SOME GENERAL RELATIONS IN STATISTICAL QUANTUM ELECTRODYNAMICS

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A number of general relations associated with the gauge invariance of quantum electrodynamics are obtained.

INTRODUCTION

As is well known, the system of equations for the Green's functions of quantum statistics[1,2] differs both in form and in content from the analogous system of equations for the Green's functions of ordinary field theory.

However, as will be shown below, a number of general relations associated with the gauge invariance of quantum electrodynamics hold not only in ordinary quantum electrodynamics[3] but also in statistical quantum electrodynamics.

It is not difficult to convince oneself of the correctness of the relations we shall obtain if one uses perturbation theory and analyzes the simplest Feynman diagrams for quantum statistics, but in this paper a more complete and rigorous proof of these relations is given by means of the exact equations for the Green's functions of quantum statistics.

1. RELATIONS OF THE WARD TYPE FOR QUANTUM STATISTICS

By using results obtained earlier[1] we can get a system of equations for the Green's functions in the presence of a source of the photon field; in the momentum representation this system can be written in the form (the notation is the same as in reference 1)

$$G^{-1}(p, p') = [-i\gamma_\nu p_\nu - \gamma_4\mu + m]\,\delta(p - p') - ie\gamma_\mu \langle A_\mu(p - p')\rangle + \Sigma(p, p'), \qquad (1)$$

$$[k^2\delta(k - k')\,\delta_{\mu\rho} - \Pi_{\mu\rho}(k, k')]\,D_{\rho\nu}(k, k') = \delta(k - k')\,(\delta_{\mu\nu} - k_\mu k_\nu / k^2), \qquad (2)$$

$$\Pi_{\mu\nu}(k k') = \frac{e^2}{(2\pi)^3\beta}\,\mathrm{Sp}\left\{\sum \int \gamma_\mu G(k + s, s_1)\,\Gamma_\nu(s_1, s_2, k') \times G(s_2, s)\,d^3s_1 d^3s_2 d^3s\right\}, \qquad (3)$$

$$\Sigma(p, p') = \frac{e^2}{(2\pi)^3\beta}\sum\int \gamma_\mu G(p + s, s_1)\,\Gamma_\nu(s_1, p', s_2) \times D(s_2, s)\,d^3s\,d^3s_1 d^3s_2, \qquad (4)$$

$$\Gamma_\mu(p, p', k) = -\frac{\delta G^{-1}(p, p')}{\delta \langle ieA_\mu(k)\rangle} = \gamma_\mu\delta(p - p' - k) - \frac{\delta\Sigma(p, p')}{\delta\langle ie A_\mu(k)\rangle}. \qquad (5)$$

In Eqs. (3) and (4) the summations are over all the fourth components of the momenta; these components are, respectively, $2\pi n/\beta$ for photons and $(2n + 1)\pi/\beta$ for fermions.* In Eq. (2) the term $k_\mu k_\nu / k^2$ must be understood in the sense of the principal value for the analytic continuation. We note that it is often more convenient to work with a different gauge for the potential A; in particular, we can drop this term.

$$\delta(k - k') = \delta(k - k')\,\delta_{(k_4 - k_4')},$$

$$\text{where } \delta_{(k_4 - k_4')} = \begin{cases} 1 & \text{for } k_4 = k_4' \\ 0 & \text{for } k_4 \neq k_4' \end{cases}$$

Using Eqs. (1) — (5) and expanding all quantities in series by perturbation theory, we can convince ourselves that the following relation holds:

$$G^{-1}(p + k, p') - G^{-1}(p, p' - k) = k_\rho \delta G^{-1}(p, p') / \delta e\langle A_\rho(k)\rangle. \qquad (6)$$

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*In the limiting case of zero temperature all the sums over the fourth components go over into integrals. (For $\beta \to \infty$ we get $\frac{1}{\beta}\sum_{p_4} \to \frac{1}{2\pi}\int dp_4$.) In particular, if we further let $\mu \to 0$, we get from (1) – (5) a system of equations equivalent to ordinary quantum electrodynamics, with the following advantages: a) the momentum space has a Euclidean metric, and therefore calculation is considerably simpler than in the usual pseudo-Euclidean space; b) in this system of equations $\mathrm{Im}\,\Pi = \mathrm{Im}\,\Sigma = \mathrm{Im}\,\Gamma = 0$, and thus the number of equations is only half as large as in ordinary quantum electrodynamics. The transition to the quantum theory in the usual representation is accomplished by analytic continuation in the fourth component of the momentum (cf. reference 1), and then all quantities become complex (there is an analogous situation in classical theory, where the dielectric constant for imaginary frequencies becomes a real quantity); c) our Green's functions in the Euclidean parameters do not have poles, and therefore division by D^{-1} and G^{-1} is unique; the poles arise in the analytic continuation of these quantities.

We shall show below that the exact system (1) — (5) (without resort to perturbation theory) also has the relation (6) as one of its consequences.

In fact, according to the left member of (6) we have

$$G^{-1}(p+k,\ p') - G^{-1}(p,\ p'-k) = i\gamma_\rho k_\rho \delta(p+k-p') + \frac{e^2 i}{(2\pi)^3\beta}\sum\!\!\int \gamma_\mu \Big[G(p+k+s,\ s_1)\,\frac{\delta G^{-1}(s_1,\ p')}{\delta e\langle A_\nu(s_2)\rangle} - G(p+s,\ s_1)\,\frac{\delta G^{-1}(s_1,\ p'-k)}{\delta e\langle A_\nu(s_2)\rangle}\Big] D_{\nu\mu}(s_2,\ s)\, d^3s d^3s_1 d^3s_2. \quad (7)$$

On the other hand, we get from (1) — (5) for the right member of Eq. (6)

$$k_\rho \frac{\delta G^{-1}(p,\ p')}{\delta e\langle A_\rho(k)\rangle} = -i\gamma_\rho k_\rho \delta(p+k-p') - \frac{e^2 i}{(2\pi)^3\beta} \times \sum\!\!\int \gamma_\mu \Big[k_\rho \frac{\delta G(p+s,\ s_1)}{\delta e\langle A_\rho(k)\rangle}\,\frac{\delta G^{-1}(s_1,\ p')}{\delta e\langle A_\nu(s_2)\rangle} - G(p+s,\ s_1)\, k_\rho \frac{\delta^2 G^{-1}(s_1,\ p')}{\delta e\langle A_\rho(k)\rangle\,\delta e\langle A_\nu(s_2)\rangle}\Big] \times D_{\nu\mu}(s_2,\ s)\, d^3s d^3s_1 d^3s_2 + \frac{e^2 i}{(2\pi)^3\beta}\sum\!\!\int \gamma_\mu G(p+s,\ s_1)\,\frac{\delta G^{-1}(s_1,\ p')}{\delta e\langle A_\nu(s_2)\rangle} \times k_\rho \frac{\delta D_{\nu\mu}(s_2,\ s)}{\delta e\langle A_\rho(k)\rangle}\, d^3s d^3s_1 d^3s_2 = R. \quad (8)$$

According to (6), the left members of (7) and (8) are equal. We shall show that indeed the right members of (7) and (8) are identically equal.

Consider the right member of (8). Using the law of the conservation of the four-dimensional current, we have (see Sec. 2)

$$k_\mu \frac{\delta D_{\nu\mu}(s_1,\ s)}{\delta e\langle A_\mu(k)\rangle} = 0.$$

According to Eq. (6)

$$k_\rho \frac{\delta^2 G^{-1}(s_1,\ p')}{\delta e\langle A_\rho(k)\rangle\,\delta e\langle A_\nu(s_2)\rangle} = \frac{\delta G^{-1}(s_1+k,\ p')}{\delta e\langle A_\nu(s_2)\rangle} - \frac{\delta G^{-1}(s_1,\ p'-k)}{\delta e\langle A_\nu(s_2)\rangle}. \quad (6a)$$

After substitution of (6) and (6a) in the right member of (8), we get the following expression:

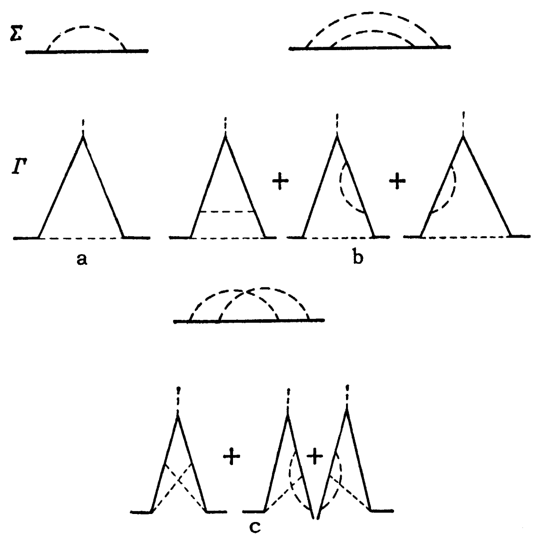
$$R = -i\gamma_\rho k_\rho \delta(p+k-p') - \frac{e^2 i}{(2\pi)^3\beta}\sum\!\!\int \gamma_\mu G(p+s,\ s_3)\,[G^{-1}(s_3+k,\ s_4) - G^{-1}(s_3,\ s_4-k)]\,G(s_4,\ s_1)\,\frac{\delta G^{-1}(s_1,\ p')}{\delta e\langle A_\nu(s_2)\rangle} \times D_{\nu\mu}(s_2,\ s)\, d^3s d^3s_1 d^3s_2 d^3s_3 d^3s_4 + \frac{e^2 i}{(2\pi)^3\beta}\sum\!\!\int \gamma_\mu G(p+s,\ s_1) \times \Big[\frac{\delta G^{-1}(s_1+k,\ p')}{\delta e\langle A_\nu(s_2)\rangle} - \frac{\delta G^{-1}(s_2,\ p'-k)}{\delta e\langle A_\nu(s_2)\rangle}\Big] D_{\nu\mu}(s_2,\ s)\, d^3s_1 d^3s_2 d^3s. \quad (8a)$$

After the reduction of similar terms in (8a) we find that the right member of (8a) agrees identically with the right member of (7). Thus we have shown that the exact equations for the Green's functions in statistical quantum electrodynamics have the relation (6) as one of their consequences.

The relation (6) is equivalent to an infinite series of relations, which are obtained from (6) by successive differentiations with respect to $\langle A\rangle$, after which one sets $\langle A\rangle = 0$. In particular, we get relations of the type of Ward's identity applicable to quantum statistics:

$$G^{-1}(p) - G^{-1}(p+k) = ik_\rho \Gamma_\rho(p,\ p-k,\ k), \qquad \frac{\partial G^{-1}(p)}{\partial p_\nu} = -i \lim_{\delta_\nu \to 0} \Gamma_\nu(p,\ p-\delta_\nu,\ \delta_\nu), \quad (9)$$

where δ_ν is a vector which has only its ν-th component different from zero (see footnote †). An analysis of the general proof of the relation (6) enables us, in particular, to indicate for each Feynman diagram of G^{-1} [more exactly, for $\Sigma(p)$] a set of diagrams of the vertex part Γ that assure the fulfillment of the relations (9). This is done in the figure, where the diagrams (up to e^4) are divided into groups a, b, c, within each of which the relations (9) hold.



Using the fact that the chemical potential μ appears in $G^{-1}(p)$ in combination with $ie\langle A_4(0)\rangle$, we get

$$\frac{\partial G^{-1}(p)}{\partial ie\langle A_4(0)\rangle} = \frac{\partial G^{-1}(p)}{\partial \mu} = -\Gamma_4(p,\ p,\ 0), \quad (10)$$

where $\Gamma_4(p, p, 0)$ must be understood as the limit of $\Gamma_4(p, p-k, k)$ when $k_4 = 0$ and $k \to 0$.

2. SOME RELATIONS FOR THE POLARIZATION OPERATOR

a) It is not hard to convince oneself that all the odd derivatives of the polarization operator with respect to $\langle A\rangle$ are equal to zero; this means that closed loops with an odd number of fermion lines are equal to zero (the analog of Furry's theorem for statistics). This can be seen easily if one takes the current in charge-symmetric form (the proof is similar to that carried out in Appendix II of reference 4).

b) The four-dimensional divergence of the polarization tensor $\Pi_{\mu\nu}$ is equal to zero. This holds also when there is an external source of the photon field, so that $\langle A \rangle \neq 0$. In fact, according to (3) and (6), we have

$$k_\nu \Pi_{\mu\nu}(p, k) = \frac{ie^2}{(2\pi)^3 \beta} \operatorname{Sp} \left\{ \sum \int \gamma_\mu G(p+s, s_1) \left[G^{-1}(s_1, s_2 - k) - G^{-1}(s_1 + k, s_2) \right] G(s_2, s) \, d^3 s \, d^3 s_1 \, d^3 s_2 \right\} = 0. \quad (11)$$

Differentiating Eq. (11) with respect to $\langle A \rangle$ and then setting $\langle A \rangle = 0$, we get an infinite sequence of relations equivalent to the operator conservation law for the total charge.

In ordinary quantum electrodynamics it follows from Eq. (11) with $\langle A \rangle = 0$ that the tensor $\Pi_{\mu\nu}$ has the form

$$\Pi_{\mu\nu} = (k_\mu k_\nu - k^2 \delta_{\mu\nu}) \Pi(k^2). \quad (12)$$

In statistics the situation is different. In this case $\Pi_{\mu\nu}(\mathbf{k})$ depends on two vectors: on the argument vector k and on the velocity vector u of the medium. Therefore it follows from Eq. (11) that in the rest system of the medium [in this system the equations (1) – (5) hold] the tensor $\Pi_{\mu\nu}$ has the form

$$\Pi_{\mu\nu} = \left(\frac{k_\mu k_\nu}{\mathbf{k}^2} - \delta_{\mu\nu} \right) A(\mathbf{k}^2, k_4^2) + \Pi_{44} \frac{k_\mu k_\nu}{\mathbf{k}^2} \frac{k_4^2}{\mathbf{k}^2},$$

$$\Pi_{\mu 4} = \Pi_{4\mu} = -\Pi_{44} k_\mu k_4 / \mathbf{k}^2, \quad \text{where } \nu, \mu = 1, 2, 3. \quad (13)$$

In a coordinate system in which the medium moves with velocity u the tensor $\Pi_{\mu\nu}$ has the following general form†

$$\Pi_{\mu\nu} = \left(\frac{k_\mu k_\nu}{k^2} - \delta_{\mu\nu} \right) A(k^2, (ku)) + \left(\frac{k_\mu k_\nu}{k^2} - \frac{k_\mu k_\nu}{(ku)} - \frac{k_\nu u_\mu}{(ku)} + \frac{u_\nu u_\mu k^2}{(ku)^2} \right) B(k^2 (ku)), \quad (14)$$

where $\nu, \mu = 1, 2, 3, 4$.

In conclusion we remark that by the method of reference 3 one can find in general form the behavior of all the Green's functions under a gauge transformation.

[1] E. S. Fradkin, Dokl. Akad. Nauk SSSR **125**, 311 (1959) Soviet Phys.-Doklady **4**, 347 (1959); JETP **36**, 1286 (1959), Soviet Phys. JETP **9**, 912 (1959).

[2] Abrikosov, Gor'kov, and Dzyaloshinskiĭ, JETP **36**, 900 (1959), Soviet Phys. JETP **9**, 636 (1959).

[3] E. S. Fradkin, JETP **29**, 259 (1955), Soviet Phys. JETP **2**, 361 (1956).

[4] E. S. Fradkin, JETP **29**, 121 (1955), Soviet Phys. JETP **2**, 148 (1956).

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†The fact that in quantum statistics all quantities have the additional dependence on the timelike vector u of the velocity of the medium has the consequence that for $\mathbf{k} \to 0$ we in general get different results depending on the way k goes to zero ($|\mathbf{k}|^2 > k_4^2$ or $|\mathbf{k}|^2 < k_4^2$). For example, from the law of the conservation of the total charge of the system it follows that $\Pi_{44}(k_4, 0) = 0$, whereas by means of the relation (10) one can show that

$$\lim_{k \to 0} \Pi_{44}(0, \mathbf{k}) = \Pi_{44}(0, 0) = e^2 (\partial / \partial \mu) \{ \operatorname{Sp} \gamma_4 G(xx) \} = -e^2 \partial \rho / \partial \mu,$$

where ρ is the charge density in the u_ν space. We have thus also shown that Eq. (3.11) of reference 1 for the Debye radius remains valid when all quantum and relativistic corrections are taken into account.