THE MOTION OF A CONDUCTING PISTON IN A MAGNETOHYDRODYNAMIC MEDIUM

I. A. AKHIEZER and R. V. POLOVIN

Physico-technical Institute, Academy of Sciences, Ukrainian S.S.R.

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The types of waves excited in a magnetohydrodynamic medium by a uniformly moving conducting plane (piston) are investigated. The question of the resolution of an initial discontinuity is discussed.

THE purpose of the present paper is the study of the character of waves excited in a magnetohydrodynamic medium of infinite conductivity in which a conducting body moves. This problem is a generalization of the piston problem in ordinary hydrodynamics.

We consider the motion of an infinite plane (piston) with constant velocity **u**; the conductivity of the piston is assumed to be infinite. The magnetohydrodynamic waves excited in this case can be either stationary shock waves of compression or self-similar rarefaction waves. A fast shock or a self-similar wave moves forward; behind it comes an Alfven discontinuity and, finally, a slow wave, shock or self-similar.[1] Taking it into account that some of the waves enumerated can be absent, we obtain 17 qualitatively different pictures of motion of the liquid, which are realized for different values of the velocity of the piston. The particular case $H_y = H_z = 0$, $u_x = 0$ (the x axis is directed along the normal to the piston) was studied by Bazer,[2] while the special case $u_y = u_z = 0$ was considered by Lyubarskiĭ and Polovin.[3]

1. The problem of the character of the waves excited by the motion of the piston can be studied in the limiting case $V \ll c$, $u \ll c$, where $\mathbf{V} \equiv \mathbf{H}\sqrt{4\pi\rho}$, **H** is the intensity of the magnetic field, ρ the density, and c the speed of sound.

Upon satisfaction of these conditions, the density discontinuities in the fast and slow shock and self-similar waves will be small, the relations between the discontinuities of magnetohydrodynamical quantities in the shock and the corresponding quantities in the self-similar waves will be the same; the difference between the shock and the self-similar waves in this approximation is only that the density increases in the shock waves ($\Delta_\rho > 0$), while it decreases in the self-similar wave ($\Delta_\rho < 0$).

The relations between the discontinuities of the magnetohydrodynamic quantities in the fast waves (shock or self-similar) have the form

$$\Delta_+ v_x = (c/\rho)\,\Delta_+\rho\,[1 + (\gamma - 3)\,\Delta_+\rho/4\rho],$$
$$\Delta_+ p = c^2\Delta_+\rho\,[1 + (\gamma - 1)\,\Delta_+\rho/2\rho],$$
$$\Delta_+ \mathbf{v}_t = -V_x\mathbf{V}_t\Delta_+\rho/c\rho, \quad \Delta_+\mathbf{H}_t = \mathbf{H}_t\Delta_+\rho/\rho, \qquad (1)$$

where $\gamma \equiv C_p/C_V$, **v** the velocity of the liquid, and p the pressure; the index t denotes the tangential component of the vector.

At the Alfven discontinuity, the relations

$$\Delta_A\mathbf{v}_t = -\Delta_A\mathbf{V}_t, \quad \Delta_A(V_t^2) = \Delta_A\rho = \Delta_A p = \Delta_A v_x = 0. \qquad (2)$$

are satisfied.

Finally, for the slow waves (shock and self-similar), we have the conditions

$$\Delta_- v_x = (V_{1x}/\rho_1)\,\Delta_-\rho, \quad \Delta_- p = c_1^2\Delta_-\rho,$$
$$\Delta_-\mathbf{v}_t = \mathbf{V}_{1t}\,[1 - (1 - 2c_1^2\Delta_-\rho/V_{1t}^2\rho_1)^{1/2}],$$
$$\Delta_-\mathbf{H}_t/\sqrt{4\pi\rho_1} = \mathbf{V}_{1t}\,[(1 - 2c_1^2\Delta_-\rho/V_{1t}^2\rho_1)^{1/2} - 1] \qquad (3)$$

(the index 1 refers to the region lying between the Alfven discontinuity and the slow wave).

Taking into consideration the fact that the medium at infinity is at rest, while on the surface of the piston the boundary condition[2,3] $\mathbf{v} = \mathbf{u}$ ($H_x \neq 0$) is satisfied, we obtain

$$\Delta_+\mathbf{v} + \Delta_A\mathbf{v} + \Delta_-\mathbf{v} = \mathbf{u}. \qquad (4)$$

Making use of Eqs. (1) – (4), we can find the values of $\Delta_+\rho$, $\Delta_-\rho$, $\Delta_A V_t$:

$$\Delta_+\rho/\rho = (u_x/c)\,[1 - (\gamma - 3)\,u_x/4c],$$
$$\frac{\Delta_-\rho}{\rho} = -\frac{u_t^2 - 2\mathbf{u}_t\mathbf{V}_t}{2c^2} + \frac{u_t^2 u_x}{2c^3}(\gamma - 2) - \frac{\mathbf{u}_t\mathbf{V}_t u_x(2\gamma - 5)}{2c^3},$$
$$\Delta_A\mathbf{V}_t = V_{1t}(\mathbf{V}_{1t} - \mathbf{u})/|\mathbf{V}_{1t} - \mathbf{u}| - \mathbf{V}_{1t}. \qquad (5)$$

If $\Delta_+\rho > 0$, then the fast wave is a shock; if $\Delta_+\rho < 0$, then it is self-similar; if $\Delta_+\rho = 0$, then the fast wave is absent. Similar relations hold for the slow waves.

2. Let us consider in more detail the case in which the magnetic field, the velocity of the piston,

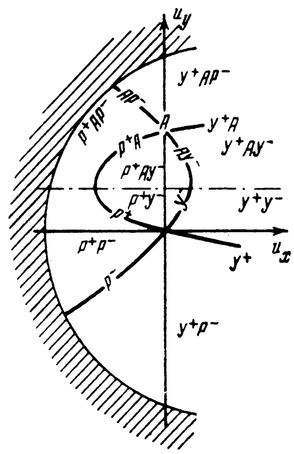

FIG. 1

and the surface normal to it lie in a single plane (the xy plane).

The picture of the motion of the liquid in this case is indicated in Fig. 1 (for definiteness we have set $H_z = 0$, $H_x > 0$, $H_y > 0$). The normal lies along the abscissa, while the tangential component of the velocity of the piston lies along the ordinate. The letters $Y^{\pm}$, $P^{\pm}$, A denote the presence of the shock wave, the rarefaction (self-similar) wave, and the Alfven wave, while the symbol "+" refers to the fast wave and the symbol "−" to the slow wave.

The equations of the curves separating the different regions in Fig. 1 can be obtained from Eqs. (5). Simple transformations lead to the following results.

Equation of the P^+-curve:

$$u_y + \frac{u_x}{c}\frac{V_xV_y}{c+{}^1/_2(\gamma-1)\,u_x} = 0, \quad -\frac{2c}{\gamma-1} < u_x < 0. \qquad (6)$$

Equation of the P^+A-curve:

$$u_y + \frac{u_x}{c}\frac{V_xV_y}{c+{}^1/_2(\gamma-1)\,u_x} - 2V_y\left(1+\frac{\gamma-1}{2}\frac{u_x}{c}\right)^{1/(\gamma-1)} = 0,$$
$$-2c/(\gamma-1) < u_x < 0. \qquad (7)$$

Equation of the Y^+-curve:

$$u_y + u_xV_xV_y/U(U-u_x) = 0; \quad u_x > 0, \quad u_y < 0, \qquad (8)$$

where U is the velocity of the fast shock wave, equal to

$$U = \tfrac{1}{4}(\gamma+1)\,u_x + \left(\tfrac{1}{16}(\gamma+1)^2u_x^2 + c^2\right)^{1/2}.$$

The equation of the Y^+A-curve:

$$u_y + u_xV_xV_y/U(U-u_x) - 2V_y\sqrt{U/(U-u_x)};$$
$$u_x > 0, \; u_y > 0. \qquad (9)$$

The curves AP^-, AY^-, Y^-, P^- are described by the equation

$$u_x + V_xu_y(u_y - 2V_y)/2c^2 = 0, \qquad (10)$$

where we have for AP^-, $u_x < 0$, $u_y > 2V_y$; for AY^-, $u_x > 0$, $V_y < u_y < 2V_y$; for Y^-, we have $u_x > 0$, $0 < u_y < V_y$, and for P^-, $u_x < 0$, $u_y < 0$.

Equations (8), and (9) have a region of applicability that is somewhat larger than $V \ll c$, $u \ll c$; to be precise, they are valid under the condition

$$V \ll c, \quad u/c \ll (c/V_y)^{2/(2-\gamma)}.$$

3. The topological structure of Fig. 1 can be obtained even without calculation, on the basis of qualitative considerations. In this case we dispense with the requirement of the smallness of the magnetic field and velocity of the piston.

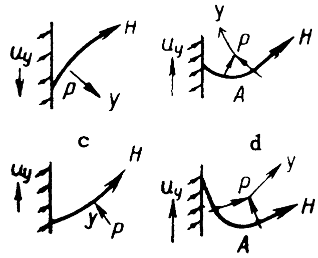

FIG. 2

We first determine through what region the axis of the ordinate passes ($u_x = 0$). Since the magnetic lines of force are "glued" to the particles of the liquid and to the piston, then for $u_y < 0$, the magnetic line of force is deformed as shown in Fig. 2a. The bending of the magnetic line of force leads to the appearance of quasi-elastic tension forces, directed to the side of the concavity (see the arrow in Fig. 2a). Since $v_x = 0$ close to the piston and at infinity, then a compressional wave (shock) arises in front of the arrow, while the rarefaction (self-similar) wave lies behind it. The Alfven wave in the given case does not appear, since H_y close to the piston and at infinity has the same sign. (We recall that the sign of H_y does not change in the shock and self-similar waves,[4] while the contrary is true in the Alfven case.) Thus, for $u_x = 0$, $u_y < 0$, the shock wave travels ahead and the self-similar wave travels behind (the combination Y^+P^-).

Similar considerations show that in the case shown in Fig. 2b, the combination P^+Y^- is realized. Upon increase in the velocity u_y, the sign of H_y close to the piston changes, which leads to the appearance of an Alfven wave (P^+AY^-) (see Fig. 2c). Upon further increase in u_y, the value of $|H_y|$ close to the piston becomes larger than H_y at infinity; in this case the resultant tensile stress is directed out from the piston (see Fig. 2d), which corresponds to the combination Y^+AP^-.

For a sufficiently large value of $|u_y|$, the amplitude of the rarefaction wave becomes so large that the density of the liquid behind the wave vanishes — cavitation sets in.

A picture of the motion of the liquid for $u_x \neq 0$, $u_y \neq 0$, $u_z = 0$ can be obtained by starting out from a consideration of the case $u_x = 0$ given above. Upon increase of u_x, the amplitude of the rarefaction wave decreases, while the amplitude of the compressional wave increases. For a certain value of u_x, the rarefaction wave is transformed into a compressional wave. In a similar way, a decrease of u_x leads to a transformation of the compressional wave into a rarefaction wave. From the qualitative consideration just given, it is also evident that the negative values of u_x make cavitation possible, while positive values hinder it. The region in which cavitation sets in in back of the rarefaction wave is shown in Fig. 1 by the shaded area. The characteristic value of the velocity at which cavitation begins is the velocity of sound.

For $u_t = 0$, $H = 0$, the cavitation condition has the form[5]

$$u_x \leqslant -2c/(\gamma - 1).$$

For $u_x = 0$, $H^2/4\pi \ll p$, numerical calculations[2] lead to the cavitation condition $u_t \geq 3.67\,c$ ($\gamma = 5/3$).

The wave Y^-, AY^-, P^+, AP^+ are possible only for not too large values of the piston velocity; this is connected with the fact that in these waves the value of $|H_y|$ falls off; therefore their amplitude is limited. The waves P^+, AP^+ cannot lead to cavitation, since the relation

$$|\Delta\rho/\rho| < |\Delta H_y/H_y| \leqslant 1$$

is satisfied in the P^+-wave.

The dividing line between the region P^+P^- and P^+AP^- is the curve on which H_y returns to zero after passage of the P^+-wave; the line dividing the regions P^+Y^- from P^+AY^- and Y^+Y^- from Y^+AY^- corresponds to the case in which the tangential magnetic field H_y vanishes in back of the Y^--wave (the special Y^--wave[6]).

4. The problem of the decay of a discontinuity in the initial conditions also reduces to the piston problem. This problem, in the case $\Delta v \ll c$, $\Delta v \ll V$ was considered in reference 7 ($\Delta\mathbf{v}$ is the velocity jump at the initial discontinuity). In the present research, the problem of the decay of an arbitrary discontinuity is investigated under the assumption $\Delta v \ll c$, $V \ll c$ for an arbitrary relation between the characteristic velocities $\Delta\mathbf{v}$ and $\mathbf{V}$. This makes it possible to consider the Alfven discontinuities, in which the angle of deflection of the magnetic field is not small; in particular, it is equal to π if $\Delta\mathbf{v}$, $\mathbf{H}$ on both sides of the discontinuity and the normal to the surface of discontinuity lie in a single plane.

At the initial instant of time $t = 0$, let arbitrary discontinuities of the magnetohydrodynamic quantities Δp, $\Delta\rho$, Δv_x, Δv_t, ΔV_t be given on the surface $x = 0$ (such a situation arises, for example, in the collision of two ionized gaseous masses in a magnetic field). If, the necessary boundary conditions[8] are not satisfied on the surface of the discontinuity, then such a discontinuity decays into seven waves.[7] Velocities of the waves generated are such that not more than three waves can be propagated on each side: in front is the fast magneto-acoustic wave (shock or self-similar); in back is a rotational discontinuity and, finally, the slow magneto-acoustic wave (shock or self-similar). The waves traveling to the left are separated from the waves traveling to the right by a contact discontinuity.

Since the liquid is at rest relative to the contact discontinuity, then the picture of the motion of the liquid on the two sides of the discontinuity will be the same as in the case when the contact discontinuity is replaced by an ideally conducting plane (piston), moving with the same velocity. In contrast with the problem of the piston considered above, the liquid at $x = \pm\infty$ is not at rest, but moves with the given velocities. The velocities of the piston $\mathbf{u}$ relative to the liquid at $+\infty$ and of $\mathbf{u}'$ relative to the liquid at $-\infty$ are related by the self-evident expression:

$$\mathbf{u}' - \mathbf{u} = \Delta\mathbf{v}. \tag{11}$$

In order to determine the velocities of the piston $\mathbf{u}$ and $\mathbf{u}'$, it is necessary to add three more equations, containing $\mathbf{u}$ and $\mathbf{u}'$, to the three equations (11). These equations follow from the condition of continuity of the magnetic field and the pressure on the contact discontinuity:

$$\Delta'_+\mathbf{H}_t + \Delta'_A\mathbf{H}_t + \Delta'_-\mathbf{H}_t = (\Delta_+\mathbf{H}_t + \Delta_A\mathbf{H}_t + \Delta_-\mathbf{H}_t) + \Delta\mathbf{H}_t,$$
$$\Delta'_+ p + \Delta'_- p = (\Delta_+ p + \Delta_- p) + \Delta p \tag{12}$$

(Δ'_+, Δ'_A, Δ'_- denote the jumps of magnetohydrodynamic quantities in the three waves moving to the left. We note that for the waves moving to the left, it is possible to make use of Eqs. (1) to (3) and (5), if $\Delta_\pm\mathbf{v}$, $\Delta_A\mathbf{v}$, $\Delta_\pm p$, $\Delta_\pm\rho$, $\Delta_\pm\mathbf{H}$, $\Delta_A\mathbf{H}$, $\mathbf{u}$ are substituted by $-\Delta'_\pm\mathbf{v}$, $-\Delta'_A\mathbf{v}$, $\Delta'_\pm p$, $\Delta'_\pm\rho$, $\Delta'_\pm\mathbf{H}$, $\Delta'_A\mathbf{H}$, $-\mathbf{u}'$).

If we express the discontinuities of all the quantities in terms of $\mathbf{u}$ and $\mathbf{u}'$ by means of Eqs. (1) to (3) and (5), and make use of Eq. (12), we get

$$u_x + u'_x = -\Delta p / c\rho, \quad \mathbf{u}_t + \mathbf{u}'_t = \Delta \mathbf{V}_t. \qquad (13)$$

Solving Eqs. (11) and (13), we find that the waves traveling to the right of the initial discontinuity will be the same as in the case of motion of a piston in an undisturbed liquid, where the velocity of the piston is equal to

$$u_x = -\tfrac{1}{2}(\Delta p / c\rho + \Delta v_x); \quad \mathbf{u}_t = \tfrac{1}{2}(\Delta \mathbf{V}_t - \Delta \mathbf{v}_t). \qquad (14)$$

Waves traveling to the left from the initial discontinuity will be the same as for a piston moving in the positive direction along the x axis with the velocity $\mathbf{u}''$:

$$u''_x \equiv -u'_x = \tfrac{1}{2}(\Delta p / c\rho - \Delta v_x);$$
$$\mathbf{u}''_t \equiv -\mathbf{u}'_t = -\tfrac{1}{2}(\Delta \mathbf{V}_t + \Delta \mathbf{v}_t) \qquad (15)$$

[in Eqs. (14) and (15), the x axis is directed into the liquid].

The discontinuity in the velocity at the contact discontinuity is shown to be equal to

$$\Delta_c \rho = \Delta \rho - \Delta p / c^2. \qquad (16)$$

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[1] Akhiezer, Lyubarskiĭ, and Polovin, JETP **35**, 731 (1958), Soviet Phys. JETP **8**, 507 (1959).

[2] J. Bazer, Astrophys. J. **128**, 685 (1958).

[3] G. Ya. Lyubarskiĭ and R. V. Polovin, Dokl. Akad. Nauk SSSR **128**, 977 (1959), Soviet Phys.-Doklady **4**, 684 (1960).

[4] R. V. Polovin and G. Ya. Lyubarskiĭ, Ukr. Phys. J. **3**, 571 (1958).

[5] L. D. Landau and E. M. Lifshitz, Механика сплошных сред (Mechanics of Continuous Media) Moscow, Gostekhizdat, 1954, p. 440.

[6] S. I. Syrovatskiĭ, JETP **35**, 1466 (1958), Soviet Phys. JETP **8**, 1024 (1959).

[7] G. Ya. Lyubarskiĭ and R. V. Polovin, JETP **35**, 1291 (1958), Soviet Phys. JETP **8**, 901 (1959).

[8] L. D. Landau and E. M. Lifshitz, Электродинамика сплошных сред (Electrodynamics of Continuous Media) Moscow, Gostekhizdat, 1957, p. 284.

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