NUCLEON-NUCLEON SCATTERING IN THE TWO-MESON APPROXIMATION FOR LARGE ORBITAL ANGULAR MOMENTA

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The method previously developed by the authors is used to calculate the nucleon-nucleon scattering amplitude in the two-meson approximation for large orbital angular momenta. Specific calculations are carried out for the singlet amplitude in the non-relativistic approximation for orbital angular momenta which are not very large, $1 \ll l \ll 4m^2/\mu^2$ (m is the nucleon mass, μ the meson mass). The results obtained indicate that the F and G phase shifts for nucleon energies $E_{lab} \lesssim 200$ Mev are given to good accuracy by the one-meson approximation. This conclusion may prove important in the phase shift analysis of nucleon-nucleon scattering.

1. CALCULATION OF THE TWO-MESON AMPLITUDE

In a previous paper[1*] it was shown that, in order to calculate the nucleon-nucleon scattering amplitude in the two-meson approximation for large orbital angular momenta, it is necessary to know the meson-nucleon scattering amplitude $f_{\alpha\beta}$. The latter amplitude, according to formulas (2.25, I), (3.1, I), (3.6, I), (3.20, I), is expressed as follows:

$$f_{\alpha\beta}(p_1,\ p_1+q;\ k+q/2,\qquad k-q/2) = f_{p\alpha\beta} + \tilde{f}'_{\alpha\beta} + \tilde{f}''_{\alpha\beta},$$

$$f_{p\alpha\beta} = -g^2\tau_\alpha\tau_\beta \frac{\hat{k}}{(k+q/2)^2 + 2p_1(k+q/2)} + g^2\tau_\beta\tau_\alpha \frac{\hat{k}}{(k-q/2)^2 - 2p_1(k-q/2)},$$

$$\tilde{f}'_{\alpha\beta} = \frac{g^2}{m}\alpha\delta_{\alpha\beta},$$

$$\tilde{f}''_{\alpha\beta} = i\frac{g^2}{\mu^2}\varepsilon_{\alpha\beta\gamma}\tau_\gamma\left(\beta_1\hat{k} + \beta_2\frac{p_1k}{m}\right), \qquad (1.1)$$

where $\alpha = 1.2$, $\beta_1 = 0.025$, and $\beta_2 = -0.029$.

We insert (1.1) into (2.29, I). Omitting the term quadratic in $\tilde{f}''_{\alpha\beta}$ [it would give a contribution to the scattering phase shift containing an extra factor $1/L$, where L is defined by (2.16, I)], we obtain

$$M^2(q^2) = \frac{g^4}{\pi m}\Big[\frac{3\alpha^2}{4}B_0 - \frac{3\alpha}{2}B_1 + \tau_\alpha^{(1)}\tau_\alpha^{(2)}B_2 + (3 + 2\tau_\alpha^{(1)}\tau_\alpha^{(2)})B_3 + (3 - 2\tau_\alpha^{(1)}\tau_\alpha^{(2)})B_4\Big], \qquad (1.2)$$

where

$$B_0 = \frac{2}{i}\int \Delta(k,\ q)\, d^4k\, (\bar{u}_{p_1+q}u_{p_1})(\bar{u}_{p_2-q}u_{p_2}),$$

$$B_1 = \frac{2m}{i}\int\Big[\frac{(\bar{u}_{p_1+q}|\hat{k}|u_{p_1})(\bar{u}_{p_2-q}u_{p_2})}{(k+q/2)^2 + 2p_1(k+q/2)} + \frac{(\bar{u}_{p_2-q}|\hat{k}|u_{p_2})(\bar{u}_{p_1+q}u_{p_1})}{(k-q/2)^2 + 2p_2(k-q/2)}\Big]\Delta(k,q)\,d^4k,$$

$$B_2 = -\frac{2m^2}{i\mu^2}\int\Big[\frac{(\bar{u}_{p_1+q}|\beta_1\hat{k} + \beta_2p_1k/m|u_{p_1})(\bar{u}_{p_2-q}|\hat{k}|u_{p_2})}{(k-q/2)^2 + 2p_2(k-q/2)} + \frac{(\bar{u}_{p_1+q}|\hat{k}|u_{p_1})(\bar{u}_{p_2-q}|\beta_1\hat{k} + \beta_2p_2k/m|u_{p_2})}{(k+q/2)^2 + 2p_1(k+q/2)}\Big]\Delta(k,q)\,d^4k,$$

$$B_3 = \frac{m^2}{i}\int\frac{(\bar{u}_{p_1+q}|\hat{k}|u_{p_1})(\bar{u}_{p_2-q}|\hat{k}|u_{p_2})\Delta(k,q)}{[(k+q/2)^2 + 2p_1(k+q/2)][(k-q/2)^2 + 2p_2(k-q/2)]}d^4k,$$

$$B_4 = -\frac{m^2}{i} \times \int\frac{(\bar{u}_{p_1+q}|\hat{k}|u_{p_1})(\bar{u}_{p_2-q}|\hat{k}|u_{p_2})\Delta(k,q)}{[(k+q/2)^2 + 2p_1(k+q/2)][(k+q/2)^2 - 2p_2(k+q/2)]}d^4k,$$

$$\Delta(k,\ q) = [(k-q/2)^2 - \mu^2]^{-1}\ [(k+q/2)^2 - \mu^2]^{-1}.$$

Introducing, as usual, the Feynman parameter x, we express B_0 in the form

$$B_0 = \frac{2w}{i}\int_0^1 dx \int d^4k\, [k^2 - \mu^2 + q^2/4 - qk(1-2x)]^{-2}, \qquad (1.3)$$

$$w = (\bar{u}_{p_1+q}u_{p_1})(\bar{u}_{p_2-q}u_{p_2}). \qquad (1.4)$$

The integral over d^4k in (1.3) extends to infinity, but the high momentum contribution appears as an additive constant, independent of q^2, and it need not be taken into account. We separate the finite term, differentiating (1.3) with respect to μ^2, then integrating over d^4k and then integrating back again with respect to μ^2. Integrating, finally, over dx we obtain the singular part of the integral in the form

$$B_0 = -w\frac{s}{\sqrt{1-s^2}}\tan^{-1}\frac{\sqrt{1-s^2}}{s}, \qquad (1.5)$$

*This paper will be referred to as I.

$$s = \sqrt{1 - q^2/4\mu^2}. \qquad (1.6)$$

For the square root (1.6) we choose the branch which is positive in the physical region. In this case, on the upper side of the cut, $s = -i\,|s|$.

We restrict the discussion below to the singlet amplitude in the nonrelativistic approximation and for this case only we shall calculate the matrix elements of the type (1.4). Expanding w in powers of p^2/m^2 and setting $\cos\theta = 1 + 2\xi^2$, we obtain, to an accuracy including up to second-order terms*

$$w = 1 + (p^2/2m^2)(1 - \cos\theta) = 1 - \varepsilon^2; \quad \varepsilon = \mu/m. \qquad (1.7)$$

Now expanding (1.5) in powers of s, and retaining the leading term, we obtain finally,

$$B_0 = -\pi(1 - \varepsilon^2)s/2. \qquad (1.8)$$

For the integration in B_1, we introduce two Feynman parameters x_1 and x_2 (taking into account that $2p_1q = -q^2$, $2p_2q = q^2$):

$$B_1 = \frac{4m}{i}\int_0^1 dx_2 \int_0^{1-x_2} dx_1 \int d^4k \left\{\frac{(\bar{u}_{p_1+q}\,|\,\hat{k}\,|\,u_{p_1})(\bar{u}_{p_2-q}\,u_{p_2})}{(k^2 - 2\bar{p}k - \Delta)^3} + \ldots\right\},$$

$$\bar{p} = -q/2 + qx_1 - p_1x_2, \quad \Delta = -q^2/4 - p_1qx_2 + \mu^2(1 - x_2), \qquad (1.9)$$

the dots indicating a similar term with p_1 replaced by p_2.

Integrating over d^4k, we obtain

$$B_1 = w\int_0^1 dx_2 \int_0^{1-x_2} dx_1 \frac{x_2}{b^2}, \quad b^2 = x_2^2 + \varepsilon^2(1 - x_2) - 4\varepsilon^2(1 - s^2)x_1(1 - x_1 - x_2). \qquad (1.10)$$

Making the substitution $x_1 = (1 - \epsilon y - z)/2$, $x_2 = \epsilon y$ and integrating over dz, we obtain

$$B_1 = \frac{w}{\sqrt{1 - s^2}}\int_0^{1/\varepsilon} \frac{y}{a}\tan^{-1}\frac{(1 - \varepsilon y)\sqrt{1 - s^2}}{a}\,dy, \qquad (1.11)$$

where

$$a^2 = s^2 + \varepsilon(1 - 2s^2)y + [1 - \varepsilon^2(1 - s^2)]y^2. \qquad (1.12)$$

For small s^2, small values of y are important in (1.11), so that $\tan^{-1}[(1 - \epsilon y)\sqrt{1 - s^2}/a]$ can be replaced by $\pi/2$ [it is shown in the Appendix that by doing this we obtain an accurate value for the singular part of the integral (1.11)]. After this is done the integral is easy to evaluate, since the upper limit gives a term, nonsingular in the neighborhood of $s^2 = 0$, which can be discarded. Consequently, expanding the result in powers of s^2 and ϵ^2 and substituting into (1.7), we obtain

$$B_1 = -\frac{\pi}{2}\left(1 + \frac{1}{2}\varepsilon^2\right)\left[s - \frac{\varepsilon}{2}\ln(\varepsilon + 2s)\right]. \qquad (1.13)$$

The integral B_1 corresponds to the interference term already discussed in Sec. 2 of reference 1, and its discontinuity across the cut coincides with (2.28, I).

As already mentioned,[1] in calculating the phase shift we are interested in those cases in which $|2s| \gg \epsilon$ for the effective region of integration in (2.15, I) (corresponding to $L \ll 4m^2/\mu^2$), so that we do not expand terms of the type $\ln(\epsilon + 2s)$ in powers of s. Since $s > 0$ in the physical region, the argument of the logarithm in (1.13) does not vanish. However, on the other sheet of the Riemann surface, where $s < 0$, this argument does vanish, and on this sheet there occurs a singular point at $4s^2 = \mu^2/m^2$, or $q^2 = 4\mu^2 - \mu^4/m^2$.

We express the integral B_2 in the form

$$B_2 = -\frac{4}{\varepsilon^2}\int_0^1 dx_2 \int_0^{1-x_2} dx_1 F,$$

$$F = \frac{1}{i}\int d^4k \left\{\frac{(\bar{u}_{p_1+q}\,|\,\hat{k}\,|\,u_{p_1})(\bar{u}_{p_2-q}\,|\,\beta_1\hat{k} + \beta_2 p_2 k/m\,|\,u_{p_2})}{(k^2 - 2\bar{p}k - \Delta)^3} + \ldots\right\}, \qquad (1.14)$$

where the quantities $\bar{p}$ and Δ are defined in (1.9), and the dots represent a similar term with p_2 replaced by p_1. Differentiating F with respect to μ^2, and then integrating over d^4k, we obtain

$$\frac{dF}{d\mu^2} = \frac{1 - x_2}{4}\left\{\frac{m^2x_2^2\beta_1 w_1 + x_2[x_2(m^2 + 2p^2) + q^2(1 - 2x_1)/4]\beta_2 w}{(\bar{p}^2 + \Delta)^2} - \frac{1}{2}\,\frac{\beta_1 w_2 + \beta_2 w_1}{\bar{p}^2 + \Delta}\right\}, \qquad (1.15)$$

where

$$w_1 = [(\bar{u}_{p_2-q}\,u_{p_2})(\bar{u}_{p_1+q}\,|\,\hat{p}_2\,|\,u_{p_1}) + (\bar{u}_{p_2-q}\,|\,\hat{p}_1\,|\,u_{p_2})(\bar{u}_{p_1-q}\,u_{p_1})]/2m,$$

$$w_2 = (\bar{u}_{p_1+q}\,|\,\gamma_\alpha\,|\,u_{p_1})(\bar{u}_{p_2-q}\,|\,\gamma_\alpha\,|\,u_{p_2}).$$

With accuracy up to terms quadratic in p^2/m^2 and ϵ^2 we have

$$w_1 = 1 + \frac{p^2}{m^2}\left(2 - \frac{1 - \cos\theta}{2}\right) = 1 + \frac{2p^2}{m^2} + \varepsilon^2,$$

$$w_2 = 1 + \frac{2p^2}{m^2}. \qquad (1.16)$$

The subsequent calculations are not difficult. Neglecting terms nonsingular in the neighborhood of $s^2 = 0$, using (1.7) and (1.16), and setting

$$\beta = -(\beta_1 + \beta_2)(1 + 2p^2/m^2 + \tfrac{5}{2}\varepsilon^2) - 2\varepsilon^2\beta_1, \qquad (1.17)$$

we obtain

*We note the fact that the next correction term to the expresion in (1.7), in the present case p^4/m^4, is absent. Similar cancellations take place in the matrix elements (1.16) and (1.23).

$$B_2 = -\frac{\pi}{4}\beta\left[s - \frac{\varepsilon}{2}\ln(\varepsilon + 2s)\right]. \quad (1.18)$$

We now turn to the calculation of the integrals B_3 and B_4, which correspond to fourth order perturbation theory. Introducing four Feynman parameters and integrating over d^4k, we obtain

$$B_3 = \frac{m^2}{4}\int dx_1 dx_2 dx_3 dx_4\, \delta\left(\sum x_i - 1\right)$$
$$\times\left[\frac{(\bar{u}_{p_2-q}\,|\,\hat{h}\,|\,u_{p_2})(\bar{u}_{p_1+q}\,|\,\hat{h}\,|\,u_{p_1})}{(h^2+\Delta)^2} - \frac{1}{2}\frac{w_2}{h^2+\Delta}\right], \quad (1.19)$$

$$B_4 = -\frac{m^2}{4}\int dx_1\, dx_2 dx_3 dx_4\, \delta\left(\sum x_i - 1\right)$$
$$\times\left[\frac{(\bar{u}_{p_2-q}\,|\,\hat{h}'\,|\,u_{p_2})(\bar{u}_{p_1+q}\,|\,\hat{h}'\,|\,u_{p_1})}{(h'^2+\Delta)^2} - \frac{1}{2}\frac{w_2}{h'^2+\Delta}\right], \quad (1.20)$$

where

$$h = p_2x_1 + p_1x_2 - q(x_1 - x_2 + x_3 - x_4)/2,$$
$$h' = p_2x_1 - p_1x_2 - q(x_1 + x_2 + x_3 - x_4)/2,$$
$$\Delta = \mu^2(x_3 + x_4) - q^2(1 - 2x_1 - 2x_2)/4.$$

Taking into account the fact that, to accuracy p^2/m^2, ϵ^2, the numerator in (1.19) can be written in the form [cf. (1.22)]

$$(u_{p_2-q}\,|\,\hat{h}\,|\,u_{p_2})(\bar{u}_{p_1+q}\,|\,\hat{h}\,|\,u_{p_1}) = m^2(x_1 + x_2)^2 w_1,$$

we transform B_3 to the following form:

$$B_3 = -\frac{1}{4}\left[\frac{w_2}{2} + w_1\frac{\partial}{\partial\gamma}\right]_{\gamma=1}\int dx_1 dx_2 dx_3 dx_4\, \delta\left(\sum x_i - 1\right)$$
$$\times [\gamma(x_1 + x_2)^2 + \varepsilon^2(x_3 + x_4 - 4(1 - s^2)x_3x_4)$$
$$+ 4(p^2/m^2 + \varepsilon^2(1 - s^2))\,x_1x_2]^{-1}.$$

Making the change of variables

$$x_1 = (\varepsilon y + \varepsilon x)/2, \quad x_2 = -x_1 + \varepsilon y, \quad x_3 = (1 - x_1 - x_2 - z)/2$$

and integrating over dz, we obtain

$$B_3 = -\frac{1}{4\sqrt{1-s^2}}\left[\frac{w_2}{2} + w_1\frac{\partial}{\partial E^2}\right]\int_0^{1/\varepsilon} dy \int_0^y \frac{dx}{b_1}\tan^{-1}\frac{(1-\varepsilon y)\sqrt{1-s^2}}{b_1},$$

$$b_1^2 = s^2 + \varepsilon(1 - 2s^2)\,y + E^2y^2 - [p^2/m^2$$
$$+ \varepsilon^2(1 - s^2)]\,x^2 \quad (E^2 = 1 + p^2/m^2).$$

Just as in the calculation of B_1 and B_2, we replace the arctangent in the integrand by $\pi/2$ and integrate over dx:

$$B_3 = -\frac{\pi}{8\sqrt{1-s^2}}\left(\frac{w_2}{2} + w_1\frac{\partial}{\partial(E^2)}\right)$$
$$\times\int_0^{1/\varepsilon}\frac{dy}{\sqrt{p^2/m^2 + \varepsilon^2(1-s^2)}}$$
$$\times\sin^{-1}\left[\frac{y}{a_1}\sqrt{p^2/m^2 + \varepsilon^2(1-s^2)}\right],$$

$$a_1^2 = s^2 + \varepsilon(1 - 2s^2)\,y + E^2y^2.$$

For p^2/m^2, $\epsilon^2 \ll 1$ the arcsine can be expanded as a series (taking two terms into account), after which integration over dy and subsequent differentiation with respect to E^2 gives, for $|s|^2 \ll 1$:

$$B_3 = -\frac{\pi}{16}\left\{\left(1 + 4\varepsilon^2\right)s - \varepsilon\left(1 + \frac{5}{2}\varepsilon^2\right)\ln(\varepsilon + 2s)\right.$$
$$\left. + \left(1 + \varepsilon^2\right)\frac{\varepsilon s}{\varepsilon + 2s}\right\}. \quad (1.21)$$

To calculate B_4 the exact value of the numerator in (1.20) is needed.

$$(\bar{u}_{p_2-q}\,|\,\hat{h}'\,|\,u_{p_2})(\bar{u}_{p_1+q}\,|\,\hat{h}'\,|\,u_{p_1}) = m^2(x_1 - x_2)^2 w_1 + m^2x_1x_2w_3, \quad (1.22)$$

$$w_3 = -\frac{4p^4}{m^4}\left(1 - \frac{1-\cos\theta}{2}\right) = -\frac{4p^2}{m^2}\left(\frac{p^2}{m^2} + \varepsilon^2\right). \quad (1.23)$$

Making the change of variables

$$x_2 = (1 - x_3 - x_4 - \varepsilon z)/2,$$
$$x_3 = (1 - \varepsilon y - x)/2, \quad x_4 = 1 - \varepsilon y - x_3$$

and integrating over dz, we obtain

$$B_4 = \frac{1}{4}\mathfrak{L}\frac{1}{E}\int_0^{1/\varepsilon} dy\int_0^{1-\varepsilon y} dx\,\frac{1}{b_2}\tan^{-1}\frac{yE}{b_2},$$

$$\mathfrak{L} = \frac{w_2}{2} + \left(w_1 - \frac{w_3}{4}\right)\frac{\partial}{\partial E^2} - \frac{w_3}{4}\frac{\partial}{\partial(p^2/m^2)},$$

$$b_2^2 = s^2 + \varepsilon(1 - 2s^2)\,y - [p^2/m^2 + \varepsilon^2(1 - s^2)]\,y^2$$
$$+ (1 - s^2)\,x^2. \quad (1.24)$$

The calculation of B_4 (see Appendix) gives the following result in the nonrelativistic approximation:

$$B_4 = -\frac{\pi}{16}\left\{(1 + 3\varepsilon^2)\,s - \varepsilon\left(1 + \frac{3}{2}\varepsilon^2\right)\ln(\varepsilon + 2s)\right.$$
$$\left. + \frac{\varepsilon}{2Ev}\ln\frac{2sE + \varepsilon v}{\xi E(1 + v)} - i\pi\frac{\varepsilon}{4v\sqrt{1-s^2}}\right\}, \quad (1.25)$$

$$v = \sqrt{1 + 4s^2/\xi^2}, \quad \xi = \mu/p. \quad (1.26)$$

The branch of the square root is defined as in (1.6) with the cut from $s^2 = -\xi^2/4$ to $-\infty$.

Characteristic of this part of the amplitude is the presence of a second singular point at $v = 0$, which corresponds to $q^2 = 4\mu^2(1 + \xi^2/4)$, with a singularity of the form

$$[4\mu^2(1 + \xi^2/4) - q^2]^{-1/2}.$$

For $\xi^2/4 \ll 1$, the second point is found in the neighborhood of the first point ($s^2 = 0$) but for $\xi^2 \to \infty$ it moves far away from the first; therefore terms in (1.25) which are singular only at the second point contribute to the phase shift only for $\xi^2/4 \lesssim 1/L$. In view of this, in calculating such terms an expansion is made in powers of ξ^2 which ensures that the phase shift is calculated with error $< 1/L$ (see Appendix). The last term in (1.25) is the only term in the whole two-meson amplitude giving a contribution to $\operatorname{Im} I_l^{(2)}$, so that it is calculated exactly for any $s^2 \le 0$, since the effective region of the integration over $|s|$ for $\operatorname{Im} I_l^{(2)}$ is in the close neighborhood of the second point, $|s| = \xi/2$.

Substituting (1.8), (1.13), (1.18), (1.21), and (1.25) into (1.2), we obtain, for the singular part of the singlet amplitude, the following expression ($|s|^2 \ll 1$, $|v|^2 \ll 1$):

$$M^{(2)}(s^2) = -\frac{3g^4}{8m}\left\{c_0 s + \varepsilon c_1 \ln(\varepsilon + 2s) + \varepsilon c_2 \frac{s}{\varepsilon + 2s} + \varepsilon \frac{3 - 2\lambda_\tau}{12}\left[\frac{1}{Ev}\ln\frac{2Es + \varepsilon v}{\xi E(1 + v)} - \frac{i\pi}{2v\sqrt{1 - s^2}}\right]\right\},$$

$$c_0 = (\alpha - 1)^2 + (^7/_2 - \alpha - \alpha^2)\varepsilon^2 + \lambda_\tau(\varepsilon^2 + 2\beta)/3,$$
$$c_1 = \alpha - 1 + (\alpha/2 - 2)\varepsilon^2 - \lambda_\tau(\varepsilon^2 + \beta)/3,$$
$$c_2 = (3 + 2\lambda_\tau)(1 + \varepsilon^2)/6, \qquad (1.27)$$

where λ_T is the eigenvalue of the operator $\tau_\alpha^{(1)}\tau_\alpha^{(2)}$. The amplitude has been calculated taking into account first order corrections in powers of ϵ^2 and $p^2/m^2 = \epsilon^2/\xi^2$.

2. CALCULATION OF THE TWO-MESON PHASE SHIFT

To calculate the two-meson singlet phase shift, it remains to integrate the scattering amplitude with respect to $|s|$ along the cut, making use of the relation (2.15, I) for each of the functions occurring in (1.27).

Taking into account that the discontinuity of the function s across the cut is $\Delta s = -2i|s|$, we obtain, for the contribution of this function to I_l,

$$[s]_l = -4\pi^{-1/2}\xi^2 Q_l(1 + 2\xi^2)L^{-3/2}. \qquad (2.1)$$

Inserting into (2.15, I) the discontinuity in the function $\ln(\epsilon + 2s)$, equal to $-2i\tan^{-1}(2|s|/\epsilon)$, we obtain for the corresponding contribution to I_l:

$$[\ln(\varepsilon + 2s)]_l = -(16\xi^2/\pi)Q_l(1 + 2\xi^2)\int_0^\infty e^{-L|s|^2} \times \tan^{-1}(2|s|/\varepsilon)|s|d|s|. \qquad (2.2)$$

We calculate the integral (2.2), first differentiating it with respect to ϵ,

$$\frac{\partial}{\partial\varepsilon}\int_0^\infty e^{-L|s|^2}\tan^{-1}\left(\frac{2|s|}{\varepsilon}\right)|s|d|s| = -\frac{1}{4}\sqrt{\frac{\pi}{L}} + \frac{\varepsilon\pi}{8}e^{\varepsilon^2 L/4}\left[1 - \Phi\left(\frac{\varepsilon\sqrt{L}}{2}\right)\right], \qquad (2.3)$$

where $\Phi(\epsilon\sqrt{L}/2)$ is the probability integral. If we restrict the discussion to not too high orbital angular momenta, then the parameter $\zeta = \epsilon\sqrt{L}/2 \ll 1$. At any energy, this makes it necessary to restrict l by the inequality

$$l \ll 4/\varepsilon^2 = 180. \qquad (2.4)$$

Now expanding (2.3) in powers of ζ, we obtain, after integrating back with respect to ϵ,

$$[\ln(\varepsilon + 2s)]_l = -4\xi^2 Q_l(1 + 2\xi^2)L^{-1}[1 - 2\pi^{-1/2}\zeta + \ldots]. \qquad (2.5)$$

Note that both terms written out in (2.5) can easily be obtained directly from (2.2), expanding the arctangent in powers of $\epsilon/2|s|$; however, it is already impossible to obtain the next term in the expansion in terms of ζ in this way, since the integral of each separate term diverges.

For the term $s/(\epsilon + 2s)$, expanding in powers of $\epsilon/2|s|$, we obtain the first term in the expansion in ζ:

$$\Delta\frac{s}{\varepsilon + 2s} = -\frac{2i\varepsilon|s|}{\varepsilon^2 + 4|s|^2} = -i\frac{\varepsilon}{2|s|} + \ldots,$$
$$\left[\frac{s}{\varepsilon + 2s}\right]_l = -\frac{2\xi^2}{\sqrt{\pi}}\varepsilon L^{-1/2}Q_l(1 + 2\xi^2). \qquad (2.6)$$

In integrating the remaining terms, we consider separately two regions of integration: $0 \le |s| \le \xi/2$ (from the first singular point to the second) and $|s| \ge \xi/2$ (beyond the second point). For the function

$$[\ln(2Es + \varepsilon v) - \ln E\xi(1 + v)]/Ev$$

in the first region we obtain

$$\Delta\frac{1}{Ev}\ln\frac{2Es + \varepsilon v}{E\xi(1 + v)} = -\frac{2i}{Ev}\tan^{-1}\frac{2E|s|}{\varepsilon v} = -\frac{2i}{Ev}\left(\frac{\pi}{2} - \frac{\varepsilon v}{2E|s|} + \ldots\right). \qquad (2.7)$$

The first term of the expansion in terms of $\epsilon/2|s|$ gives a contribution to I_l:

$$-\frac{8\xi^2}{E}Q_l(1 + 2\xi^2)\int_0^{\xi/2} e^{-L|s|^2}\frac{|s|d|s|}{\sqrt{1 - 4|s|^2/\xi^2}} = -\frac{4\xi^3}{E\sqrt{L}}Q_l(1 + 2\xi^2)e^{-\xi^2 L/4}\int_0^{\xi\sqrt{L}/2} e^{z^2}dz. \qquad (2.8)$$

In the second region of integration

$$\Delta\frac{1}{Ev}\ln\frac{2E|s| + \varepsilon v}{E\xi(1 + v)} = \frac{2i}{E|v|}\left[\ln(\varepsilon|v| + 2E|s|) - \ln 2E|s|\right] = \frac{i\varepsilon}{E^2|s|} + \ldots. \qquad (2.9)$$

Combining the contributions of the first term of the expansion (2.9) and of the second term in (2.7), we obtain

$$4\pi^{-1/2}\xi^2\varepsilon L^{-1/2}(1 - p^2/m^2)Q_l(1 + 2\xi^2). \qquad (2.10)$$

In (2.10) we set $E^2 = 1 + p^2/m^2$ and include terms of first order in p^2/m^2.

Ratio of two-meson to one-meson phase shifts

E_{lab}, Mev	2,5	10	40	90	160	360	650
^1P	-0.08	-0.15	-0.2				
^1D	0.02	0.04	0.1	0.25			
^1F	$-3\cdot 10^{-3}$	$-6\cdot 10^{-3}$	-0.02	-0.04	-0.05	-0.1	
^1G	10^{-3}	$3\cdot 10^{-3}$	0.01	0.04	-0.1	0.25	
^1H	-10^{-4}	$-4\cdot 10^{-4}$	$-3\cdot 10^{-3}$	$-7\cdot 10^{-3}$	-0.015	-0.035	-0.08

The final term in the amplitude (1.27) is the only one contributing to $\operatorname{Im} I_l^{(2)}$. Its cut only begins at the second point,* so that we cannot use formula (2.15, I), obtained for the end-point $t = 1 + 2\xi^2$. For the end point $t_\xi = 1 + 2\xi^2(1 + \xi^2/4)$ which corresponds to $v = 0$ $[q^2 = 4\mu^2(1 + \xi^2/4)]$, similar considerations give the formula

$$I_l = \frac{\xi^4}{\pi i} Q_l(t_\xi) \int_0^\infty e^{-L_\xi |v|^2} \Delta M(v^2)\, d|v|^2,$$

$$L_\xi = (l+1)\xi^3 / \sqrt{(1+\xi^2/4)[1+\xi^2(1+\xi^2/4)]}. \quad (2.11)$$

Substituting $i/v\sqrt{1-s^2} \approx i/v\sqrt{1+\xi^2/4}$ in (2.11) we obtain for the corresponding contribution to I_l:

$$[i/v\sqrt{1-s^2}]_l = 2i\xi^3 (\pi L)^{-1/2} Q_l(t_\xi). \quad (2.12)$$

Taking account of (2.1), (2.5), (2.6), (2.10), and (2.12), we obtain, for the two-meson singlet integral I_l:

$$I_l^{(2)} = \frac{3g^4}{2m} \frac{\xi^2}{\sqrt{\pi} L^{1/2}} \Big\{ Q_l(1+2\xi^2)[c_0 + \zeta d_1 + \zeta^2 d_2] + i\pi \frac{3 - 2\lambda_\tau}{12\varepsilon} \zeta^2 \xi Q_l(t_\xi) \Big\}, \quad (2.13)$$

where

$$d_1 = 2\sqrt{\pi}\,[c_1 + \tfrac{1}{6}(3 - 2\lambda_\tau)(1 - p^2/2m^2)\,\psi(z)],$$

$$d_2 = 2c_2 - 4c_1 - (1 - \tfrac{2}{3}\lambda_\tau)(1 - p^2/m^2),$$

$$\psi(z) = ze^{-z^2} \int_0^z e^{x^2} dx, \quad z^2 = L\xi^2/4$$

Inserting numerical values for α, β, and $\epsilon^2 = 0.0223$, we obtain

$$c_0 = 0.06 + 0.01\lambda_\tau,$$

$$d_1 = 0.6 - 0.03\lambda_\tau + 0.6\lambda_\tau^2 (1 - p^2/2m)\,\psi(z),$$

$$d_2 = -0.7 + p^2/m^2 + \lambda_\tau(1.4 - 2p^2/3m^2). \quad (2.14)$$

The singlet phase shift is related to the integral I_l in the following way:

$$p \operatorname{Re} I_l = 2\sin 2\delta_l \approx 4\delta_l, \qquad p \operatorname{Im} I_l = 2(1 - \cos 2\delta_l) \approx 4\delta_l^2.$$

Hence it follows that for large orbital angular momenta (when $|\delta_l| \ll 1$), the two-meson phase shift $\delta_l^{(2)} = (\epsilon m/4\xi)\operatorname{Re} I_l^{(2)}$. $\operatorname{Im} I_l^{(2)}$ is proportional to the square of the one-meson phase shift $\delta_l^{(1)}$, since $\operatorname{Im} I_l^{(1)} = 0$, and the square of the "total" phase-shift is $(\delta_l)^2 \approx [\delta_l^{(1)}]^2$.

The results (2.13) and (2.14), which we have obtained, show that there exists a strong compensation between the contributions from perturbation theory (fourth-order diagrams) and the terms obtained with the help of dispersion relations (containing α and β). Furthermore, there is mutual compensation of terms containing β_1 and β_2 (the final result involves their linear combination $\beta \ll \beta_1, \beta_2$), as a result of which the part $\tilde{f}''_{\alpha\beta}$ of the meson-nucleon scattering amplitude [cf. (1.1)] gives a negligible contribution to (2.13).

Formula (2.13) is the main part of an asymptotic expansion in the parameter $1/L$ (cf. Sec. 2 of reference 1) and therefore its accuracy should be, generally speaking, of order $1/L$. In the result given above, the cancellations in the main term may increase the importance of higher terms in the asymptotic expansion. For a rough order of magnitude estimate we can apply the formula obtained also to cases when the expansion parameter is not extremely small, $1/L \lesssim 1$.

The table shows values of $\delta_l^{(2)}/\delta_l^{(1)}$ calculated for several l and ξ, using formulas (2.13) and (2.9, I). We see that, to good accuracy, one can use the one-meson ^1D phase shift for energies $E_{lab} \lesssim 40$ Mev, and the one-meson ^1F and ^1G phase shifts for $E_{lab} \lesssim 150$ Mev. The estimates of the order of magnitude should also be valid for the triplet phase shifts. Hence it follows, that for the phase shift analysis of nucleon-nucleon scattering for $E_{lab} \lesssim 150$ Mev, all phase shifts corresponding to $l \geq 3$ can be taken to be the one-meson ones, and only the S, P, and D phase shifts determined from experiment.

3. CONCLUSIONS

The results we have obtained indicate that already for moderate orbital angular momenta the nucleon-nucleon elastic scattering phase shifts are determined by the one-meson interaction. This

***It is interesting to note that this result does not depend on the approximation we have applied (expansion in 1/L) and it is valid for the exact scattering amplitude, i.e. values of $t \geqslant t_\xi$ give a contribution to $\operatorname{Im} I_l$.**

circumstance may be of importance in carrying out phase shift analyses of nucleon scatterings since (as has already been noted[2,3]) this makes it unnecessary to treat every significant phase shift as an arbitrary variable parameter. If for given l and E the two-meson amplitude (2.13) appears appreciably smaller than the one-meson one (2.9, I), then with good justification one can take account of the corresponding phase shift with the one-meson approximation. The absence of a given angular momentum state from the whole analysis of nucleon-nucleon scattering apparently makes impossible a unique shift analysis of the experimental data. The best available data on p-p scattering at an energy of 310 Mev give eight sets of phase shifts[4] of which only two have phase shifts for large l agreeing with the one-meson ones. Undoubtedly the use of the one-meson "tail," within the limits indicated in the present paper, should facilitate phase shift analyses.

The results of this paper depend strongly on the dispersion relations for momentum transfer near $4\mu^2$. Although there is no reason to doubt the applicability of the dispersion relations under these conditions, nevertheless experimental verification of the results obtained might shed light on the region of applicability of the dispersion relations. Since we have considered only the singlet scattering, naturally we have not obtained the complete matrix (in spin space) of the scattering operator responsible for the two-meson exchange. However, the calculations for the triplet state do not introduce any difficulties of principle and have been carried out by Grashin and Kobzarev.[6]

In all the above discussion, we have been making an expansion in powers of $1/L$, retaining only the first nonvanishing term, so that the accuracy of our results should be of order $1/L$. It is not difficult to see, however, that the basic formulas (2.12, I) and (2.20, I) must hold for appreciably weaker restrictions on the value of the orbital angular momentum. For these to be applicable the inequality

$$1 \gg \frac{Q_l\left(1+\frac{9}{2}\xi^2\right)}{Q_l(1+2\xi^2)} \approx \begin{cases} \exp\{-(l+1/2)\xi\} & \text{for } \xi^2 \ll 1 \\ (4/9)^{l+1} & \text{for } \xi^2 \gg 1 \end{cases},$$

must hold, in order to make it possible to neglect three-meson states in the sum over intermediate states in $A_1(E, q^2)$. Under these conditions formula (2.21, I) remains. In this case, of course, one cannot restrict the calculation of $\tilde{f}_{\alpha\beta}$ to the point $\omega = 0$ and $q^2 = 4\mu^2$, but it is necessary to know the meson-nucleon scattering amplitude in some limiting region round the point $\omega = 0$, $q^2 = 4\mu^2$. Therefore the problem of analytic continuation of the meson-nucleon scattering amplitude becomes much more complicated. If, however, such an analytic continuation is feasible, then it may be possible to obtain with sufficient accuracy an expression for the two-meson nucleon-nucleon scattering phase shift for quite small l. It should be emphasized that such an extension to small l depends fundamentally on Mandelstam's results,[5] while the results obtained in this paper depend essentially only on the following: (1) the nearest singular point (apart from the one-meson pole) lies at $q^2 = 4\mu^2$; (2) near $q^2 = 4\mu^2$ there are no other singular points [except for $q^2 = 4\mu^2(1+\xi^2/4)$].

As a result of the strong compensation of the leading terms, which we discussed above, errors in the determination of $L^{(1)}(0, 4\mu^2) = \alpha$ (Sec. 3 of reference 1) may appreciably change (for example by two degrees) the final two-meson phase shifts given in the table. However, if the ratio $\delta_l^{(2)}/\delta_l^{(1)}$ is, for example, less than 10%, these errors do not alter the conclusion that in the given case the scattering phase shift is basically determined by the one-meson interaction.

It has proved possible to express the nucleon-nucleon scattering, caused by two-meson exchange, in terms of the pion-nucleon scattering and, in this way, a connection has been established between these two distinct processes. This is not accidental. If it were possible to do the calculation for small l, then the nucleon-nucleon scattering would be expressed in terms of the meson-nucleon scattering amplitude, the amplitude for the process $\pi + n \rightarrow 2\pi + n$, etc.

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APPENDIX

1. Calculation of B_1. For calculating the integral B_1, we introduce the auxiliary function

$$B_1(u) = \frac{1}{\sqrt{1-s^2}} \int_0^{1/\varepsilon} \frac{y}{a} \tan^{-1} \frac{(1-\varepsilon y)\sqrt{1-s^2}+u}{a} dy,$$

$$a^2 = s^2 + \varepsilon(1-2s^2)y + [1-\varepsilon^2(1-s^2)]y^2. \quad \text{(A.1)}$$

The desired integral is equal to the value of this function for $u = 0$, which can be written in the form

$$B_1(0) = B_1(\infty) - \int_0^\infty \frac{\partial B_1(u)}{\partial u} du, \quad \text{(A.2)}$$

$$B_1(\infty) = \frac{\pi}{2\sqrt{1-s^2}} \int_0^{1/\varepsilon} \frac{y\,dy}{a}, \tag{A.3}$$

$$\frac{\partial B_1(u)}{\partial u} = \frac{1}{\sqrt{1-s^2}} \times \int_0^{1/\varepsilon} \frac{y\,dy}{1-\varepsilon y + y^2 + u^2 + 2(1-\varepsilon y)\sqrt{1-s^2}}; \quad u \geqslant 0. \tag{A.4}$$

We consider the analytic properties of the functions (A.2), (A.3), (A.4) of the variable s^2. The original integral was analytic in the whole plane with a cut from $s^2 = 0$ to $-\infty$ (corresponding to $q^2 \geq 4\mu^2$, $t \geq 1 + 2\xi^2$). In the integral (A.3) there appears an additional singularity at the point $s^2 = 1$ ($q^2 = 0$), so that for (A.3) it is necessary to make a second cut from $s^2 = 1$ to $+\infty$ ($q^2 \leq 0$, $t \leq 1$). The integral (A.4) is analytic as a function of the variable $\sqrt{1-s^2}$ in the right half-plane, and as a function of s^2 in the whole plane with a cut from $s^2 = 1$ to $+\infty$, since we took $\sqrt{1-s^2} > 0$ for $s^2 < 1$. Since for the calculation of the phase shift we need to integrate along the cut $s^2 \leq 0$ ($q^2 \geq 4\mu^2$), the contribution of the second term in (A.2) vanishes identically and it need not be taken into account. The calculation of the remaining integral is elementary. Thus the method we have indicated has enabled us to calculate exactly the singular part of the original integral (1.11), giving a contribution to (2.15, I), while "spoiling" the behavior of the function in other regions which do not contribute to the subsequent integration (2.15, I).

2. Calculation of B_4. In contrast to the previous integral, the integral B_4, which corresponds to the fourth-order Feynman diagram shown in the figure,

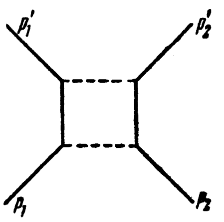

Diagram for the integral B_4

possesses singularities such that the integrand in (1.20) as a function of its parameters is singular for any real q^2. It follows from this that the real axis for the integral $B_4(q^2)$, considered formally for any complex values of q^2, is a singular line. However, direct calculation shows that the function B_4 may be analytically continued from the physical region to all densities with a cut from $q^2 = 4\mu^2$ to $+\infty$, which corresponds to the analytic continuation used in Sec. 2 of reference (1). This will agree with the original Feynman integral only in the upper half-plane. To calculate the phase shift we need just this function and we will not discuss its singularities.

To calculate the remaining integral over two parameters in (1.24), we introduce the function

$$f(u) = \int_0^{1/\varepsilon} dy \int_0^{1-\varepsilon y} dx \frac{1}{b_2} \tan^{-1}\frac{yE+u}{b_2},$$

$$b_2^2 = s^2 + \varepsilon(1-2s^2)y - [p^2/m^2 + \varepsilon^2(1-s^2)]y^2 + (1-s^2)x^2 - i0. \tag{A.5}$$

The desired integral is equal to the value of $f(u)$ for $u = 0$:

$$f(0) = f(\infty) - \int_0^{\infty} \frac{\partial f(u)}{\partial u} du, \tag{A.6}$$

$$f(\infty) = \frac{\pi}{2}\int_0^{1/\varepsilon} dy \int_0^{1-\varepsilon y} \frac{dx}{b_2} = \frac{\pi}{2\sqrt{1-s^2}} \times \int_0^{1/\varepsilon} [\ln(R_1 + \sqrt{1-s^2}(1-\varepsilon y)) - \ln R]\,dy,$$

$$R^2 = s^2 + \varepsilon(1-2s^2)y - [p^2/m^2 + \varepsilon^2(1-s^2)]y^2 - i0,$$

$$R_1^2 = 1 - \varepsilon y - p^2y^2/m^2 - i0, \tag{A.7}$$

$$\frac{\partial f(u)}{\partial u} = \int_0^{1/\varepsilon} dy \int_0^{1-\varepsilon y} dx \Big[s^2 + \varepsilon(1-2s^2)y + \Big(E^2 - \frac{p^2}{m^2} - \varepsilon^2(1-s^2)\Big)y^2 + (1-s^2)x^2 + 2Eyu + u^2\Big]^{-1}. \tag{A.8}$$

The integrand in (A.7) is singular, as before, on the whole real axis, so that for the following integration over dy it is necessary to take s^2 real and analytically continue the integrated result to the unphysical region. The integrand in (A.8) is already singular only for $s^2 \leq 0$ ($q^2 \geq 4\mu^2$), i.e., the second term in (A.6) is an analytic function in the whole plane with a cut from $s^2 = 0$ to $-\infty$. After integrating with respect to x (A.8) takes the form

$$\frac{\partial f(u)}{\partial u} = \frac{1}{\sqrt{1-s^2}} \int_0^{1/\varepsilon} \frac{dy}{a_2} \tan^{-1}\frac{\sqrt{1-s^2}(1-\varepsilon y)}{a_2},$$

$$a_2^2 = s^2 + \varepsilon y(1-2s^2) + [E^2 - p^2/m^2 - \varepsilon^2(1-s^2)]y^2 + 2Eyu + u^2. \tag{A.9}$$

From this integral we select the part which is singular on the cut, introducing another auxiliary parameter in the same way as in (A.1), (A.2):

$$\frac{\partial f(u)}{\partial u} = \frac{\partial f(u,\infty)}{\partial u} - \int_0^{\infty} \frac{\partial^2 f(u,v)}{\partial u\,\partial v} dv, \tag{A.10}$$

$$\frac{\partial f(u,\infty)}{\partial u} = \frac{\pi}{2\sqrt{1-s^2}} \int_0^{1/\varepsilon} \frac{dy}{a_2}, \tag{A.11}$$

$$\frac{\partial^2 f(u,v)}{\partial u\,\partial v} = \int_0^{1/\varepsilon} \left[1 - \varepsilon y + \left(E^2 - \frac{p^2}{m^2}\right) y^2 + 2Eyu + u^2 + v^2 + 2(1-\varepsilon y)\, v\sqrt{1-s^2}\right]^{-1} dy, \qquad u,\ v \geqslant 0. \tag{A.12}$$

The second term in (A.10) need not be taken into account since it is an analytic function for Re $s^2 < 1$ (Re $q^2 > 0$). In the remaining singular part (A.11) there appears an additional singularity at $s^2 = 1$ and a second cut from $s^2 = 1$ to $+\infty$ ($q^2 \leq 0$). We now insert (A.10) and (A.11) into (A.6) and integrate over du. At the upper limit the integral diverges, but the dependence on s^2 disappears, so that we need only take account of the lower limit. Thus the singular part, contributing to (2.15, I), takes the form

$$f = f(\infty) + \frac{\pi}{2\sqrt{1-s^2}} \int_0^{1/\varepsilon} \ln[R_2 + Ey]\, dy,$$

$$R_2^2 = s^2 + \varepsilon y(1-2s^2) + [E^2 - p^2/m^2 - \varepsilon^2(1-s^2)]\, y^2 - i0. \tag{A.13}$$

The integral over dy remaining in (A.13) is calculated exactly, since integrating by parts and removing the irrationality in the denominator of the integral leads to the elementary result:

$$f(\infty) = i\pi \frac{2p^2/m^2 + \varepsilon^2}{4pP^2/m} \ln \frac{\varepsilon - 2ip/m}{\varepsilon} - \frac{\pi K_1}{4\sqrt{1-s^2}} \ln(-K_1) + \frac{\pi K_2}{4\sqrt{1-s^2}} \ln K_2 + i\pi^2 \frac{\varepsilon(1-2s^2)}{8P^2\sqrt{1-s^2}} - \pi \frac{\varepsilon(1-2s^2)}{8P^2\sqrt{1-s^2}} \times \ln\left(\frac{4p^2}{m^2} + \varepsilon^2\right) - \frac{\pi K}{8P^2\sqrt{1-s^2}} \ln \frac{2\sqrt{1-s^2}\,p/m - iK}{2\sqrt{1-s^2}\,p/m + iK} + \frac{\pi\varepsilon(1-2s^2)}{8P^2\sqrt{1-s^2}} \ln\left[\frac{1}{4P^2}(F_1 - KF_2)(F_1 + KF_2)\right] + \frac{\pi K}{8P^2\sqrt{1-s^2}} \ln \frac{F_1 - KF_2}{F_1 + KF_2}. \tag{A.14}$$

Here we introduce

$$F_1 = 2(2p^2/m^2 + \varepsilon^2)(\sqrt{1-s^2} + 1 - s^2) + K^2,$$

$$F_2 = \varepsilon(1 + 2\sqrt{1-s^2}),$$

$$P^2 = p^2/m^2 + \varepsilon^2(1-2s^2), \qquad K = \sqrt{4s^2p^2/m^2 + \varepsilon^2},$$

$$K_1 = [K + \varepsilon(1-2s^2)]/2P^2, \qquad K_2 = [K - \varepsilon(1-2s^2)]/2P^2.$$

The root K is defined so that it is positive in the physical region with a cut running from $s^2 = -\xi^2/4$ to $-\infty$ [$q^2 \geq 4\mu^2(1+\xi^2/4)$]. The function (A.14) is analytic in the whole plane with the indicated cut. Leaving only terms singular for $s^2 < 1$ ($q^2 > 0$), we obtain

$$f(\infty) = \frac{\pi}{4\sqrt{1-s^2}}[-K_1 \ln(-K_1) + K_2 \ln K_2]. \tag{A.15}$$

The other term in (A.13) is calculated similarly:

$$\int_0^{1/\varepsilon} dy\, \ln[R_2 + Ey] = \frac{1}{\varepsilon}[\ln(\sqrt{E^2 - p^2/m^2} + E) - 1] + \frac{K_1}{2}\ln(-K_1) - \frac{K_2}{2}\ln K_2 + K_2 \ln(1/\varepsilon + K_2) - \frac{i\pi K_1}{2} + \frac{K}{2P^2}\ln(2Es + K) - \frac{E\varepsilon(1-2s^2)}{2WP^2}\ln[2Ws + \varepsilon(1-2s^2)] + \frac{K_1 + K_2}{2}\ln(2P^2\varepsilon) - \frac{K_1}{2}\ln\{2(E + \sqrt{E^2 - p^2/m^2})[K\sqrt{E^2 - p^2/m^2} + \varepsilon E(1-2s^2)] + \varepsilon K(K - \varepsilon)\} - \frac{K_2}{2}\ln\{2(E - \sqrt{E^2 - p^2/m^2})[K\sqrt{E^2 - p^2/m^2} + \varepsilon E(1-2s^2)] + \varepsilon K(K + \varepsilon)\} + \frac{\varepsilon E(1-2s^2)}{2WP^2}\ln\left(\frac{2W}{\varepsilon}\sqrt{E^2 - p^2/m^2} + 2\frac{E^2 - p^2/m^2}{\varepsilon} - \varepsilon\right),$$

where we have introduced $W^2 = E^2 - P^2$. Leaving out terms nonsingular for $s^2 < 1$, and also the last term, singular for $W = 0$ (which corresponds to $q^2 = 4m^2$), and combining with (A.15), we obtain finally the singular part of the original integral (for $4\mu^2 \leq q^2 < 4m^2$):

$$f = \frac{\pi}{2\sqrt{1-s^2}}\left\{-\frac{E\varepsilon(1-2s^2)}{2WP^2}\ln[2Ws + \varepsilon(1-2s^2)] + \frac{K}{2P^2}\ln(2Es + K) - \frac{i\pi K}{4P^2} + K_2\ln\left(2\frac{p^2}{m^2} + \varepsilon^2 + \varepsilon K\right) - \frac{K_1}{2}\ln\{2(E + \sqrt{E^2 - p^2/m^2})[K\sqrt{E^2 - p^2/m^2} + \varepsilon E(1-2s^2)] + \varepsilon K(K - \varepsilon)\} - \frac{K_2}{2}\ln\{2(E - \sqrt{E^2 - p^2/m^2})[K\sqrt{E^2 - p^2/m^2} + \varepsilon E(1-2s^2)] + \varepsilon K(K + \varepsilon)\}\right\}. \tag{A.16}$$

The singularities of the functions $f(s^2)$ and $B_4(s^2)$, obtained from $f(s^2)$ by the application of some differential operator [cf. (1.24)], consist of two singular points: the point $s^2 = -\xi^2/4$, which corresponds to the value $K = 0$ [$q^2 = 4\mu^2(1+\xi^2/4)$]. For $\xi^2/4 \ll 1$ the second point is found in the neighborhood of the first, but for $\xi^2 \to \infty$ it moves away to an extremely large distance from the first. It is evident that terms singular only at the second point contribute to the integration (2.15, I) only for $\xi^2/4 \lesssim 1/L$ since for them the cut begins for $|s|^2 = \xi^2/4$ and compared with terms singular at the first point they will have, after integration, an additional factor $\exp(-\xi^2 L/4)$. In view of this, such terms in (A.16) can be expanded in powers of ξ^2 and the leading term in the expansion retained. To this accuracy the phase shift will depend on the parameter $\xi^2 \exp(-\xi^2 L/4)$, vanishing as $\xi^2 \to 0$ (high energy) and as $\xi^2 \to \infty$ (low energy). The largest error will be in the intermediate region $\xi^2/4 \approx 1/L$, where this parameter takes its maximum value $4/2.7\,L \approx 1/L$. But even in this region the expansion in ξ^2 provides just the same accuracy as comes from our asymptotic expansion in $1/L$. Besides this, it is necessary to consider only the first term of the expansion in powers of K^2, just as we considered only the first term in the expansion in powers of s^2. Taking this into account we simplify the last three terms in (A.16), after which we obtain, in the nonrelativistic approximation, for $|s^2| \ll 1$, $|K|^2 \ll \epsilon^2$:

$$f = \frac{\pi}{4P^2}\left\{-\frac{E\varepsilon}{W}\ln\left[2Ws + \varepsilon(1-2s^2)\right] + K\ln\left[\frac{p}{m}\frac{2Es+K}{E(\varepsilon+K)}\right] - \frac{i\pi K}{2\sqrt{1-s^2}}\right\}. \quad (A.17)$$

The next correction terms in (A.17) vanish in this case, and the largest of the discarded terms takes the form $K\xi^4/P^2$, $K(p/m)^4/P^2$. Operating on (A.17) with the differential operator $\mathfrak{L}(1/4E)$ [cf. (1.24)], we obtain formula (1.26).

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