# ON THE STRUCTURE OF THE PHOTON GREEN'S FUNCTION

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It is shown that the so-called Redmond procedure is not unambiguous. This conclusion is not changed when the requirements of renormalization invariance are taken into account.

ATTENTION has recently been called to the possibility of using dispersion relations (d.r.) for the elimination of the fictitious pole of the boson Green's function in quantum field theory.[1] A simple analysis (see Sec. 1) shows, however, that this procedure does not possess the necessary property of uniqueness. As can be seen from the results of Sec. 2, this conclusion remains valid also when one takes into account the requirements of renormalization invariance.

## 1. ON THE AMBIGUITY OF THE REDMOND PROCEDURE

We confine ourselves to a consideration of quantum electrodynamics. It is not hard to show that if the Källén-Lehmann d.r.

$$d(z) = 1 + \frac{z}{\pi}\int\limits_0^\infty \frac{\operatorname{Im} d(\zeta)\, d\zeta}{\zeta(\zeta - z - i\varepsilon)}, \tag{1}$$

holds, where

$$(k_\mu k_\nu - z\delta_{\mu\nu})\, d(z) = -D_{\mu\nu}(k)\, z^2, \quad z \equiv k_0^2 - \mathbf{k}^2,$$

then an analogous dispersion relation also holds for the polarization operator

$$\Pi(z) = \frac{z}{\pi}\int\limits_0^\infty \frac{\operatorname{Im}\Pi(\zeta)\, d\zeta}{\zeta(\zeta - z - i\varepsilon)},$$

$$d^{-1}(z) = 1 + \Pi(z). \tag{2}$$

For the proof it suffices to note that, because of the condition $\operatorname{Im} d(z) \le 0$, the function $d(z)$ has no zeroes either in the complex $z$ plane or on the negative semiaxis $\operatorname{Re} z < 0$.

An important point is that in general the converse proposition is not true: for the d.r. (2) for $\Pi(z)$ to have as a consequence the d.r. (1), we have as the necessary and sufficient condition

$$\frac{1}{\pi}\int\limits_0^\infty \frac{\operatorname{Im}\Pi(\zeta)}{\zeta}\, d\zeta \leqslant 1. \tag{3}$$

In fact, as can be seen from Eq. (2), and also from the condition $\operatorname{Im}\Pi = -\operatorname{Im} d/|d|^2 \ge 0$, the inequality (3) assures the absence of poles of the function $d(z)$ outside the semiaxis $\operatorname{Re} z > 0$.

It is important to emphasize that the condition (3) is entirely identical with the well known inequality (8) of the paper of Lehmann, Symanzik, and Zimmermann.[2] The function $F(z)$ contained in that inequality is connected with $\Pi(z)$ by the relation $F(z) = 2z \operatorname{Im}\Pi(z)$. Though it does satisfy the d.r. (2), the asymptotic expression for $d(z)$ obtained in the "three-gamma" approximation (cf., e.g., reference 3) is nevertheless in contradiction with the condition (3) $[\operatorname{Im}\Pi(z) = \pi\alpha,\ \alpha = e^2/3\pi]$. Therefore a fictitious pole appears in $d(z)$ and the d.r. (1) is violated.

The recently proposed procedure,[1] having as its purpose the removal of this difficulty, consists in using a summation of the "main" terms of the perturbation-theory series to calculate only the quantity $\operatorname{Im} d(z)$. The function $d(z)$ itself is reconstructed by means of the relation (1). By what has been said, the d.r. (2) and the condition (3) then hold, as can also be verified directly. This procedure, however, is not unambiguous. In fact, any function $\operatorname{Im}\Pi$ that satisfies (3) and goes over for $\alpha \to 0$ into the corresponding expression of perturbation theory (to a given order in $\alpha$) can be used to reconstruct the photon Green's function by means of Eq. (2). The function so obtained will obey the d.r. (1) and agree with perturbation theory.

As an example let us consider the following expression:

$$\operatorname{Im}\Pi(z) = \pi\alpha/(1 + z/z_0), \tag{4}$$

where by the condition (3) $z_0 \le m^2 \exp(1/\alpha)$. Simple manipulations give

$$d^{-1}(z) = 1 - \frac{\alpha}{z + z_0}\left\{z_0 \ln\left(1 - \frac{z}{m^2}\right) + z \ln \frac{z_0}{m^2}\right\}; \quad z_0 \gg m^2. \tag{5}$$

For correspondence with perturbation theory it is enough to require that as $\alpha$ decreases $z_0$ increase faster than any finite power of $\alpha^{-1}$. If, in particular, we set $z_0 = m^2 \exp(\alpha^{-1/2})$, we arrive at an expression that has no resonance properties and does not lead to strong coupling $(d^{-1}(\infty) = 1 - \alpha^{1/2})$.

In the removal of this ambiguity a large part could be played by general requirements of causality and unitarity of the theory. In this connection it is important to emphasize that the conditions expressed by the d.r. (1) are only necessary, but by no means sufficient to secure causality and unitarity.

A treatment of this circle of questions is impossible, however, in the language of one-particle Green's functions; it is necessary to bring into the argument Green's functions of higher orders, in terms of which Im $\Pi(z)$ is expressed. It cannot be excluded that the expression obtained for the Green's function in reference 1, which has its real part nonanalytic in $\alpha$ and its imaginary part analytic in $\alpha$, may be in contradiction with the conditions of unitarity and causality, which closely relate the real and imaginary parts of matrix elements.

## 2. ON THE RENORMALIZATION INVARIANCE OF THE PHOTON GREEN'S FUNCTION

The condition of renormalization invariance (r.i.) of the photon Green's function $d(z, \lambda, \alpha_\lambda)$ is of the form[4]

$$\alpha_\lambda d(z, \lambda, \alpha_\lambda) = \alpha_{\lambda'} d(z, \lambda', \alpha_{\lambda'}), \tag{6}$$

where $\lambda$ is the square of the normalized momentum $[d(\lambda, \lambda, \alpha_\lambda) = 1]$, $z = k_0^2 - |\mathbf{k}|^2$, and $\alpha_\lambda = e_\lambda^2/3\pi$ is the corresponding coupling constant.

Starting with Eq. (6) and assuming that for $z \gg m^2$ the function d depends only on $z/\lambda$ and $\alpha_\lambda$ (the perturbation-theory series has this property), it has been proved[5,4] that the renormalized d function, i.e., the function corresponding to $\lambda = 0$, must have the form

$$d(z, \alpha_0) = \alpha_0^{-1} F(\ln(z/m^2) + \varphi(\alpha_0)). \tag{7}$$

Here F and $\varphi$ are mutually inverse functions. From this a number of important conclusions have been drawn: that the shape of the effective charge distribution of the electron is independent of $\alpha_0$, that the value of the bare charge is independent of $\alpha_0$ and strong coupling inevitably appears (in the case of a finite charge renormalization), and so on. We would like to emphasize that these conclusions do not have the force of inevitability, being connected not only with the requirement of r.i., but also with definite assumptions about the structure of the Green's function.

In themselves these propositions are by no means obligatory (particularly in the case of finite charge renormalization). Thus the use of the dispersion relations in finding the photon Green's function[1,6] leads to the appearance of terms nonanalytic in $\alpha$, which change decidedly the behavior of the d function in the high-momentum region. Although even in this case the perturbation-theory series depends for $z \gg m^2$ on the combination $z/\lambda$ only, the exact expression $d(z, \lambda, \alpha_\lambda)$ does not have this property.

It is therefore important to ascertain whether the requirement of r.i. by itself imposes any limitations on the structure of the d function. For this purpose we turn to the general solution of the functional equation (6). It is easily verified that this equation can be written in the form

$$\alpha_\lambda d(z, \lambda, \alpha_\lambda) = \alpha_0(\alpha_\lambda, \lambda)\, d(z, \alpha_0(\alpha_\lambda, \lambda)), \tag{8}$$

where the renormalization-invariant (unchanged by a change of $\lambda$) function $\alpha_0(\alpha_\lambda, \lambda)$ is given by the relation*

$$\alpha_0^{-1} = \alpha_\lambda^{-1} + \ln(-\lambda/m^2). \tag{9}$$

It is obvious that the function $d(z, \alpha_0)$ that appears here is identical with the renormalized Green's function.

Thus from any given expression for the renormalized Green's function one can reconstruct a renormalization-invariant expression $d(z, \lambda, \alpha_\lambda)$ that for $\lambda \to 0$ goes over into the original expression. According to Eqs. (8) and (9) this task reduces simply to the introduction of the factor $\alpha_0/\alpha_\lambda$ and the replacement of $\alpha_0$ by an invariant combination of $\alpha_\lambda$ and $\lambda$.†

In particular we get a renormalization-invariant expression for the Green's function (5) considered above. Confining ourselves for simplicity to the region $m^2 \ll |\lambda| \ll z_0$, we have from Eq. (9)

$$\alpha_0^{-1} = \alpha_\lambda^{-1} + \ln(-\lambda/m^2). \tag{10}$$

Using Eq. (8), we get ($z \gg m^2$ throughout)

$$d^{-1}(z, \lambda, \alpha_\lambda) = 1 - \frac{\alpha_\lambda}{z + z_0}\left(z_0 \ln\left(\frac{z}{\lambda}\right) + z \ln\left(-\frac{z_0}{\lambda}\right)\right). \tag{11}$$

From here on we must express $z_0$ and $\alpha_0$ in terms of $\alpha_\lambda$ and $\lambda$ by means of Eq. (10). In particular, setting $z_0 = m^2 \exp(1/\alpha_0^{1/2})$, we get

$$d^{-1}(z, \lambda, \alpha_\lambda) = \frac{1 - \alpha_\lambda \ln(z/\lambda) + (\alpha_\lambda/\alpha_0)(1 - \sqrt{\alpha_0})(z/z_0)}{1 + z/z_0}. \tag{12}$$

*The relations (8) and (9) are in complete agreement with the results of Ovsyannikov.[7]

†In the general case the requirement of r. i. imposes one relation on the arguments of $d(z, \lambda, \alpha_\lambda)$. Therefore the function $d(z, \alpha_0)$ is, generally speaking, an arbitrary function of two arguments.

This expression is in contradiction with Eq. (7), and at the same time satisfies the requirement of r.i. and the Källén-Lehmann equation and goes over into the perturbation-theory series for $\alpha \rightarrow 0$. Only for the special choice $z_0 = Cm^2 \exp(1/\alpha_0)$, $(C \leq 1)$ does one get an expression consistent with Eq. (7).

Summarizing, we can say that the requirement of r.i. does not in itself impose any restrictions on the renormalized Green's function. Even the further requirement that for $\alpha \rightarrow 0$ the function must go over into the perturbation-theory series does not lead with necessity to the relation (7).

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