn^{218} (4.5%) by Coulomb excitation alone, one would have to assume that for Rn^{218} one has the value B(E2; $0 \to 2$) = 40 $\times 10^{-48}$ cm^4, which exceeds considerably the expected value (1 to 2×10^{-48} cm^4).

The reduced probability for radiative E2 decay is known also for the odd nucleus Pb^{207} [B(E2; $1/2 \to 5/2$) = 0.028×10^{-48} cm^4]. In the decay of Po^{211} ($9/2^-$), which leads to the formation of this nucleus, we have, besides the intensive transition to the state ($1/2^-$, $l' = 4$), also transitions to the excited states ($5/2^-$) and ($3/2^-$). According to formula (27) we find for the relative intensity of the electric excitation of the level the value 4.5×10^{-5}, which is approximately one hundredth of the experimental value ($l' = 4$, $l = 2$; 4; 6; $J_1 + J_2 = 1.2$; 1.8; 0.8).

Let us now consider the Coulomb excitation of levels with odd parity. In the decay of the even isotopes of Ra we observe, besides the intensive transitions to levels of the same parity, relatively rare transitions to excited states (1^-) of the daughter nuclei Rn ($E_{1-} \approx 0.6$ Mev). For the decay $Ra^{224} \to Rn^{220}$ we have $J_1 \approx 0.25$ and $J_2 \approx 0.07$. According to formula (29), the observed intensity of the decay ($Ra^{224} \to Rn^{220}$, 1^-) would correspond to the value B(E1; $0^+ \to 1^-$) = $1/50 \times 10^{-24}$ cm^2. For the decay $Ra^{226} \to Rn^{222}$ ($J_1 = 0.10$, $J_2 = 0.02$) we obtain B(E1: $0^+ \to 1^-$) = $1/10 \times 10^{-24}$ cm^2. Both these values are close to the one-particle values of B(E1), but are considerably (10 to 50 times) higher than the value of the reduced probability expected for nuclei with octupole-deformed ("pear-shaped") surfaces.[15]

Decays to levels with odd parity are also observed in heavier even nuclei. Let us take as an example the decay $Th^{228} \to Ra^{224}$. The experimental intensity of the α decay to the level 1^- leads, according to formula (29), to $q_1 = 0.45$ ($\lambda = 1$, dipole excitation) or $q_3 = 9$ ($\lambda = 3$, octupole excitation). This value of q_1 is approximately 10 times larger than the experimental value[16] and the value expected for "pear-shaped" nuclei. According to formula (29), the intensity of the Coulomb excitation of the level 3^- of Ra^{224} amounts to about $1/10$ of the intensity of the decay to the level 1^- for dipole excitation and to $1/3$ of that intensity in the case of octupole excitation (the experimental value is $\sim 1/10$). In the case of Ra^{226}, where the probability for α decay to the level 1^- is relatively smaller than in the decay $Th^{228} \to Ra^{224}$, we find values of q_1 and q_3, which are about one half of the former ones.

Excitation of rotational levels with odd parity is also observed in the decay of Am^{241}:

$$(5/2,\ K = 5/2^+) \to (5/2,\ K = 5/2^-, \quad \text{ground state}).$$

It can be assumed that this decay is the result of electric de-excitation of the daughter nucleus by α particles of the principal group:

$$(5/2,\ 5/2^+) \to (5/2,\ 5/2^+, \quad \text{excited state}).$$

The probability of such an electric dipole process was computed by a different method in reference 11. It appeared that one cannot explain the observed intensity of the decay of Am^{241} to the ground state of Np^{237} by electric dipole excitation: this would require a value for the dipole moment which disagrees with the experimental estimates. Formula (31) leads to the same result. On the other hand, the interpretation of the decay of Am^{241} to the ground state of Np^{237} as an electric octupole excitation would require the value (3 to 4) R^3 for the octupole moment. This is close to the value expected for octupole-deformed nuclei (the calculations were based on formula (31); we included the terms with $j' = 5/2$, $7/2$, $9/2$, $l' = 0.2$, and l = 1, 3, 5).

We note that in deformed nuclei only $\frac{1}{8}$ of the observed intensity of α decay to the state 2^+ could be explained by electric quadrupole excitation. The most important effect is the deformation of the nuclear potential.

The above-quoted numerical values of the intensity of the Coulomb excitation of weak lines are true if the Coulomb excitation is the only or the predominant mechanism. In the α decays under consideration this is actually not the case [possible exceptions are the decays of Am^{241} ($\lambda = 3$) and of the even isotopes of Ra ($\lambda = 1$)]. The amplitude of the Coulomb excitation must be added to the amplitude of the nuclear excitation. In the case of vibrational levels we combine the amplitude (8) with the amplitude of the Coulomb excitation (26) and (28), and obtain for the excitation amplitude of the level 2^+:

$$\psi_{\text{Coul.}} + k\psi_{\text{surf. oscill.}}$$

The coefficient k in front of the nuclear excitation amplitude takes account of the circumstance that only part of the total quadrupole momentum can have an effect on the "surface" excitation. Comparing the intensity calculated in this way with the experimental value, we find $k = 0.25$ for Pb^{206} and $k = 0.5$ for Po^{210}. These values of k are close to the values of the coefficients of the quadrupole polarization of the nuclear core as determined by the experimental data on the quadrupole moments of nuclei close to the magic numbers.[17,18]

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20