INTERNAL STRUCTURE OF SUPER-DENSE STARS

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The peculiarities of a "condensed state — plasma" transition in ultra-compressed matter are considered, and it is concluded that the cores of dense white dwarfs may be in a condensed state. As a result, the nuclear processes have much lower rates than in a plasma, and the possible hydrogen concentration in the matter of white dwarfs may be much higher.

1. INTRODUCTION

WHITE dwarfs (w.d.), stars of exceedingly high density and small radius and luminosity, do not lack interest for many branches of physics. The exceedingly high density of matter, approaching 10^9 g/cm^3, the presence of relativistic degenerate electron gas, the specific features of the nuclear processes — this is a far from complete list of the distinguishing properties of w.d. (see reference 1).

In the present article we consider the aggregate state of matter in w.d. One usually assumes that it is in a plasma state; accordingly, for example, the speeds of nuclear reactions in w.d. are calculated with the aid of relations applicable only to ionized gases.

Such a representation, which is true for a majority of stars, necessitates when applied to w.d., generally speaking, a review. The point is that owing to the high density of the w.d. the Coulomb coupling between nuclei may be so rigid, that even stellar temperatures ($\sim 10^7$ deg) are found to be insufficiently high for the "evaporation" of the condensate.*

A consistent examination of this problem requires the calculation of thermodynamic potentials of the condensed and plasma phases and the determination of which is the smaller. It is important to emphasize that, for uncompressed matter, even a qualitative examination of this problem would be very difficult owing to the important role of electron shells. The situation is considerably simplified, however, at high compressions, corresponding to a small value of the parameter $RZ^{1/3}/a_0$ (see below for notation). We deal essentially with a practically ideal and homogeneous electron gas, which plays the role of a background that compensates for the positive charge of the nuclei.

In spite of this simplification, the problem remains exceedingly complex. We shall therefore employ here a simpler and clearer, although rougher approach.* There are grounds for assuming that not far from the phase curve the results obtained will be sufficiently reliable.

The presence of a condensed core in the w.d. influences greatly the rate of the nuclear processes (p-p reaction) and the chemical composition of the w.d. What is radically changed (compared with the plasma) is the kinetic mechanism of the process: the factors that come to the forefront are the height of the barrier, (which causes the binding of the particle in the condensate), the frequency of particle vibration, etc.

Wildhack[2] and Zel'dovich[3] have considered the reaction in cold crystalline hydrogen, due to the tunnel effect. The reaction yield was found to be large and incompatible with the small luminosity and the long lifetime of the w.d. This conclusion becomes even more aggravated at a temperature different from 0. Thus, even in a condensed w.d. a hydrogen content on the order of several times ten percent becomes impossible.†

Hydrogen concentrations on the order of several percent ($< 10^{-3}$% according to the plasma model) are possible, however, if a configuration is produced, in which the protons are uniformly

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*We make no special distinction between the solid and liquid state (see, incidentally, Section 2), since for nuclear reactions only "near order" is significant.

*This approach is used in the theory of the metallic bond to derive the equation of state of matter, etc.

†We note that even a few years ago[5] the question of the discrepancy between the values of the hydrogen concentration, obtained on the basis of the mass and radius (50–70%) and on the basis of luminosity ($<10^{-5}$, obtained for Sirius B), was quite acute. At the present time, owing to the more precise determination of the radius, this question apparently has become less acute.[1]

"frozen" in the bulk of heavy nuclei. The reaction is then suppressed because the protons must overcome not only the mutual repulsion, but also the potential barrier produced by the neighboring inactive nuclei. We note that the presence of even such a relatively small hydrogen concentration may be of prime significance for the theory of stellar evolution and the theory of novas and supernovas (in this connection see reference 4).

Nuclear reactions are apparently the only factor influenced by the presence of condensate. Such quantities as the pressure, conductivity, etc are determined as before essentially by the electronic component.*

To conclude this section, we give numerical values of the parameters of w.d., which will be used henceforth. The density ρ is chosen to be 10^6 g/cm^3. Only indirect data are available on the temperature; we assume a probable value of 10^7 deg (see references 1 and 5). Finally, the average atomic number Z is taken to be on the order of 10 (reference 6).

2. PHASE TRANSITION IN SUPERDENSE MATTER

We shall start with a consideration of the condensed phase and find the range of temperatures and densities in which it is stable.

We consider first matter containing nuclei of one kind. We separate an isolated neutral spherical cell with a nucleus at the center. The remaining matter is ignored in this approach; its influence manifests itself only in boundary effects, which prevent the expansion of the separated cell.

The cell radius R, which has the significance of the average distance between nuclei, is equal to

$$R = (3\eta ZM/4\pi\rho)^{1/3}, \tag{1}$$

where M is the proton mass and η is the ratio of the atomic weight to Z.

Introducing the Bohr radii of the electron, $a_0 = \hbar^2/me^2$, and of the nucleus $A_0 = \hbar^2/\eta MZ^3e^2$, we have

$$R/a_0 = 3.8\cdot 10^{-2} \ll 1, \quad R/A_0 = 1.4\cdot 10^5 \gg 1, \tag{2}$$

i.e., the system is dense from the point of view of electrons and rarefied from the point of view of nuclei.

An elementary calculation gives the following expressions for the potential energy of the nucleus in the cell

$$U(r) = -3Z^2e^2/2R + Z^2e^2r^2/2R^3, \tag{3}$$

where r is the distance from the center of the cell (r < R). The nucleus is thus in an oscillator potential well. The corresponding oscillation frequency is

$$\omega = (Z^2e^2/\hbar R)(A_0/R)^{1/2}. \tag{4}$$

At 0 temperature the level occupied by the nucleus corresponds to an energy (reckoned from the bottom of the well)

$$E_0 = \tfrac{3}{2}\hbar\omega = (3Z^2e^2/2R)(A_0/R)^{1/2}. \tag{5}$$

By virtue of (2) this quantity is considerably less than the depth of the potential well, which also is evidence of the strong bond between the nuclei in the cold condensate, and thereby of the stability of the latter.*

As R decreases the value of E_0 increases more rapidly than U, and at extremely high compression when R becomes of the order of A_0, the nucleus cannot be held by the potential well. In spite of the 0 temperature, the condensate is destroyed by a process that can be called "cold evaporation." It must be noted, incidentally, that electron capture begins long before that and leads to a transition of the matter into a neutron state; in view of this, the upper part of the figure given here is only tentative.

We now consider the influence of the temperature, an effect that reduces to an increase in the average energy of the nucleus. The corresponding factor for an oscillator is $\coth(\hbar\omega/2kT)$. Thus, the ratio of the nuclear energy to the depth of the well is of the form

$$\xi = (A_0/R)^{1/2}\coth[(Z^2e^2/2RkT)(A_0/R)^{1/2}]. \tag{6}$$

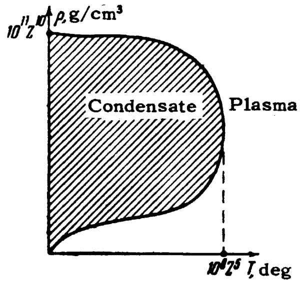
It plays the role of a criterion of phase transition: when $\xi \ll 1$ and $\xi \gg 1$ we deal with condensed and plasma states respectively. If $\xi \sim 1$, no definite conclusions can be made, owing to the inapplicability of the approach itself. The figure shows the approximate course of the curve $\xi = 1$. When $T > T_{cr} \sim 10^8 Z^5$ deg, the condensed state is altogether impossible, and when $T < T_{cr}$ the condensate region is bounded both from below (the usual thermal evaporation) and from above ("cold evaporation").

Under the conditions of w.d., $Z^2e^2/RkT = 84 \gg 1$, $\hbar\omega/2kT = 0.11 \ll 1$. Hence

$$\xi \approx 2RkT/Z^2e^2 = 0.024 \ll 1.$$

*We note that the usual arguments that lead to the equation of equilibrium of a w.d. become, strictly speaking, unsuitable for w.d. heavier than Sirius B, owing to the Klein paradox. This question will be considered separately.

*With this, the amplitude of the zero oscillations of the nucleus is approximately one order of magnitude less than R (cf. the basic premise of the Lindeman melting theory).



The latter estimate makes it possible to speak with a great degree of confidence on the condensed state of the matter in w.d.

Let us determine the displacement of the nucleus during a time t in the diffusion process; according to Seit[7]

$$l \sim R\,(\omega t)^{1/4} \exp(-W/2kT).$$

The activation energy W cannot be obtained from an examination of an isolated cell; therefore the estimates given below are quite tentative.

Putting $W \sim Z^2e^2/R$, we find that during the entire time of existence of w.d. ($\sim 10^{17}$ sec) the nucleus shifts merely by a distance on the order R. One can therefore speak of a solid (possibly amorphous) state of the matter in w.d.

Let us make two remarks. We have seen earlier that for the condensate to exist the system of nuclei must be rarefied [see Eq. (2)]. Under this condition, an important role is played by the internuclear correlation, which, in the final analysis, causes the plasma condensation. A "crystallization" of similar kind of a rarefied degenerate electron gas was considered by Wigner.[8]

We note furthermore that during the process of formation of the w.d. the latent heat of condensation should be liberated. This source of energy can apparently play a noticeable role and should be taken into account in the theory of stellar evolution.

We proceed to consider a model in which, in addition to the heavy component with average $Z = 10$, there exist also protons. The heavy nuclei, as noted above, form a more or less stable skeleton of the system; with this, at least "in the small," one can speak of a crystalline lattice. The latter will be considered, for the sake of being definite, as face-centered cubic (close packing). We shall also assume that one proton belongs to each crystalline cell of this lattice. This makes the concentration of hydrogen by weight on the order of $[\eta(4Z+1)]^{-1} \approx 1\%$ (for heavy w.d. the volume of condensate is close to the total volume of the w.d.).

Let us separate an individual crystal cell and replace it by an equivalent sphere, where the charge of the heavy nuclei is uniformly distributed over the surface of the sphere. Designating the corresponding quantity with a subscript 1, we shall have $R_1 = 4^{1/3}R$. The potential energy of the proton in the cell is

$$U_1(r) = -(4Z+3)/2R_1 + (4Z+1)\,r^2/2R_1^3, \quad (7)$$

whence $\omega_1 = \sqrt{2}\,\omega$.

We have furthermore $\xi_1 = [2 \times 4^{1/3}Z^2/(4Z+3)]\,\xi = 0.17$. Thus, the protons are more weakly bound in the condensate than the heavy nuclei; however, the character of their motion differs greatly from thermal. The protons are displaced relatively slowly over the condensate and vibrate with high frequency about the equilibrium positions.

Let us find the diffusion path l (see above) for protons. Assuming that W is not less than the depth of the potential well, and taking accordingly $W \sim 2Z/R_1$, we obtain $l \sim 10^5$ cm, which is approximately four orders less than the radius of the w.d. Thus, (if this estimate is confirmed by more accurate calculation), there is no need to fear that the greater part of the protons will diffuse into the plasma periphery of the w.d., and will be taken out of play there by the nuclear reaction.

3. RATE OF NUCLEAR REACTION IN WHITE DWARFS

In the introduction we already indicated that, other conditions being equal, the rate of the principal reaction for w.d., $p + p = d + e^+ + \tilde{\nu}$ in the solid phase should be less than in the plasma phase.

We shall estimate now the rate of this reduction for the model considered in the end of the last section (we shall henceforth drop the subscript 1).

The rate of reaction in the plasma phase is

$$q' \sim \overline{\sigma' v}\, n^2, \quad (8)$$

where σ' is the reaction cross section (allowing for the penetration coefficient) and n is the proton concentration.

In the condensed phase, as can be readily seen, the rate of reaction is independent of the rate of diffusion of nuclei (the change in the number of nuclei which the given nucleus encounters per unit time is exactly compensated for by the opposite change in the fraction of the time spent near them). If ω is the frequency of the oscillation (the number of approaches of the nucleus to the barrier per unit time) then the rate of reaction in the condensate is

$$q \sim b(\sigma/R^2)\,\omega n \sim b\sigma\omega R n^2, \qquad (9)$$

where b = 6 is the number of neighboring protons. In practice $b\omega R/v = 1$ and

$$q/q' \sim \sigma/\sigma'$$

(the velocity v is taken at the saddle point, see below). Thus, in spite of the different kinetic mechanism, the rates of reaction in the plasma and condensed phases differ only in the characteristics of the elementary act (more accurately speaking, in the penetration factors).

Let us proceed to calculate the latter. If the interaction potential between protons has the form $V(r) = e^2\varphi(x)/r$, $x = r/R$, then the usual procedure (see, for example, reference 1) leads to the following expression for the coefficient of penetration:

$$K(\varphi) \sim \exp\{-\tau(\varphi - {}^2/_3\, x_0\varphi')/(\varphi - x_0\varphi')^{1/2}\}.$$

The saddle point x_0 is determined by the equation

$$x_0^{3/2} = \alpha(\varphi - x_0\varphi').$$

Here $\tau = 3(\pi^2 e^2/4A_0kT)^{1/2}$, $\alpha = (2/\pi)(A_0/R)^{1/2} \times (e^2/RkT)$, the functions φ and φ' refer to the argument x_0. For pure Coulomb interaction $K(1) = e^{-\tau}$. In finding φ we shall assume that the protons react while in different (neighboring) cells.* For plasma one usually assumes the following expression for V:[1]

$$V(r) = (e^2/r)(1 - \tfrac{3}{2}\, r/R_0 + \tfrac{1}{2}\, r^3/R_0^3) \qquad \text{for } r < R_0,$$

$$V(r) = 0 \qquad \text{for } r > R_0.$$

Here $R_0 = R[\eta(4Z+1)]^{-1/3}$ is the radius of a neutral sphere containing the proton. Thus, allowance for the screening of the protons by the electron gas yields

$$\varphi(x) = \begin{cases} 1 - 3Z^{1/3}x + 4Zx^3, & x < 1/2Z^{1/3}, \\ 0, & x > 1/2Z^{1/3}. \end{cases}$$

In the case of a condensate it is necessary to add a term that takes into account the proton bond in the lattice. Considering that the protons are on the average arranged symmetrically with respect to the boundary that separates their cells, we obtain with the aid of (7)

$$\delta\varphi = Zx(2 - x)^2.$$

This term corresponds to the potential well referred to in Section 2, and leads to an effective increase in the repulsion between protons.

The sought reduction in reaction rate is given by the ratio $A = K(\varphi + d\varphi)/K(\varphi)$. Numerical calculations yield $\alpha = 0.0315$, $x_0 = 0.094$, and

$$A \sim 10^{-9}. \qquad (10)$$

Since this quantity has the significance of a probability of emergence from the potential well and depends relatively little on ρ, one can assume this estimate to be valid for a greater portion of the core of a dense w.d. The smallness of A is evidence that the factor that limits the hydrogen content is not the reaction rate, but the model selected by us (see end of Sec. 2). Models with greater hydrogen contents require a special investigation; in any case, a concentration on the order of several percent is quite possible.

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*The statistical weight of the configuration, in which a noticeable number of cells contain two or more protons, is negligible.

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102