of corresponding constants of the motion. For example, for a system with central symmetry there is conservation of the components of the angular momentum M, which generate the rotation group. Gauge transformations with a constant phase are generated by the charge operator $\hat{Q}$. Such constants of the motion do not exist, however, for the general group of gauge transformations,

$$\psi \rightarrow e^{i\Lambda}\psi, \qquad A_\mu \rightarrow A_\mu - \partial\Lambda/\partial x_\mu . \qquad (1)$$

We shall show that by the introduction of additional variables into the Hamiltonian one can construct an infinite set of constants of the motion, which generate the transformations (1), and thus can include the gauge transformations in the general scheme of canonical transformations.

Let us write the Hamiltonian of quantum electrodynamics in the form

$$H = \sum_{p,\sigma} E_p (a^+_{p\sigma} a_{p\sigma} + b^+_{p\sigma} b_{p\sigma}) + \sum_k \omega\, (c^+_{1k} c_{1k} + c^+_{2k} c_{2k} - c^+_{3k} c_{4k} - c^+_{4k} c_{3k}) + \sum_k \sum_{\lambda=1}^{4} \frac{e}{\sqrt{2\omega}} \{c_{\lambda k} (e^\lambda j^+_k) + c^+_{\lambda k} (e^\lambda j_k)\}; \qquad (2)$$

$$j_k \equiv (\rho_k, \mathbf{j}_k) = \int \bar{\psi} \gamma_\mu \psi e^{-i\mathbf{kx}} d^3x, \qquad (3)$$

where a and b are the operators of the electron-positron field, and c are the operators of the electromagnetic field; j_k are the Fourier components of the current vector. The integral in Eq. (3) is taken over unit volume. The polarization vectors e^λ are chosen in the following way: e^1 and e^2 are unit space vectors perpendicular to **k**, and

$$e^3 = (1, \mathbf{k}/\omega)/\sqrt{2}, \qquad e^4 = (1, -\mathbf{k}/\omega)/\sqrt{2}.$$

For the photons one introduces an indefinite metric. In accordance with this, the operators c satisfy the commutation relations

$$[c^+_{3k}, c_{4k'}] = [c^+_{4k}, c_{3k'}] = \delta_{kk'}, \qquad [c^+_{3k}, c_{3k'}] = [c^+_{4k}, c_{4k'}] = 0$$

(the remaining commutation relations are the usual ones). We now introduce "supplementary" variables α_k and β_k, which satisfy the commutation relations

$$[\alpha^+_k, \beta_{k'}] = [\beta^+_k, \alpha_{k'}] = \delta_{kk'}, \quad [\alpha^+_k, \alpha_{k'}] = [\beta^+_k, \beta_{k'}] = 0 \quad (4)$$

and commute with all the other quantities, and add to the Hamiltonian (2) the quantity

$$H_{\alpha\beta} = -\sum_k \omega\, (\alpha^+_k \beta_k + \beta^+_k \alpha_k).$$

It is easy to verify that the "total" Hamiltonian $H + H_{\alpha\beta}$ commutes with the quantities

$$R_k = \alpha^+_k (e\rho_k/2\omega^{3/2} - c_{4k}), \qquad R^+_k = \alpha_k (e\rho^+_k/2\omega^{3/2} - c^+_{4k}). \quad (5)$$

In terms of the operators (5) the gauge transformations (1) can be expressed in the form of a unitary operator

$$U_\Lambda = \exp\left\{i \sum_k \left(\lambda_k R_k + \lambda^+_k R^+_k\right)\right\}, \qquad (6)$$

where λ_k are arbitrary numbers. The function $\Lambda(x)$ for the transformation (6) is

$$\Lambda(x) = \sum_k \frac{1}{2\omega^{3/2}} (\lambda_k \alpha_k \exp\{i(\mathbf{kx} - \omega t)\} + \lambda^+_k \alpha^+_k \exp\{-i(\mathbf{kx} - \omega t)\}),$$

and in virtue of the relations (4) $\Lambda(x)$ can be regarded as a numerical function.

In our representation the supplementary condition $(\partial A_\mu/\partial x_\mu)\Phi = 0$ can be written in the form

$$(c_{4k} - e\rho_k/2\omega^{3/2})\Phi = 0, \qquad (c^+_{4k} - e\rho^+_k/2\omega^{3/2})\Phi = 0. \qquad (7)$$

Comparing Eqs. (7) and (5), we see that for the allowed states the quantities R_k and $R^\dagger_k$ are equal to zero:

$$R_k \Phi = R^+_k \Phi = 0. \qquad (8)$$

Obviously the conditions (8) single out the Maxwellian electrodynamics from among all the theories described by the Hamiltonian (2).

The variables α and β are of the nature of two additional components of the electromagnetic field. Since these components do not interact with charges, this scheme is entirely equivalent to the usual electrodynamics.

Translated by W. H. Furry
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THE RELATIVISTIC PHOTOEFFECT IN THE L SHELL

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Submitted to JETP editor July 14, 1959

J. Exptl. Theoret. Phys. (U.S.S.R.) **38**, 309-311 (January, 1960)

THE problem of a theory of the nonrelativistic photoeffect in the L shell was solved a long time ago.[1,2] The relativistic aspects of this problem, however, have only been remarked upon. In view of the successful development of β-ray spectrom-

$h\nu/mc^2$	0	1/8	1/4	1/2	1	4	∞
$\sigma^{nr}_{L_I}/\sigma_{L_I}$	1	0.95	0.91	0.79	0.54	0.07	0
$\sigma^{nr}_{L_{II}+L_{III}}/\sigma_{L_{II}+L_{III}}$	1	0.72	0.61	0.38	0.16	0.007	0
$\sigma_{L_{II}}/\sigma_{L_{III}}$	0.500	0.535	0.546	0.558	0.531	0.366	0.281

etry in recent years it becomes now desirable to make a more careful analysis of the contribution of the L shell to the photoelectric absorption at high energies.[3,4] As it is impossible to obtain exact analytic expressions for the relativistic cross sections, we attempt to find their approximate form for the light elements.* Thus we determine the cross sections for the L_I subshell which are accurate up to and including the first order of αZ; in the case of the L_{II} and L_{III} subshells we only obtained approximations of zeroth order in αZ.

As usual, the calculation is based on the central Coulomb field approximation for the different electrons, with Z changed to $Z_S = Z - 4.5$ (see reference 1, §69 α). The spinor corresponding to the final state of the electron is treated in the Born approximation. The integration in the matrix elements is performed in momentum space. The calculations are analogous to those in the case of the K shell, described earlier by the author.[5] To find the differential cross section for one of the L subshells, we must sum the contributions from all electrons of this subshell, taking into account the two possible spin orientations in the final state. Mathematically this involves very laborious calculations of traces.

As a result we obtain for the differential cross sections the expressions

$$d\sigma_{L_I} = \frac{1}{8}(d\sigma_K)_s, \tag{1}$$

$$\begin{aligned} d\sigma_{L_{II}} = {} & \frac{1}{24}\lambda_0^2\alpha^8 Z_s^7(\gamma^2-1)^{1/2}\gamma^{-4}(\gamma-1)^{-5} \\ & \times\Big\{\frac{1}{4}(3\gamma+1)\Theta^{-4} - \frac{1}{16}\gamma(9\gamma^2+30\gamma-7)\Theta^{-3} \\ & + \frac{1}{8}\gamma^2(\gamma^3+6\gamma^2+11\gamma-2)\Theta^{-2} \\ & - \frac{1}{16}\gamma^3(\gamma-1)(\gamma+7)\Theta^{-1} + \sin^2\theta\cos^2\varphi(\gamma+1)\gamma^{-1}\big[2\Theta^{-5} \\ & - 2\gamma\,\Theta^{-4}\frac{1}{8}\gamma^2(3\gamma+1)(\gamma-1)\Theta^{-3}\big]\Big\}\,d\omega, \end{aligned} \tag{2}$$

$$\begin{aligned} d\sigma_{L_{III}} = {} & \frac{1}{12}\lambda_0^2\alpha^8 Z_s^7(\gamma^2-1)^{1/2}\gamma^{-4}(\gamma-1)^{-5}\Big\{-\frac{1}{4}(3\gamma-1)\Theta^{-4} \\ & + \frac{1}{2}\gamma(3\gamma^2-1)\Theta^{-3} + \frac{1}{2}\gamma^2(\gamma^3-2\gamma^2+2\gamma+1)\Theta^{-2} \\ & - \frac{1}{4}\gamma^3(\gamma-2)(\gamma-1)\Theta^{-1} + \sin^2\theta\cos^2\varphi(\gamma+1)\gamma^{-1}[2\Theta^{-5} \\ & - \gamma(3\gamma-1)\Theta^{-4} + \gamma^2(\gamma-1)\Theta^{-3}]\Big\}\,d\omega, \end{aligned} \tag{3}$$

where $(d\sigma_K)_S$ is given by formula (92) of reference 5, with Z replaced by Z_S, the angles θ and φ are determined in reference 5, and,

$$\gamma = 1/(1-\beta^2)^{1/2}, \quad \Theta = 1 - \beta\cos\theta, \quad \lambda_1 = \hbar/mc.$$

The corresponding total cross sections are

$$\sigma_{L_I} = \frac{1}{8}(\sigma_K)_s, \tag{4}$$

$$\begin{aligned} \sigma_{L_{II}} = {} & \frac{1}{256}\alpha^6 Z_s^7\varphi_0\frac{(\gamma^2-1)^{1/2}}{(\gamma-1)^5} \\ & \times\Big\{9\gamma^3 - 5\gamma^2 + 24\gamma - 16 - \frac{\gamma^2+3\gamma-8}{(\gamma^2-1)^{1/2}}\ln[\gamma+(\gamma^2-1)^{1/2}]\Big\}, \end{aligned} \tag{5}$$

$$\begin{aligned} \sigma_{L_{III}} = {} & \frac{1}{32}\alpha^6 Z_s^7\varphi_0\frac{(\gamma^2-1)^{1/2}}{(\gamma-1)^5} \\ & \Big\{4\gamma^3 - 6\gamma^2 + 5\gamma + 3 - \frac{\gamma^2-3\gamma+4}{(\gamma^2-1)^{1/2}}\ln[\gamma+(\gamma^2-1)^{1/2}]\Big\}, \end{aligned} \tag{6}$$

where σ_K is given by formula (98) of reference 5. The formulas for the cross sections are valid for light nuclei, since we must fulfill the condition $(\pi\alpha Z_S/\beta)^2 \ll 1$ for the L_I subshell and the condition $\pi\alpha Z_S/\beta \ll 1$ for the L_{II} and L_{III} subshells.

In the extreme relativistic limit $\beta \to 1$ we easily find that the total cross sections (4) to (6) are proportional to $mc^2/h\nu$. In the limiting case of small energies we can show by neglecting terms of order β^2 that the angular distributions given by formulas (1) to (3) go over into the corresponding nonrelativistic expressions of Schur (see reference 1, §72 β). The value of the ratio $(d\sigma_{L_{II}}/d\sigma_{L_{III}})_{nr} = 1/2$ is equal to the ratio of the number of electrons in these subshells.

Formulas (4) to (6) predict a slower decrease of the total cross sections with increasing photon energy than the nonrelativistic formulas. In the case of the light elements this can be seen from the table, in which we list the ratios of the relativistic (σ) over the nonrelativistic (σ^{nr}) cross sections (to lowest order in αZ_S). There we also give the values of the relativistic ratio $\sigma_{L_{II}}/\sigma_{L_{III}}$. It can be assumed that the error associated with the ratios given in the table is of order $(\alpha Z_S)^2$.

In conclusion the author expresses his grati-

tude to Acad. S. Titeica for his interest in this work and to Prof. R. H. Pratt for clarifying comments on a number of questions touched upon in this paper.

*Recently Pratt calculated the total cross section up to terms of order αZ in the extreme relativistic case.

[1] H. A. Bethe and E. Salpeter, Handbuch der Physik **35**, Part I, Berlin (1957).

[2] H. Hall, Revs. Modern Phys. **8**, 358 (1936).

[3] E. P. Grigor'ev and A. V. Zolotavin, JETP **36**, 393 (1959), Soviet Phys. JETP **9**, 272 (1959).

[4] Novakov, Hultberg, and Andersson, Arkiv Fysik **13**, 117 (1958).

[5] M. Gavrila, Phys. Rev. **113**, 514 (1959).

Translated by R. Lipperheide
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ERRATA TO VOLUME 10

page	reads	should read
Article by A. S. Khaĭkin		
1044, title	. . . resonance in lead	. . . resonance in tin
6th line of article	~ 1000 oe	~ 1 oe
Article by V. L. Lyuboshitz		
1223, Eq. (13), second line	$\ldots-\mathrm{Sp}_{1,2}\,\mathscr{E}(e_1)$	$\ldots-\mathrm{Sp}_{1,2}\,\mathscr{E}(e_2)\ldots$
1226, Eq. (26), 12th line	$\cdots\{(p+q, p$	$\cdots\{(p+q, p)-(p+q, n)\cdot\cdot(pn)\}$
1227, Eqs. (38), (41), (41a) numerators and denominators	(p^2-q)	$(p^2-q^2)^2$
1228, top line	$m_2=\frac{q_1-p_1}{q_1-p_1}$	$m_2=[m_3\, m_1]$

ERRATA TO VOLUME 12

page	reads	should read
Article by Dzhelepov et al.		
205, figure caption	54	5.4
Article by M. Gavrila		
225, Eq. (2), last line	$-2\gamma\Theta^{-4}\,1/8$	$-2\gamma\Theta^{-4}-1/8$
Article by Dolgov-Savel'ev et al.		
291, caption of Fig. 5, 4th line	$p_0 = 50 \times 10^{-4}$ mm Hg	$p_0 = 5 \times 10^{-4}$ mm Hg.
Article by Belov et al.		
396, Eq. (24) second line	$\ldots-(4-2\eta)\,\sigma_1+\ldots$	$\ldots+(4-2\eta)\,\sigma_1+\ldots$
396, 17th line (r) from top	. . less than 0.7	. . less than 0.07
Article by Kovrizhnykh and Rukhadze		
615, 1st line after Eq. (1)	$\omega_{0e}^2 = 2\pi e^2 n_e/m_e$,	$\omega_{0e}^2 = 4\pi e^2 n_e/m_e$,
Article by Belyaev et al.		
686, Eq. (1), 4th line	$\ldots b_{\rho_2 m_2}(s_2')+\cdots$	$\ldots b_{\rho_1 m_1}(s_1')+\cdots$
Article by Zinov and Korenchenko		
798, Table X, heading of last column	$\sigma_{\pi^-\to\pi^+}=$	$\sigma_{\pi^-\to\pi^-}=$
Article by V. M. Shekhter		
967, 3d line after Eq. (3)	$\varepsilon\equiv 2m_pE+m_p^2$	$\varepsilon\equiv(2m_pE+m_p^2)^{1/2}$
967, Eq. (5), line 2	$+(B_V^2+B_A^2)\ldots$	$+(B_V^2+B_A^2)\,Q\ldots$
968, Eq. (7)	$\ldots(C_V^2+C_A^2).$	$\ldots C_V^2+C_A^2-Q^2(B_V^2+B_A^2).$
968, line after Eq. (7)	for $C_V^2+C_A^2\equiv\ldots$	for $C_V^2+C_A^2-Q^2(B_V^2+B_A^2)\equiv\ldots$
Article by Dovzhenko et al.		
983, 11th line (r)	$\gamma=1.8\pm$	$\gamma=1.8\pm0.2$
Article by Zinov et al.		
1021, Table XI, col. 4	−1,22	1,22
Article by V. I. Ritus		
1079, line 27 (1)	$-\Lambda_{\pm}(t)$,	$\Lambda_{\pm}(t)$,
1079, first line after Eq. (33)	$\frac{1}{2}(1+\beta)$.	$\frac{1}{2}(1\pm\beta)$.
1079, 3d line (1) from bottom	$\ldots\mathfrak{N}(q'p;\ pq')\ldots$	$\ldots\mathfrak{N}(p'q;\ pq')\ldots$
Article by R. V. Polovin		
1119, Eq. (8.2), fourth line	$U_{0x}u_xg(\gamma)-[\gamma\cdots$	$-U_{0x}u_xg(\gamma)\,[\gamma\cdots$
1119, Eq. (8.3)	$\cdots\mathrm{sign}\ u.$	$\cdots\mathrm{sign}\ u_g.$
Article by V. P. Silin		
1138, Eq. (18)	$\cdots+\frac{4}{5}c^2k^2$	$\cdots+\frac{6}{5}c^2k^2.$