THE POSSIBILITY OF THE DETERMINATION OF THE SCATTERING AMPLITUDES OF UNSTABLE PARTICLES AT ZERO ENERGY

V. N. GRIBOV

Leningrad Physico-technical Institute, Academy of Sciences, U.S.S.R.

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The possibility of determining the scattering amplitudes of unstable particles by analyzing the energy and angular distributions of reactions in which they are produced is considered. Reactions are discussed involving the formation of two particles ($\pi + N \rightarrow \Lambda + K$, $\Sigma + K$) and three particles ($N + N \rightarrow N + \Lambda + K$), two of which interact by resonance at low energy.

1. INTRODUCTION

OF the elementary particles known at the present time, the majority are unstable. This circumstance greatly complicates the study of their interaction, inasmuch as it is impossible to carry out direct experiments on the scattering of unstable particles by each other. Therefore, it is important to find indirect means for the measurement of quantities characterizing the interaction of unstable particles.

In the recent papers of Chew,[1] Chew and Low,[2] and Pomeranchuk and Okun'[3] the possibility was considered of measuring the scattering amplitudes of unstable particles by each other by means of an analytic continuation of the cross sections of processes taking place with stable particles or by measurement of the phases of the scattering corresponding to large orbital momentum. This mode of measurement is extremely attractive inasmuch as it can give information on the interaction of unstable particles over a wide range of energies and transferred momenta. However, practical application of this method is at present complicated because of the non-uniqueness of the analytic continuation of cross sections and the difficulties of measurement of phases with large momenta.

Another method of measurement of quantities characterizing the interaction of unstable particles is based on a study of reactions with production of these particles close to threshold or at the limit of the spectrum.[4-6] This method is more limited by reason of the fact that it affords a possibility of determining the scattering amplitudes of unstable particles only for zero energy of their relative motion. However, it is entirely unambiguous and much simpler than the method of Chew et al. Up to recent times, this method has been used only in the case of reactions with production of three particles, two of which interact by resonance at low energy (Migdal and Watson[4]).

In references 5 and 6, reactions were considered with the production of three particles of low energy which interact in nonresonant fashion. For the determination of the scattering amplitudes of the generated particles in certain reactions, for example, $\pi^- + p \rightarrow n + \pi^+ + \pi^-$, $n + \pi^0 + \pi^0$, $p + \pi^- + \pi^0$, it was shown to be sufficient to measure their energy distribution, and in the others ($K^+ \rightarrow 2\pi^0 + \pi^-$, $2\pi^0 + \pi^+$) it was necessary to measure the correlation between directions of their emission.

The possibility of determining amplitudes in these cases is brought about for different physical reasons, as will be made clear in what follows.

In the first case, the possibility of determining the amplitudes is based on the fact that if we consider the matrix element of a reaction (taking place in a certain volume of radius r_0) as a function of the momenta of the produced particle $\mathbf{p}_i$, then the terms linear in $p_i r_0$ are completely determined by the amplitudes of pair scattering. In turn this is explained by the fact that the wave function of the three particles at $p_i \neq 0$, in the region of distances of the order of r_0 between particles, differs from the function for zero energy ($p_i = 0$), with accuracy up to terms linear in $p_i r_0$, only by a factor that is determined by the behavior of the wave function at large distances between the particles.

However, in a number of cases (for example in the reactions $K^+ \rightarrow 2\pi^+ + \pi^-$, $2\pi^0 + \pi^+$) in squaring the matrix element terms linear in $p_i r_0$ drop out from the expression for the cross section. In these cases, measurement of the energy distribution does not allow us to determine the amplitudes of pair scattering. None the less, as was shown in detail in reference 5, thanks to the effect of the centrifugal barrier, correlations between the directions of flight of the particles are fundamentally determined by pair interactions.

In the present work, the possibilities are discussed (Sec. 2) of the determination of scattering amplitudes of unstable particles (Λ, K; Σ, K) from a study of reactions with the formation of two particles ($\pi + N \rightarrow \Lambda + K$, $\pi + N \rightarrow \Sigma + K$). The basic idea used in the study of reactions with the formation of three particles also becomes clear from this section.

In Secs. 3 and 4 we study in detail reactions with formation of three particles (of the type $N + N \rightarrow N + \Lambda + K$) under the assumption that the interaction of two of the particles (N, Λ) has a resonance character for low energies.

In contrast to Migdal,[4] we consider not only the resonance interaction but also the interaction of a third particle (for example, K) with two resonant interacting particles (N, Λ). Account of these interactions leads to corrections to the energy distribution of the produced particles [see, for example, Eq. (52)]. These corrections are of the order $p_i r_0$ and are expressed only by amplitudes of pair interactions and the effective radius of the resonant interaction.

Study of the energy distribution of the particles produced allows us in this case to measure the scattering amplitudes of all three formed particles on each other and also the effective radius of resonance interaction.

2. REACTIONS WITH FORMATION OF TWO LOW-ENERGY PARTICLES

Let us consider a reaction of the type

$$A + B \rightarrow A' + B', \tag{1}$$

where $m_A + m_B < m_{A'} + m_{B'}$ (m = mass of the corresponding particle). The matrix element of such a reaction

$$M = \langle \Psi_E^{(-)}(A', B') \mid \Psi_E^{(+)}(A, B) \rangle, \tag{2}$$

depends on the energy E in the center of mass system (c.m.s.) as an explicit function of the initial and also the final state.

However, as was noted in reference 6, one can show that the matrix element

$$\langle \Psi_{E_0}^{(-)}(A', B') \mid \Psi_E^{(+)}(A, B) \rangle,$$

where E_0, the threshold energy of the reaction (1), is an analytic function of E near $E = E_0$, with the exception of special cases. It can therefore, with accuracy up to terms of order kr_0, be replaced by the matrix element $M_0 = \langle \Psi_{E_0}^{(-)}(A', B') \mid \Psi_{E_0}^{(+)}(A, B) \rangle$, where r_0 is the effective interaction radius, $k^2 = 2\mu(E - E_0)$ and μ is the reduced mass of the particles A′ and B′. Therefore, accurate to terms linear in kr_0, the dependence of the matrix element M on the energy is determined as a function of the final state. This dependence is usually easily found because of the fact that the Schrödinger equation for the function $\Psi_E^{(-)}(A', B')$ contains only k^2 and, consequently, the dependence on k appears only as a result of the boundary condition at infinity. Since only the S-state gives a contribution to (2) at low energy, then (for distances between particles $r \gg r_0$),

$$\Psi_E^{(-)}(A', B') = e^{-i\delta} \frac{\sin(kr + \delta)}{kr},$$

δ is the phase of S-scattering of the particles A′ and B′. For $r_0 \ll r \ll 1/k$,

$$\Psi_E^{(-)}(A', B') = e^{-i\delta} \frac{\sin\delta}{kr} [1 + k\cot\delta + O(k^2r^2)]$$
$$= e^{-i\delta} \frac{\sin\delta}{kr} \left[1 + \frac{r}{a} + O(k^2r^2)\right], \tag{3}$$

where $a = \delta/k|_{k=0}$ is the scattering amplitude, whence it follows that, with accuracy up to terms of order kr_0, the function $\Psi_E^{(-)}$ differs from the function $\Psi_{E_0}^{(-)}$ by the factor

$$F = e^{-i\delta} \frac{\sin\delta}{ka} \text{ и } M = \langle \Psi_E^{(-)}(A', B') \mid \Psi_{E_0}^{(+)}(A, B) \rangle$$
$$= F^* \langle \Psi_{E_0}^{(-)} \mid \Psi_{E_0}^{(+)} \rangle. \tag{4}$$

However, if $a \sim r_0$ (nonresonant case), then we may expand $e^{-i\delta}\sin\delta$ in a series with the accuracy considered thus far. Then $F = 1 - ika$ and, consequently, the cross section of reaction (1) is

$$\sigma(A + B \rightarrow A' + B') = k(1 + 2k \operatorname{Im} a) \mid M_0 \mid^2. \tag{5}$$

Thus, by measuring the cross section of the reaction (1), one can determine $\operatorname{Im} a = (k/4\pi)\sigma'$; σ' is the total interaction cross section of particles A′ and B′, which, because of the possibility of the reaction $A' + B' \rightarrow A + B$, is proportional to 1/k. If other channels than $A' + B' \rightarrow A + B$ are possible in the interaction of A′ and B′, then σ' is not expressed directly in terms of σ.

Evidently the most interesting reactions of this type are

$$\pi^- + p \rightarrow \Lambda^0 + K^0, \tag{6}$$

$$\pi^+ + p \rightarrow \Sigma^+ + K^+. \tag{6'}$$

Since the reactions

$$\Lambda^0 + K^0 \rightarrow N + \pi, \quad N + 2\pi,$$
$$N + 3\pi, \quad N + 4\pi, \tag{7}$$

are possible at zero energies of Λ^0 and K^0, then σ' is not expressed directly in terms of σ. Furthermore, from the unitarity property of the S-matrix, considering that the contribution of the last two reactions of (7) is small because of the smallness of the phase volume, we find that

$$\operatorname{Im} a = (k/4\pi)\{\sigma(\Lambda^0, K^0 \to N, \pi) + \sigma(\Lambda^0, K^0 \to N, \pi, \pi)\}. \quad (8)$$

Thus, by measuring the dependence of the cross section of reaction (5) on energy, we measure the cross section of the reaction $\Lambda^0 + K^0 \to N + \pi + \pi$ at low energy of Λ^0 and K^0.

The reaction (6′) can be considered in an analogous manner. One deficiency of this method is that only the imaginary part of the scattering amplitude is determined. As will be seen below, the real part of a can be determined from a study of the reactions

$$p + p \to \Lambda^0 + K^+ + p, \quad p + p \to \Sigma^+ + K^+ + n,$$
$$p + p \to \Sigma^+ + K^0 + p, \qquad p + p \to \Sigma^0 + K^+ + p. \quad (9)$$

Other interesting possibilities of measurement of scattering amplitudes of unstable particles are uncovered if several channels with neighboring thresholds are possible in a reaction of type (1). As an example, let us consider a reaction with the formation of Σ and K particles:

$$\pi^- + p \to \Sigma^- + K^+, \quad (10a)$$
$$\pi^- + p \to \Sigma^0 + K^0. \quad (10b)$$

In this case it is immediately advantageous to make use of isotopic invariance. The state (π^-, p) is the superposition of states with isotopic spin $T = 1/2$ and $T = 3/2$. For matrix elements with the definite isotopic spins $M_{1/2}$ and $M_{3/2}$, one can write down relations that are analogous to (4):

$$M_{1/2} = (1 + ika^*_{1/2})\, M^0_{1/2}, \quad M_{3/2} = (1 + ika^*_{3/2})\, M^0_{3/2}, \quad (11)$$

$a_{1/2}$ and $a_{3/2}$ are the scattering amplitudes of Σ and K in the states with isotopic spin 1/2 and 3/2.

From this it is easy to obtain the following expression for the cross section of the reactions (10a) and (10b):

$$\sigma_{-+} = \frac{1}{3} k\, |M^0_{3/2}|^2 (1 + 2\sqrt{2}\, x \cos\varphi + 2x^2)$$
$$\times \left[1 - 2k \operatorname{Im} \frac{(a^*_{3/2} + \sqrt{2} a^*_{1/2} e^{i\varphi})(1 + \sqrt{2} x e^{-i\varphi})}{1 + 2\sqrt{2} x \cos\varphi + 2x^2}\right], \quad (12)$$

$$\sigma_{00} = \frac{2}{3} k\, |M^0_{3/2}|^2 \left(1 - \sqrt{2}\, x \cos\varphi + \frac{x^2}{2}\right)$$
$$\times \left[1 - 2k \frac{(a^*_{3/2} + x a^*_{1/2} e^{i\varphi} / \sqrt{2})(1 - x e^{-i\varphi} / \sqrt{2})}{1 - \sqrt{2}\, x \cos\varphi + x^2/2}\right], \quad (13)$$

where x, φ are the modulus and phase relation of the matrix elements $M^0_{1/2}/M^0_{3/2}$. The values of x and φ can be determined, for example, from the ratio of the cross section of reactions (10a) and (10b) to the cross section of the reaction (6′) (in which the same $M_{3/2}$ and $a_{3/2}$ enter). Then, by measuring the energy dependence of the cross section one can, by means of (12) and (13), find two relations between $a_{1/2}$ and $a_{3/2}$. Inasmuch as $\operatorname{Im} a_{3/2}$ is determined independently from the cross section of the reaction (6′), then three unknown quantities enter into these two relations. For a unique determination of all the quantities, it is necessary to study simultaneously reactions with the production of three particles.

3. REACTIONS WITH THE FORMATION OF THREE LOW-ENERGY PARTICLES, TWO OF WHICH UNDERGO RESONANCE INTERACTION

a) Nonresonant interaction

To establish the energy distribution in a reaction of the type

$$A + B \to A' + B' + C' \quad (14)$$

at energies close to threshold, one can proceed in a fashion similar to what was done for binary reactions. The wave function $\Psi_E^{(-)}$ of the S-state of the three particles, A′, B′, C′ in the center-of-mass system (in what follows we shall enumerate these with the indices 1, 2, 3) has the form

$$\Psi_E^{(-)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \Phi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) + \frac{a_{12}}{\rho_{12}} \exp(-ik_{12}\rho_{12}) \frac{\sin p_3\rho_3}{p_3\rho_3}$$
$$+ \frac{a_{13}}{\rho_{13}} \exp(-ik_{13}\rho_{13}) \frac{\sin p_2\rho_2}{p_2\rho_2} + \frac{a_{23}}{\rho_{23}} \exp(-ik_{23}\rho_{23}) \frac{\sin p_1\rho_1}{p_1\rho_1} \quad (15)$$

at large distances between all particles. The spin variables are omitted everywhere in what follows for simplicity; ρ_{ik} is the distance between particles "i" and "k", ρ_l is the distance from the particle l to the center of mass of the two other particles,

$$\mathbf{k}_{il} = (m_l \mathbf{p}_i - m_i \mathbf{p}_l)/(m_i + m_l), \quad (15a)$$

$\mathbf{p}_i$ are the momenta of the particles in the c.m.s., a_{il} are the amplitudes of scattering of pairs of particles at zero energy, m_l are the masses of the particles, and $\Phi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ is the wave function of three free particles with a total orbital momentum equal to zero and with a fixed energy for each particle. The function $\Phi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ is the mean value of the function

$$\exp(i\mathbf{p}_1\mathbf{r}_1 + i\mathbf{p}_2\mathbf{r}_2 + i\mathbf{p}_3\mathbf{r}_3) = \exp(i\mathbf{k}_{12}\boldsymbol{\rho}_{12} + i\mathbf{p}_3\boldsymbol{\rho}_3) = \ldots \quad (15b)$$

over all orientations of the plane in which the vectors $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$ lie. If the state of the three particles is to be characterized not by the momenta $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$, but by their magnitudes p_1, p_2, p_3, and the Euler angles θ_i of the coordinate system that has two axes in the plane of these vectors, then

$$\Phi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \Phi_{p_1, p_2, p_3}(\rho_{12}, \rho_{13}, \rho_{23})$$
$$= \frac{1}{8\pi^2}\int \exp(i\mathbf{p}_1\mathbf{r}_1 + i\mathbf{p}_2\mathbf{r}_2 + i\mathbf{p}_3\mathbf{r}_3)\,d\theta_i. \quad (16)$$

Equation (16) is almost self-evident, and can easily be obtained from equations which will be presented below. We note only that it is valid under the conditions

$$\rho_{il} \gg r_0, \quad \rho_{il} \gg a_{il}. \quad (17)$$

We assume as in the case of two particles that the distances ρ_{il} are such that

$$1/k_{il} \gg \rho_{il} \gg a_{il},\ r_0. \quad (18)$$

Then, expanding in a series and limiting ourselves to terms that are linear in a/ρ, ka, we obtain

$$\Psi_E^{(-)} = 1 + a_{12}/\rho_{12} + a_{13}/\rho_{13} + a_{23}/\rho_{23} - ik_{12}a_{12}$$
$$- ik_{13}a_{13} - ik_{23}a_{23} = (1 - ik_{12}a_{12} - ik_{13}a_{13} - ik_{23}a_{23})$$
$$\times (1 + a_{12}/\rho_{12} + a_{13}/\rho_{13} + a_{23}/\rho_{23}) \quad (19)$$

in place of (15). Inasmuch as terms linear in kr_0 cannot appear from the interaction at small distances, as has already been noted, then even at small distances between the particles, the function $\Psi_E^{(-)}$ differs from the function $\Psi_{E_0}^{(-)}$ only in the factor $1 - ik_{12}a_{12} - ik_{13}a_{13} - ik_{23}a_{23}$. This result was used in references 5 and 6.

However, as is well known, in the interaction of nucleons, the scattering amplitude at zero energy $a \gg r_0$ and the condition $ka \ll 1$ are observed only in a very narrow range of energies. Therefore it is necessary to obtain a result which is free from the restriction $ka \ll 1$ (the restriction $kr_0 \ll 1$, naturally, remains).

In the case in which all the amplitudes $a_{il} \gg r_0$, the problem becomes very complicated[7] and therefore we shall limit ourselves to the case in which only a single amplitude $a_{12} \gg r_0$, while the amplitudes a_{13}, $a_{23} \sim r_0$. This case takes place for reactions of the type $N + N \rightarrow N + N + \pi$, and in the reactions

$$N + N \rightarrow N + \Lambda + K, \quad N + N \rightarrow N + \Sigma + K, \quad (20)$$

if we assume that all the baryons interact in resonant fashion at low energies. Since the simple formula (15) is no longer useful in this case, more detailed analysis of the wave function of the three-particle system is necessary to obtain results.

b) The System of Equations for the Wave Function of Three Particles

For the investigation of the wave function of three particles, it is convenient to introduce the Jacobi coordinates $\boldsymbol{\rho}_{12}$, $\boldsymbol{\rho}_3$, or $\boldsymbol{\rho}_{13}$, $\boldsymbol{\rho}_2$, or $\boldsymbol{\rho}_{23}$, $\boldsymbol{\rho}_1$. In the variables $\boldsymbol{\rho}_{12}$ and $\boldsymbol{\rho}_3$, the Schrödinger equation has the form

$$\Big[-\frac{1}{2\mu_{12}}\nabla^2_{\rho_{12}} - \frac{1}{2\mu_3}\nabla^2_{\rho_3} + V_{12}(\rho_{12}) + V_{13}(\rho_{13})$$
$$+ V_{23}(\rho_{23}) + V_{123}\Big] + \Psi_E = E\Psi_E, \quad (21)$$
$$1/\mu_{12} = 1/m_1 + 1/m_2,\ 1/\mu_3 = 1/m_3 + 1/(m_1 + m_2).$$

We note that the assumption as to the existence of interaction potentials is introduced only for simplicity and final results are not dependent on it.

In order to take the asymptotic conditions into account explicitly (since we are interested in functions of the final state, Ψ_E at infinity must have the form of an incident plus an outgoing wave), it is convenient to write down the equation in integral form:

$$\Psi_E(\boldsymbol{\rho}_{12}, \boldsymbol{\rho}_3) = \Phi(\boldsymbol{\rho}_{12}, \boldsymbol{\rho}_3)$$
$$- \int G(\boldsymbol{\rho}_{12} - \boldsymbol{\rho}'_{12}, \boldsymbol{\rho}_3 - \boldsymbol{\rho}'_3)\,V_{12}(\rho'_{12})\,\Psi_E(\boldsymbol{\rho}'_{12}, \boldsymbol{\rho}'_3)\,d^3\rho'_{12}\,d^3\rho'_3$$
$$- \int G(\boldsymbol{\rho}_{13} - \boldsymbol{\rho}'_{13}, \boldsymbol{\rho}_2 - \boldsymbol{\rho}'_2)\,V_{13}(\rho'_{13})\,\Psi_E(\boldsymbol{\rho}'_{13}, \boldsymbol{\rho}'_2)\,d^3\rho'_{13}\,d^3\rho'_2$$
$$- (1 \rightarrow 2) - \int G(\boldsymbol{\rho}_{12} - \boldsymbol{\rho}'_{12}, \boldsymbol{\rho}_3 - \boldsymbol{\rho}'_3)\,V_{123}(\boldsymbol{\rho}'_{12}, \boldsymbol{\rho}'_3)$$
$$\times \Psi_E(\boldsymbol{\rho}'_{12}, \boldsymbol{\rho}'_3)\,d^3\rho'_{12}\,d^3\rho'_3. \quad (22)$$

$\Phi(\boldsymbol{\rho}_{12}, \boldsymbol{\rho}_3)$ is a function defined by (16),

$$G(\rho_{12}, \rho_3) = G(\rho_{13}, \rho_2) = G(\rho_{23}, \rho_1)$$
$$= \int \frac{d^3p\,d^3k}{(2\pi)^6}\,\frac{\exp\{i(\mathbf{p}\boldsymbol{\rho}_3 + \mathbf{k}\boldsymbol{\rho}_{12})\}}{p^2/2\mu_3 + k^2/2\mu_{12} - E + i\varepsilon}. \quad (23)$$

The following expressions will also be convenient:

$$G(\rho_{12}, \rho_3) = \frac{\mu_{12}}{2\pi}\int \frac{d^3p}{(2\pi)^3}\,e^{i\mathbf{p}\boldsymbol{\rho}_3}\,\frac{e^{-ik_p\rho_{12}}}{\rho_{12}}$$
$$= -\frac{i}{8\pi^2}(\mu_{12}\mu_3)^{1/2}\,\frac{\varkappa^2 H_2^{(2)}\big(\varkappa\sqrt{\gamma_{12}\rho_{12}^2 + \gamma_3\rho_3^2}\big)}{\gamma_{12}\rho_{12}^2 + \gamma_3\rho_3^2},$$
$$k_p = \sqrt{2\mu_{12}(E - p^2/2\mu_3)},$$
$$\varkappa = \sqrt{2(\mu_{12}\mu_3)^{1/2}E},\ \gamma_{12} = 1/\gamma_3 = (\mu_{12}/\mu_3)^{1/2}, \quad (24)$$

$H_2^{(2)}$ are the Hankel functions. All the quadratic roots of the type k_p entering into what follows are determined for $p^2/2\mu_3 > E$ as $-i\sqrt{p^2/2\mu_3 - E}$.

In the matrix element of production of three particles in which we are interested, the wave function Ψ_E enters in the region $\rho_{12} \sim \rho_3 \sim r_0$. We want to prove that in this region, with accuracy up to terms linear in κr_0, Ψ_E differs from Ψ_0 only by an energy dependent factor; we want to find an expression for this factor in terms of the amplitude of the pair interaction ($E_0 = 0$).

In order to accomplish this, it suffices to show that Ψ_E differs from Ψ_0 by a factor in region

$$1/\varkappa^2 \gg \rho_{12}^2 + \rho_3^2 \gg r_0^2. \quad (25)$$

In this region one can always neglect the contribution of the three particle interaction [the last term in (22)]. Actually, by making use of the explicit expression of the Green's function (24), we obtain the result that the last term is equal to

$$\frac{-i(\mu_{12}\mu_3)^{1/2}}{(\gamma_{12}\rho_{12}^2+\gamma_3\rho_3^2)^2}\int V_{123}(\rho_{12}', \rho_3')\,\Psi(\rho_{12}', \rho_3')\,d^3\rho_{12}'\,d^3\rho_3'$$

$$\sim \frac{r_0^4}{(\gamma_{12}\rho_{12}^2+\gamma_3\rho_3^2)^2}\overline{\Psi}(r_0, r_0),$$

if $V_{123}r_0^2(\mu_{12}\mu_3)^{1/2} \sim 1$. If the interactions of particles 1, 3; 2, 3 and 1, 2, 3 are nonresonant, then $\overline{\Psi}(r_0, r_0) \sim a_{12}/r_0$. Consequently, it is of the order $a_{12}r_0^3/(\gamma_{12}\rho_{12}^2+\gamma_3\rho_3^2)^2$, which, as will be seen, is much smaller than the contribution from the two particle interactions. Omitting this term, we consider in more detail the remaining terms in Eq. (22). In this case, it is easy to make note of the following important circumstance: if the variables on the left hand side are changed in the region (25) or even simply in the region $\rho_{12}^2+\rho_3^2 \gg r_0^2$, then the function appears in one of the three regions $\rho_{12} \sim r_0$, $\rho_3 \gg r_0$; $\rho_{13} \sim r_0$, $\rho_2 \gg r_0$; $\rho_{23} \sim r_0$, $\rho_1 \gg r_0$, on the right hand side under the integral. Therefore, if we were to know Ψ in all these regions, then we would find Ψ in the region $\rho_{12}^2+\rho_3^2 \gg r_0^2$ and, consequently, in the region (25).

We shall show that in each of these regions, respectively, the function Ψ_E has the form $\varphi_{12}(\rho_{12})\Psi_3(\rho_3)$, $\varphi_{13}(\rho_{13})\Psi_2(\rho_2)$, $\varphi_{23}(\rho_{23})\Psi_1(\rho_1)$ with accuracy up to terms linear in κr_0, where φ_{il} is the wave function of the particles "i" and "l" at zero energy, while the functions $\Psi_i(\rho_i)$ satisfy some set of equations in which as unknown parameters there appear only the amplitudes of pair interactions. For solution of these equations it is shown that if $\rho_i \ll 1/\kappa$, then all the functions $\Psi_i(\rho_i)$ differ from their values for $E = 0$ by one and the same factor. The latter also leads to the result that the entire function $\Psi(\rho_{12}, \rho_3)$ differs from its value at $E = 0$ by the same factor. Simultaneously, the functions $\Psi_i(\rho_i)$ are found; these can be used in a number of other problems.

In order to prove these assertions, we consider as an example the region $\rho_{12} \sim r_0$, $\rho_3 \gg r_0$.

In this region we want to expand the right hand side of (22) in powers of κr_0 and r_0/ρ_3. For this purpose, it is convenient to represent the function $G(\rho_{12}-\rho_{12}', \rho_3-\rho_3')$ in the form

$$G(\rho_{12}-\rho_{12}', \rho_3-\rho_3') = \delta(\rho_3-\rho_3')\frac{\mu_{12}}{2\pi|\rho_{12}-\rho_{12}'|} + \frac{\mu_{12}}{2\pi}\int\frac{d^3p}{(2\pi)^3}e^{ip(\rho_3-\rho_3')}\frac{\exp\{ik_p|\rho_{12}-\rho_{12}'|\}-1}{|\rho_{12}-\rho_{12}'|}. \quad (26)$$

We can now carry out the indicated expansion in all terms except that containing $\delta(\rho_3-\rho_3')$. In first approximation, we obtain $(\rho_{12} \sim r_0,\ \rho_3 \gg r_0)$:

$$\Psi(\rho_{12}, \rho_3) = \Psi_3(\rho_3) - \frac{\mu_{12}}{2\pi}\int\frac{V_{12}(\rho_{12}')}{|\rho_{12}-\rho_{12}'|}\Psi(\rho_{12}', \rho_3)\,d^3\rho_{12}', \quad (27)$$

$$\Psi_3(\rho_3) = \frac{\sin p_3\rho_3}{p_3\rho_3} - \frac{\mu_{12}}{2\pi}\int G_{12}(\rho_3-\rho_3')\Psi(\rho_{12}', \rho_3')V_{12}(\rho_{12}')\cdot$$
$$\times d^3\rho_{12}'d^3\rho_3' - \int G\left(\rho_3, -\frac{\mu_{13}}{m_1}\rho_3-\rho_2'\right)V_{13}(\rho_{13}')\Psi(\rho_{13}', \rho_2')d^3\rho_{13}'d^3\rho_2'$$
$$-\int G\left(\rho_3, -\frac{\mu_{23}}{m_2}\rho_3-\rho_1'\right)V_{23}(\rho_{23}')\Psi(\rho_{23}', \rho_1')\,d^3\rho_{23}'d^3\rho_1', \quad (28)$$

where

$$G_{12}(\rho_3-\rho_3') = -i\int\frac{d^3p}{(2\pi)^3}e^{ip(\rho_3-\rho_3')}\sqrt{2\mu_{12}(E-p^2/2\mu_3)}$$

and the relations among ρ_{13}, ρ_2, ρ_{23}, ρ_1 and ρ_{12}, ρ_3, and the smallness of ρ_{12} are taken into account in the latter terms of (28). But it follows immediately from (27) that $\Psi(\rho_{12}, \rho_3) = \varphi_{12}(\rho_{12})\Psi_3(\rho_3)$ for $\rho_{12} \sim r_0$, $\rho_3 \gg r_0$, where $\varphi_{12}(\rho_{12})$ satisfies the equation

$$\varphi_{12}(\rho_{12}) = 1 - \frac{\mu_{12}}{2\pi}\int\frac{V_{12}(\rho_{12}')}{|\rho_{12}-\rho_{12}'|}\varphi_{12}(\rho_{12}')\,d^3\rho_{12}', \quad (29)$$

i.e., it is the wave function of particles 1 and 2 for zero energy.

Repeating these discussions for two other regions, we obtain

$$\Psi(\rho_{13}, \rho_2) = \varphi_{13}(\rho_{13})\Psi_2(\rho_2) \text{ for } \rho_{13} \sim r_0,\ \rho_2 \gg r_0,$$
$$\Psi(\rho_{23}, \rho_1) = \varphi_{23}(\rho_{23})\Psi_1(\rho_1) \text{ for } \rho_{23} \sim r_0,\ \rho_1 \gg r_0 \quad (30)$$

with the expressions for $\Psi_1(\rho_1)$ and $\Psi_2(\rho_2)$ obtained from (28) by substitution of indices.

However, if we now take it into account that the wave function enters the right hand side of (28) in precisely the regions under consideration, then we can substitute (29) and (30) in (28). In this case we obtain the following relation among the functions Ψ_1, Ψ_2, and Ψ_3:

$$\Psi_3(\rho_3) = \frac{\sin p_3\rho_3}{p_3\rho_3} + a_{12}\int G_{12}(\rho_3-\rho_3')\Psi_3(\rho_3')\,d^3\rho_3'$$
$$+ \frac{2\pi}{\mu_{13}}a_{13}\int G\left(\rho_3, -\frac{\mu_{13}}{m_1}\rho_3-\rho_2'\right)\Psi_2(\rho_2')\,d^3\rho_2'$$
$$+ \frac{2\pi}{\mu_{23}}\int G\left(\rho_3, -\frac{\mu_{23}}{m_2}\rho_3-\rho_1'\right)\Psi_1(\rho_1')\,d^3\rho_1',$$

$$a_{il} = -\frac{\mu_{il}}{2\pi}\int V_{il}(\rho_{il}')\varphi_{il}(\rho_{il}')\,d^3\rho_{il}', \quad (31a)$$

where a_{il} is the scattering amplitude of the particles "i" and "l" at zero energy.

Substituting (29) — (30) in the expressions for $\Psi_2(\rho_2)$ and $\Psi_1(\rho_1)$ (which we have not written out), we obtain

$$\Psi_2(\rho_2) = \frac{\sin p_2\rho_2}{p_2\rho_2} + a_{13}\int G_{13}(\rho_2-\rho_2')\Psi_2(\rho_2')\,d^3\rho_2'$$
$$+ \frac{2\pi}{\mu_{12}}\int G\left(\rho_2, -\frac{\mu_{12}}{m_1}\rho_2-\rho_3'\right)\Psi_3(\rho_3')\,d^3\rho_3'$$
$$+ \frac{2\pi}{\mu_{23}}\int G\left(\rho_2, -\frac{\mu_{23}}{m_3}\rho_2-\rho_1'\right)\Psi_1(\rho_1')\,d^3\rho_1', \quad (31b)$$

$$\Psi_1(\rho_1) = \frac{\sin p_1\rho_1}{p_1\rho_1} + a_{23}\int G_{23}(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_1')\,\Psi_1(\rho_1')\,d^3\rho_1'$$

$$+\frac{2\pi}{\mu_{12}}\int G\left(\rho_1, -\frac{\mu_{12}}{m_2}\boldsymbol{\rho}_1 - \boldsymbol{\rho}_3'\right)\Psi_3(\rho_3')\,d^3\rho_3'$$

$$+\frac{2\pi}{\mu_{13}}\int G\left(\rho_1, -\frac{\mu_{13}}{m_3}\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2'\right)\Psi_2(\rho_2')\,d^3\rho_2'. \quad (31c)$$

Equations (31a), (31b), and (31c) represent a system of three integral equations with functions $\Psi_1(\rho)$, $\Psi_2(\rho)$, and $\Psi_3(\rho)$. This system of equations is identical with the system of equations obtained by Skornyakov and Ter-Martirosyan[7] with the help of boundary conditions for the wave functions in the case in which all three interactions have a resonance character. As is seen from the foregoing, these equations follow naturally from the initial Schrödinger equation without any assumptions on the resonance of all interactions. In order to estimate the order of magnitude of the individual terms in each of these equations, and also the order of magnitude of the terms neglected in their derivation, we consider their solution by the method of successive approximations. For this purpose, we substitute, on the right hand side of (31a) for example,

$$\Psi_1^{(0)} = \frac{\sin p_1\rho_1}{p_1\rho_1}, \quad \Psi_2^{(0)} = \frac{\sin p_2\rho_2}{p_2\rho_2}, \quad \Psi_3^{(0)} = \frac{\sin p_3\rho_3}{p_3\rho_3}. \quad (32)$$

By making use of (24), we obtain

$$\Psi_3(\rho_3) = \frac{\sin p_3\rho_3}{p_3\rho_3}(1 - ik_{12}a_{12}) + \frac{a_{13}}{\rho_3}e^{ik_{13}\rho_3}\frac{\sin(p_2\rho_3\mu_{13}/m_1)}{p_2\rho_3\mu_{13}/m_1}$$

$$+\frac{a_{23}}{\rho_3}e^{ik_{23}\rho_3}\frac{\sin(p_1\rho_3\mu_{23}/m_2)}{p_1\rho_3\mu_{23}/m_2}, \quad (33)$$

from which it follows that if all the interactions are nonresonant ($a_{il} \sim r_0$), then the correction terms $\sim \kappa r_0$, $r_0/\rho_3 \ll 1$ and the application of successive approximations is valid. The neglected terms in this case are of the order $(\kappa r_0)^2$, $r_0\kappa r_0/\rho_3$, r_0^2/ρ_3^2.

If $r_0 \ll \rho_3 \ll 1/\kappa$, then

$$\Psi_3(\rho_3) = (1 - ik_{12}a_{12} - ik_{13}a_{13} - ik_{23}a_{23})(1 + a_{13}/\rho_3 + a_{23}/\rho_3) \quad (34)$$

in agreement with (20).

When all three interactions are resonant ($a_{il} \gg r_0$, $\kappa a_{il} \sim 1$), none of the correction terms is small, and it is necessary to find a method of exact solution of Eqs. (31). The neglected terms in this case are of the order of κr_0 and r_0/ρ_3.

In our present work, we are interested in the case in which only one of the interactions has a resonance character ($a_{12} \gg r_0$, $a_{13} \sim a_{23} \sim r_0$). In this case, we can neglect the last two terms in Eq. (31a) in the zeroth approximation. This equation is then easily solved. If we set $\Psi_3^{(0)} = A(\sin p_3\rho_3)/p_3\rho_3$, then it follows from Eq. (31a) that

$$A = (1 + ik_{12}a_{12})^{-1}. \quad (35a)$$

To find $\Psi_2(\rho_2)$, $\Psi_1(\rho_1)$ in this same approximation, it suffices to substitute $\Psi_3^{(0)}(\rho_3)$ in the third term on the right in (31b) and (31c), and to neglect the second and fourth terms. We then have

$$\Psi_1^{(0)}(\rho_1) = \frac{\sin p_1\rho_1}{p_1\rho_1} + \frac{a_{12}}{1+ik_{12}a_{12}}\frac{e^{-ik_{12}\rho_1}}{\rho_1}\frac{\sin(\mu_{12}p_3\rho_1/m_2)}{\mu_{12}p_3\rho_1/m_2} \quad (35b)$$

and, similarly,

$$\Psi_2^{(0)}(\rho_2) = \frac{\sin p_2\rho_2}{p_2\rho_2} + \frac{a_{12}}{1+ik_{12}a_{12}}\frac{e^{-ik_{12}\rho_2}}{\rho_2}\frac{\sin(\mu_{12}p_3\rho_2/m_1)}{\mu_{12}p_3\rho_2/m_1}. \quad (35c)$$

If we assume in (35b) and (35c) that $r_0 \ll \rho_1 \ll 1/\kappa$ and $r_0 \ll \rho_2 \ll 1/\kappa$, respectively, then we obtain

$$\Psi_1^{(0)}(\rho_1) = \frac{1}{1+ik_{12}a_{12}}\left(1 + \frac{a_{12}}{\rho_1}\right)$$

$$\Psi_2^{(0)}(\rho_2) = \frac{1}{1+ik_{12}a_{12}}\left(1 + \frac{a_{12}}{\rho_2}\right), \quad (36)$$

i.e., in these regions the functions differ from their values for zero energy by the same factor.

In order to find the form of $\Psi^{(0)}(\rho_{12}, \rho_3)$ in this region, where the distance between an arbitrary pair of paricles is much larger than r_0, but much smaller than the wavelength, Eqs. (35a) — (35c) can be substituted in the initial equation (22). Direct calculation gives

$$\Psi^{(0)}(\boldsymbol{\rho}_{12}, \boldsymbol{\rho}_3) = \frac{1}{1 + ik_{12}a_{12}}\left(1 + \frac{a_{12}}{\rho_{12}}\right), \quad (37)$$

i.e., it differs from $\Psi^{(0)}$ at $E = 0$ by the same factor.

Thus, in all the regions (25), the functions $\Psi_E^{(0)}(\rho_{12}, \rho_3)$ and $\Psi_0^{(0)}(\rho_{12}, \rho_3)$ differ only by the factor $(1+ik_{12}a_{12})^{-1}$. Consequently, such a situation also prevails in the region $\rho_{12}^2 + \rho_3^2 \sim r_0^2$. Therefore the matrix element of the reaction (15) for the presence of resonance has the form

$$\langle\Psi_E^{(-)}(A', B', C')\,|\,\Psi_E^{(+)}(A, B)\rangle =$$

$$= \frac{1}{1 - ik_{12}a_{12}^*}\langle\Psi_0^{(-)}(A', B', C')\,|\,\Psi_0^{(+)}(A, B)\rangle, \quad (38)$$

in zeroth approximation in κr_0; this is the well-known result of reference 4.

The purpose of the present work is to obtain corrections to (38) of the order of κr_0 which contain the amplitude a_{13} and a_{23}.

a) Corrections to the Matrix Element of Order κr_0

To find corrections of order κr_0, it is first necessary in the solution of Eqs. (31a) — (31c) to take into consideration terms containing a_{13} and a_{23}, and in the second place to consider corrections of order κr_0 from the interaction of particles 1 and 2, which were not treated in the development of these equations (in the derivation of (31a) — (31c), terms of order κr_0 relative to those considered were discarded). However, since the interaction of particles 1 and 2 is a resonant one ($\kappa a_{12} \sim 1$), corrections to it are of the order of the contribution from the other interactions.

Corrections of the second type are easily taken into account since in their computation the interaction with a third particle is unimportant (consideration of this interaction would give a correction to a correction), the interaction of the particles 1 and 2 changes only the factor in the wave function, with accuracy up to terms linear in κr_0 [see Eq. (3)]. Therefore, account of corrections of this type reduces to replacing the factor $(1+ik_{12}a_{12})^{-1}$ in (35a) by the quantity

$$e^{-i\delta_{12}}\frac{\sin\delta_{12}}{k_{12}a_{12}} = (1+ik_{12}a_{12})^{-1}\left\{1-\frac{k_{12}^2 r_0 a_{12}}{2(1+ik_{12}a_{12})}\right\}, \quad (39)$$

r_0 is the effective radius of interaction of particles 1 and 2.

To find corrections of the first type we shall solve (31a) — (31c) by successive approximations relative to a_{13} and a_{23}. In Eq. (31a) we set

$$\Psi_3(\rho_3) = (1+ik_{12}a_{12})^{-1}\frac{\sin p_3\rho_3}{p_3\rho_3} + \Psi_3'(\rho_3). \quad (40)$$

Then the following equation holds for $\Psi_3'(\rho_3)$:

$$\Psi_3'(\rho_3) = a_{12}\int G(\boldsymbol{\rho}_3-\boldsymbol{\rho}_3')\Psi_3'(\rho_3')\,d^3\rho_3' + \Phi_{13}(\rho_3) + \Phi_{23}(\rho_3), \quad (41)$$

where $\Phi_{13}(\rho_3)$ and $\Phi_{23}(\rho_3)$ are the results of substitution in the last two terms of (31a) of the functions $\Psi_1^{(0)}(\rho_1)$ and $\Psi_2^{(0)}(\rho_2)$ from (35b) and (35c);

$$\Phi_{13}(\rho_3) = \Phi_{13}'(\rho_3) + \Phi_{13}''(\rho_3), \quad (41')$$

where $\Phi_{13}'(\rho_3)$ and $\Phi_{13}''(\rho_3)$ are the result of substitution of the first and second terms, respectively, from (35b) in Eq. (31a):

$$\Phi_{13}'(\rho_3) = \frac{a_{13}}{\rho_3}e^{-ik_{13}\rho_3}\frac{\sin p_2'\rho_3}{p_2'\rho_3} \equiv \frac{a_{13}}{2p_2'}\int_{k_{13}-p_2'}^{k_{13}+p_2'} dz\,\frac{e^{-iz\rho_3}}{\rho_3}, \quad (42)$$

$$\Phi_{13}''(\rho_3) = \frac{a_{12}a_{13}}{1+ik_{12}a_{12}}\frac{1}{2p_3'}\int_{k_{12}-p_3'}^{k_{12}+p_3'} dz\int\frac{d^3q}{(2\pi)^3}\frac{\exp\left[-iq\rho_3\frac{\mu_{13}}{m_1}-ik_q\rho_3\right]}{\rho_3(q^2-z_\varepsilon^2)},$$

$$p_2' = \mu_{13}p_2/m_1, \quad p_3' = \mu_{12}p_3/m_1,$$

$$k_q = \sqrt{2\mu_{13}(E-q^2/2\mu_2)}, \quad z_\varepsilon = z-i\varepsilon, \quad \varepsilon\to 0. \quad (43)$$

The function $\Phi_{23}(\rho_3)$ differs from $\Phi_{13}(\rho_3)$ by a permutation of indices.

Equation (41) is easily solved if we proceed to the momentum representation. If

$$\Psi_3'(\rho_3) = 4\pi\int\frac{\sin p\rho_3}{p\rho_3}\Psi_3'(p)\,p^2dp, \quad (44)$$

then

$$\Psi_3'(p) = [\Phi_{13}(p)+\Phi_{23}(p)]/(1+ik_pa_{12}),$$

$$k_p = \sqrt{2\mu_{12}(E-p^2/2\mu_3)} \quad (45)$$

$$\Phi_{13}(p) = \Phi_{13}'(p) + \Phi_{13}''(p), \quad \Phi_{13}'(p) = \frac{a_{13}}{4\pi^2p_2'}\int_{k_{13}-p_2'}^{k_{13}+p_2'} dz\,\frac{1}{p^2-z_\varepsilon^2},$$

$$\Phi_{13}''(p) = \frac{a_{12}a_{13}}{1+ik_{12}a_{12}}$$

$$\times\frac{1}{2p_3'}\int_{k_{12}-p_3'}^{k_{12}+p_3'} dz\int\frac{d^3q}{4\pi^4}\frac{1}{(q^2-z_\varepsilon^2)\left[(\mathbf{p}-\mu_{13}\mathbf{q}/m_1)^2-k_q^2+i\varepsilon\right]} \quad (46)$$

and similarly for $\Phi_{23}(p)$.

As in the previous cases, we are interested in $\Psi_3'(\rho_3)$ for $r_0 \ll \rho_3 \ll 1/\kappa$. In this region, the expression for $\Psi_3'(\rho_3)$ is greatly simplified. We shall first consider the contribution from $\Phi_{13}'(p)$. It can be written in a form that is suitable for further work by separating out the component that does not vanish at $E = 0$:

$$\frac{a_{13}}{\pi p_2'}\int_{k_{13}-p_2'}^{k_{13}+p_2'} dz\int_0^\infty p^2dp\,\frac{\sin p\rho_3}{p\rho_3}$$

$$\times\left[\frac{1}{(p^2-z_\varepsilon^2)(1+ik_pa_{12})}-\frac{1}{p^2(1+\gamma_{12}a_{12}p)}\right]$$

$$+\frac{a_{13}}{\rho_3}\frac{2}{\pi}\int_0^\infty\frac{\sin p\rho_3\,dp}{p(1+\gamma_{12}a_{12}p)}. \quad (47)$$

The first integral vanishes at $E = 0$ while the second does not depend on E. Moreover the first converges at $\rho_3 = 0$. Therefore, in the region $r_0 \ll \rho_3 \ll 1/\kappa$ one can set $\rho_3 = 0$, neglecting terms of order κr_0, r_0/a_{12}. In this case the contribution from $\Phi_{13}'(p)$ takes the form

$$a_{13}F_{13}\left(\kappa a_{12}, \frac{p_2}{\kappa}\right) + \frac{a_{13}}{\rho_3}\frac{2}{\pi}\int_0^\infty\frac{\sin x\,dx}{x(1+x\gamma_{12}a_{12}/\rho_3)}, \quad (48)$$

where

$$F_{13}\left(\kappa a_{12}, \frac{p_2}{\kappa}\right)$$

$$= \frac{1}{\pi p_2'}\int_{k_{13}-p_2'}^{k_{13}+p_2'} dz\int_0^\infty dp\,\frac{a_{12}p^2(\gamma_{12}p-ik_p)+z^2(1+ik_pa_{12})}{(p^2-z_\varepsilon^2)(1+ik_pa_{12})(1+p\gamma_{12}a_{12})}.$$

The contribution from $\Phi_{13}''(p)$ can be similarly transformed, but by a somewhat more awkward method. As shown in the appendix, it has the form

$$\frac{a_{12}a_{13}}{1+ik_{12}a_{12}}G_{13}\left(\kappa a_{12}, \frac{p_3}{\kappa}\right)$$

$$-\frac{ik_{12}a_{12}}{1+ik_{12}a_{12}}\frac{a_{13}}{\rho_3}\frac{2}{\pi}\int_0^\infty\frac{\sin x\,dx}{x(1+x\gamma_{12}a_{12}/\rho_3)}$$

$$+\frac{a_{12}a_{13}C}{(1+ik_{12}a_{12})\rho_3^2}\frac{2}{\pi}\int_0^\infty\frac{\sin x\,ax}{1+x\gamma_{12}a_{12}/\rho_3},$$

$$C = \frac{m_1}{\mu_{13}}\tan^{-1}\sqrt{\frac{\mu_{12}\mu_3}{m_1^2}}, \quad (49)$$

where

$$G_{13}\left(\varkappa a_{12},\frac{p_3}{\varkappa}\right)=-\frac{i}{2\pi p_3'}\int\limits_{k_{12}-p_3'}^{k_{12}+p_3'}dz\int\limits_{-1}^{1}dx\int\limits_{0}^{\infty}p^2dp\left\{\frac{z}{1+ik_pa_{12}}\right.$$

$$\times\left(z^2+2pzx\frac{\mu_{12}}{m_1}+\frac{\mu_{12}}{\mu_{13}}p^2-2\mu_{12}E\right)^{-1}\frac{\mu_{12}}{\mu_{13}}-\frac{z}{p^2(1+p\gamma_{12}a_{12})}$$

$$+\frac{\mu_{12}}{\mu_{13}}\left[2\mu_{12}\left(E-\frac{p^2}{2\mu_{13}}\right)+\frac{\mu_{12}^2}{m_1^2}p^2x^2\right]^{-1/2}$$

$$\times\left[1+\frac{z^2}{\left(\sqrt{2\mu_{12}(E-p^2/2\mu_{13})+\mu_{12}^2p^2x^2/m_1^2}+\mu_{12}px/m_1\right)^2-z_\varepsilon^2}\right]$$

$$\left.\times\frac{1}{1+ik_pa_{12}}-\frac{i}{p(1+p\gamma_{12}a_{12})}\sqrt{\frac{\mu_{12}}{\mu_{13}}}\frac{1}{\sqrt{1-\frac{\mu_{12}\mu_{13}}{m_1^2}x^2}}\right\}.$$

Taking (48) and (49) into account, we obtain the result that for $r_0 \ll \rho_3 \ll 1/\kappa$,

$$\Psi_3'(\rho_3)=\frac{1}{1+ik_{12}a_{12}}\left[a_{13}a_{12}G_{13}+a_{23}a_{12}G_{23}+\frac{a_{13}+a_{23}}{\rho_3}\frac{2}{\pi}\right.$$

$$\left.\times\int\limits_0^\infty\frac{\sin x dx}{x(1+x\gamma_{12}a_{12}/\rho_3)}+\frac{a_{13}+a_{23}}{\rho_3}C\frac{a_{12}}{\rho_3}\frac{2}{\pi}\int\limits_0^\infty\frac{\sin dx}{1+x\gamma_{12}a_{12}/\rho_3}\right]$$

$$+a_{13}F_{13}+a_{23}F_{23}. \qquad (50)$$

Also, considering the correction (39), we conclude that, with the initial accuracy, $\Psi(\rho_{12},\rho_3)$ can be written for $\rho_{12}\sim r_0$, $\rho_3\gg r_0$, and $\rho_3 \ll 1/\kappa$ in the form

$$\Psi(\rho_{12},\rho_3)=L\varphi_{12}(\rho_{12})\left[1+\frac{a_{13}+a_{23}}{\rho_3}\frac{2}{\pi}\int\limits_0^\infty\frac{\sin x dx}{x(1+x\gamma_{12}a_{12}/\rho_3)}\right.$$

$$\left.+\frac{C(a_{13}+a_{23})a_{12}}{\rho_3^2}\frac{2}{\pi}\int\limits_0^\infty\frac{\sin x dx}{1+x\gamma_{12}a_{12}/\rho_3}\right], \qquad (51)$$

where

$$L=\frac{1}{1+ik_{12}a_{12}}\left[1-\frac{1}{2}\frac{k_{12}^2r_0a_{12}}{1+ik_{12}a_{12}}+a_{12}a_{13}G_{13}+a_{12}a_{23}G_{23}\right]$$

$$+a_{13}F_{13}+a_{23}F_{23}. \qquad (52)$$

Thus we have found the correction of order κr_0 to the factor $(1+ik_{12}a_{12})^{-1}$ in the wave function and the correction to the wave function in the region $\rho_{12}\sim r_0$, $\rho_3\gg r_0$ of order r_0/ρ_3, which can be of interest in a number of problems.

To find the corrections to the function $\Psi_2(\rho_2)$, we set

$$\Psi_2(\rho_2)=\Psi_2^{(0)}(\rho_2)+\Psi_2'(\rho_2).$$

To determine $\Psi_2'(\rho_2)$, it is necessary to substitute $\Psi_3'(\rho_3)$ from (44) in the third term on the right-hand side of (31b), with account of the correction (39), and in the second and fourth terms, $\Psi_2^{(0)}(\rho_2)$ and $\Psi_1^{(0)}(\rho_1)$ from (35b) and (35c), respectively.

The expression for $\Psi_2(\rho_2)$ resulting from such substitutions has the form

$$\Psi_2(\rho_2)=L\frac{a_{12}}{\rho_2}\left\{1+\frac{\rho_2}{a_{12}}+\frac{a_{13}}{\rho_2}f_1\left(\frac{a_{12}}{\rho_2}\right)+\frac{a_{23}}{\rho_2}f_2\left(\frac{a_{12}}{\rho_2}\right)\right\},$$

$$f_i\left(\frac{a_{12}}{\rho_2}\right)=\frac{m_i}{\mu_{12}}\frac{2}{\pi}\int\limits_0^\infty\frac{\sin xe^{-\alpha_i x}}{1+\alpha_i x\frac{a_{12}}{\rho_2}}\left(\frac{1}{x}+\frac{Ca_{12}}{\rho_2}\right)dx,\qquad \alpha_i=\frac{m_i}{\sqrt{\mu_{12}\mu_3}}, \qquad (53)$$

for $r_0\ll\rho_2\ll 1/\kappa$, i.e., it contains the same constant factor.

Similar results are obtained for $\rho_{23}\sim r_0$, $r_0 \ll \rho_1 \ll 1/\kappa$ and $r_0\ll\rho_{12}, \rho_{13}, \rho_{23}, \ll 1/\kappa$.

Therefore, as in the previous cases, it can be proved that the matrix element of the reaction (14) has the form

$$\langle\Psi_E^{(-)}(A',B',C')\,|\,\Psi_E^{(+)}(A,B)\rangle$$

$$=L^*\langle\Psi_0^{(-)}(A',B',C')\,|\,\Psi_0^{(+)}(A,B)\rangle. \qquad (54)$$

with accuracy up to terms linear relative to κr_0.

The quantities G_{13}, G_{23}, F_{13}, F_{23} entering into the expression for L depend both on the energy distribution among the particles and on the resonance amplitude of scattering a_{12}. If we introduce the variables

$$u=p_1^2/2\mu_1E,\quad v=p_2^2/2\mu_2E,\quad w=p_3^2/2\mu_3E, \qquad (55)$$

which determine the fraction of the entire kinetic energy which is associated with the relative motion of one of the particles (1, 2, and 3 respectively) and the center of mass of the other two, then it is easy to show that

$$G_{13}=\varkappa_{12}\varkappa_3G(\beta_1,w,a),\quad G_{23}=\varkappa_{12}\varkappa_3G(\beta_2,w,a),$$

$$F_{13}=\varkappa_3F(\beta_1,v,a),\quad F_{23}=\varkappa_3F(\beta_2,u,a), \qquad (56)$$

where

$$\varkappa_{12}=\sqrt{2\mu_{12}E},\quad \varkappa_3=\sqrt{2\mu_3E},\quad \beta_1=\mu_{13}/\mu_3=\mu_{12}/\mu_2,$$

$$\beta_2=\mu_{23}/\mu_3=\mu_{12}/\mu_1,\quad a=\varkappa_{12}a_{12}.$$

The functions F and G, after computation of the part of the intervals entering into (48) and (49), can be written in the form

$$F(\beta,u,a)=\frac{1}{2\pi\sqrt{u(1-\beta)}}\int\limits_{z_1}^{z_2}\frac{zdz}{1+a^2(1-z^2)}\left[2a\sqrt{1-z^2}\arccos z\right.$$

$$\left.-i\pi-\frac{4az}{\sqrt{1+a^2}}\left(\sinh^{-1}a+\frac{i\pi}{2}\right)\right]-\frac{2}{\pi a\sqrt{1+a^2}}$$

$$\times\left[\ln\frac{1}{2}(1+\sqrt{1+a^2})-\frac{i\pi}{2}(\sqrt{a^2+1}-1)\right], \qquad (57)$$

here $z_{1,2}=\sqrt{(1-u)\beta}\mp\sqrt{u(1-\beta)}$;

$$G(\beta,w,a)=\frac{1}{2\pi\sqrt{\beta(1-\beta)}(z_2-z_1)}\int\limits_{z_1}^{z_2}dz\left\{\int\limits_0^1dy\frac{-iy}{1+ia\sqrt{1-y^2}}\right.$$

$$\times\ln\left|\frac{(y\sqrt{1-\beta}+\sqrt{\beta(1-y^2)})^2-z^2\beta}{(y\sqrt{1-\beta}-\sqrt{\beta(1-y^2)})^2-z^2\beta}\right|+\int\limits_1^\infty ydy\left[\frac{1}{1+a\sqrt{y^2-1}}\right.$$

$$\left.\times\tan^{-1}\frac{2\sqrt{\beta(1-\beta)y^2(1-y^2)}}{\beta z^2-y^2(1-2\beta)-\beta}-\frac{1}{1+ay}\tan^{-1}\frac{2\sqrt{\beta(1-\beta)}}{2\beta-1}\right]$$

$$-i\int\limits_0^\infty ydy\frac{1}{1+ia\sqrt{1-y^2}}\left[\ln\frac{\beta z^2+2zy\sqrt{1-\beta}+y^2-\beta}{\beta z^2-2zy\sqrt{1-\beta}+y^2-\beta}\right.$$

$$\left.\left.-\frac{4z\sqrt{1-\beta}}{y(1+ay)}\right]+\pi\theta(1-\beta-z^2\beta)z\int\limits_{y_0}^1\frac{ydy}{1+ia\sqrt{1-y^2}}\right\}, \qquad (58)$$

where

$$z_{1,2} = \sqrt{\beta(1-w)} \mp \sqrt{w(1-\beta)},$$

$$y_0 = \sqrt{\beta(1-z^2\beta)} + z\sqrt{\beta(1-\beta)},$$

$$\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}, \qquad 0 < \tan^{-1} x < \pi.$$

In spite of the crudeness of the expressions for F and G, these functions can be tabulated without great difficulty.

The differential cross section of the reaction (14), as a function of the energy of the third particle (which interacts in nonresonant fashion) for example, and of its angle of emission, which is determined by the angle between the direction of the momentum of the incident particles and the plane in which the momenta of the particles produced lie, has the form

$$\frac{d^2\sigma(E, w, \cos\vartheta)}{dw\, d\cos\vartheta} = |M_0|^2 |L|^2 \sqrt{w(1-w)}\,(E-E_0)^2, \tag{59}$$

$$L = (1 + ik_{12}a_{12})^{-1}\{1 - k_{12}^2 r_0 a_{12}/2(1 + ik_{12}a_{12})$$
$$+ \varkappa_{12}a_{12}[\varkappa_3 a_{13} G(\beta_1, w, a) + \varkappa_3 a_{23} G(\beta_2, w, a)]\}$$
$$+ \varkappa_3 a_{13} F(\beta_1, v, a) + \varkappa_3 a_{23} F(\beta_2, u, a),$$

where ϑ is the angle between $\mathbf{p}_3$ and $\mathbf{k}_{12}$.

For given w and ϑ,

$$v = w(1-\beta_1) + \beta_1(1-w) + 2\sqrt{w(1-w)\beta_1(1-\beta_1)}\cos\vartheta,$$
$$u = w(1-\beta_2) + \beta_2(1-w) - 2\sqrt{w(1-w)\beta_2(1-\beta_2)}\cos\vartheta. \tag{60}$$

Thus the terms in the variables w and ϑ in the square brackets of (59) are responsible only for the energy distribution, while the terms containing $F(\beta_1, v, a)$ and $F(\beta_2, u, a)$ also involve the correlation between the momenta $\mathbf{p}_3$ and $\mathbf{k}_{12}$.

4. ENERGY DISTRIBUTION $p + p \to p + \Lambda^0 + K^+$, $p + p \to N + \Sigma + K$

Let us consider the energy distribution in the following reactions:

$$p + p \to p + \Lambda^0 + K^+, \tag{I}$$

$$p + p \to p + \Sigma^0 + K^+, \tag{IIa}$$

$$p + p \to p + \Sigma^+ + K^0, \tag{IIb}$$

$$p + p \to n + \Sigma^+ + K^+, \tag{IIc}$$

and assume that the interactions (Λ, N) and (Σ, N) are resonant.

In order that this be done, it is necessary to take into account the spin and isotopic variables of the particles generated. In the case of reaction (I), in view of the impossibility of charge exchange, the presence of the isotopic spin is not important. Account of the usual spin leads only to the result that p and Λ^0 can be produced both in the singlet and in the triplet states. In this case a system of three particles of small energy (p, Λ^0, K^+) can depend on the internal parity of the system (Λ^0, K^+) relative to the proton, either in the state $(0^+, 1^+)$, or in the states $(0^-, 1^-)$.

However, inasmuch as two protons, as fermions, cannot be found in the state 1^+, then, depending on the parity of (Λ, K^+) in the final state, either only the singlet state (p, Λ^0) or the singlet plus the triplet states are possible. In the first case, the expression (59) is completely applicable for the reaction (I) if by a_{12} one means the amplitude of scattering of p on Λ^0 in the singlet state — a_0. In the second case, since the states 0^- and 1^- do not interfere, after averaging over the directions of the momenta of the incident particles, the reaction cross section has the form

$$\frac{d^2\sigma(E, w, \cos\vartheta)}{dw\, d\cos\vartheta} = \sqrt{w(1-w)}\,(E-E_0)^2$$
$$\times\left\{\frac{1}{4}|M_0|^2 |L_0|^2 + \frac{3}{4}|M_1|^2 |L_1|^2\right\}, \tag{61}$$

where $|M_0|^2$ and $|M_1|^2$ are the squares of the moduli of the matrix elements of creation in the singlet and triplet states for zero energy. The quantities $|L_0|^2$ and $|L_1|^2$ are expressions entering into (59), in which, in place of a_{12}, there are substituted respectively a_0 and a_1 — the scattering amplitudes of (p, Λ^0) in the singlet and triplet states.

In the case of the reactions (IIa, b, c) the situation is entirely similar in the behavior of the spin variables; however consideration of the isotopic spin materially changes the final result. In this case, the equations (31a, b, c) are more complicated, inasmuch as for $\rho_{il} \sim r_0$ the wave function $\varphi_{il}(\rho_{il})$ of the particles i and l is a superposition of functions with different isotopic spin T_{il} ($\frac{1}{2}$, $\frac{3}{2}$ for N, Σ; 0, 1 for N and K; $\frac{1}{2}$, $\frac{3}{2}$ for Σ and K). We shall not repeat the calculations but merely write down the final result which is almost self-evident.

In place of the matrix element of the reactions (IIa, b, c), which we denote by M_{+0+}, M_{++0}, M_{0++}, we introduce the matrix elements $M_{1/2}$, $M_{3/2}$ of transitions into states with a total isotopic spin T = 1 and isotopic spin of N and Σ, $T_{12} = \frac{1}{2}$ and $\frac{3}{2}$:

$$M_{+0+} = (1/\sqrt{3})\,M_{1/2} - (1/\sqrt{6})\,M_{3/2}, \quad M_{++0} = (\sqrt{3}/2)\,M_{3/2},$$
$$M_{0++} = -\sqrt{2/3}\,M_{1/2} - (1/2\sqrt{3})\,M_{3/2}. \tag{62}$$

We can now write

$$M_{T_{12}} = \sum_{T'_{12}} \langle T_{12} | L | T'_{12}\rangle^* M^0_{T'_{12}} \tag{63}$$

in place of Eq. (54).

To obtain $\langle T_{12} | L | T'_{12} \rangle$ from (52), it suffices to note the following. The term

$$(1 + ik_{12}a_{12})^{-1} [1 - k_{12}^2 r_0 a_{12} / 2 (1 + ik_{12}a_{12})]$$

depends only on the interaction of N, Σ, for which T_{12} coes not change. Therefore, it gives a contribution to $\langle T_{12} | L | T'_{12} \rangle$ of the form

$$(1 + ik_{12}a_{T_{12}})^{-1} [1 - k_{12}^2 r_0^{T_{12}} a_{T_{12}} / 2 (1 + ik_{12}a_{T_{12}})] \times \delta_{T_{12}T'_{12}} \equiv \alpha_{T_{12}} \delta_{T_{12}T'_{12}}, \quad (64)$$

$a_{T_{12}}$, $r_0^{T_{12}}$ are the scattering amplitude and the effective radius in the state with isotopic spin T_{12}. The term $\kappa_3 a_{13} F(\beta_1, v, a) + \kappa_3 a_{23} F(\beta_2, u, a)$, as is easily seen from the derivation, is a contribution from processes in which the interaction of particles 1 and 2 enter, and then the interaction of particles 1 and 3 or 2 and 3. The interaction of particles 1 and 3 or 2 and 3 changes the isotopic spin of the particles 1 and 2 (T_{12}). Therefore a_{13} and a_{23} are replaced by

$$\langle T_{12} | b | T'_{12} \rangle = \sum_{T_{13}} S(T_{12}, T_{13}) b_{T_{13}} S(T_{13}, T'_{12}),$$

$$\langle T_{12} | c | T'_{12} \rangle = \sum_{T_{23}} \widetilde{S}(T_{12}, T_{23}) c_{T_{23}} \widetilde{S}(T_{23}, T'_{12}), \quad (65)$$

where $b_{T_{13}}$ and $c_{T_{23}}$ are the scattering amplitudes of (K, N) and (K, Σ), respectively, in states with a definite isotopic spin.

The coefficients $S(T_{12}, T_{13})$ and $\widetilde{S}(T_{12}, T_{23})$ are determined by the rules of addition of moments and are simply connected with the Racah coefficients, for example

$$S(T_{12}, T_{13}) = \sqrt{(2T_{12}+1)(2T_{13}+1)} W(1, 1/2, 1, 1/2; T_{12}, T_{13}).$$

Simultaneously,

$$F(\beta_1, v, a) \to F(\beta_1, v, \varkappa_{12} a_{T'_{12}}).$$

The remaining terms correspond to the successive interactions of particles 1 and 2, 1 and 3, or 2 and 3, and then again 1 and 2. Therefore, they are replaced by

$$\frac{\varkappa_{12} a_{T_{12}}}{1 + ik_{12}a_{T_{12}}} [\varkappa_3 \langle T_{12} | b | T'_{12} \rangle G(\beta_1, w, \varkappa_{12} a_{T'_{12}}) + \varkappa_3 \langle T_{12} | c | T'_{12} \rangle G(\beta_2, w, \varkappa_{12} a_{T'_{12}})].$$

By making use of all of the above, it is easy to obtain the following relations between $M_{1/2}$ or $M_{3/2}$ and their values at zero energy, $M^0_{1/2}$ or $M^0_{3/2}$,

$$M_{1/2} = \Big\{\alpha_{1/2} + \varkappa_3 \frac{1}{3}(2b_1 + b_0) f_1\Big(\frac{1}{2}, \frac{1}{2}\Big) + \varkappa_3 \frac{1}{9}(8c_{3/2} + c_{1/2}) f_2\Big(\frac{1}{2}, \frac{1}{2}\Big)\Big\} M^0_{1/2} + \varkappa_3 \Big\{\frac{\sqrt{2}}{3}(b_1 - b_0) f_1\Big(\frac{1}{2}, \frac{3}{2}\Big) + \frac{2\sqrt{2}}{9}(c_{3/2} - c_{1/2}) f_2\Big(\frac{1}{2}, \frac{3}{2}\Big)\Big\}^* M^0_{3/2}, \quad (66)$$

$$M_{3/2} = \Big\{\alpha_{3/2} + \varkappa_3 \frac{1}{3}(b_1 + 2b_0) f_1\Big(\frac{3}{2}, \frac{3}{2}\Big) + \varkappa_3 \frac{1}{9}(c_{3/2} + 8c_{1/2}) f_2\Big(\frac{3}{2}, \frac{3}{2}\Big)\Big\}^* M^0_{3/2} + \varkappa_3 \Big\{\frac{\sqrt{2}}{3}(b_1 - b_0) f_1\Big(\frac{3}{2}, \frac{1}{2}\Big) + \frac{2\sqrt{2}}{9}(c_{3/2} - c_{1/2}) f_2\Big(\frac{3}{2}, \frac{1}{2}\Big)\Big\}^* M^0_{1/2},$$

where

$$f_1(T_{12}, T'_{12}) = F(\beta_1, v, \varkappa_{12} a_{T'_{12}}) + \frac{\varkappa_{12} a_{T_{12}}}{1 + ik_{12}a_{T_{12}}} G(\beta_1, w, \varkappa_{12} a_{T'_{12}}),$$

$$f_2(T_{12}, T'_{12}) = F(\beta_2, u, \varkappa_{12} a_{T'_{12}}) + \frac{\varkappa_{12} a_{T_{12}}}{1 + ik_{12}a_{T_{12}}} G(\beta_2, w, \varkappa_{12}, a_{T'_{12}}). \quad (67)$$

Equations (62) and (66) allow us to find the cross section of the three reactions (II) depending on the ratio $M^0_{1/2}/M^0_{3/2}$ and the amplitudes of pair interactions.

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APPENDIX

Calculation of the contribution from Φ''_{13} [Eq. (49)]. To obtain (49) it is convenient first to integrate over the modulus q in (46). In this case we obtain

$$\Phi''_{13}(p) = \frac{a_{12}a_{13}}{1 + ik_{12}a_{12}} \frac{-i}{2p'_3} \int_{k_{12}-p'_3}^{k_{12}+p'_3} dz \int_{-1}^{1} \frac{dx}{4\pi^2} \times \Big\{\frac{z}{z^2\mu_{13}/\mu_{12} + 2pzx\mu_{13}/m_1 + p^2 - 2\mu_{13}E + i\varepsilon} + \frac{\mu_{12}}{\mu_{13}} \frac{1}{\sqrt{2\mu_{12}(E - p^2/2\mu_{13}) + \mu_{12}^2 p^2 x^2 / m_1^2}} \times \Big[1 + \frac{z^2}{(\sqrt{2\mu_{12}(E - p^2/2\mu_{13}) + \mu_{12}^2 p^2 x^2/m_1^2} + \mu_{12}px/m_1)^2 - z_\varepsilon^2}\Big]\Big\}, \quad \text{(a)}$$

where the relation $\mu_{12}\mu_3/m_1^2 + 1 = \mu_3/\mu_{13} = \mu_2/\mu_{12}$ is employed.

The contribution from $\Phi''_{13}(p)$ in $\Psi_3(\rho_3)$ has the form

$$4\pi \int \frac{\sin p\rho_3}{p\rho_3} \frac{\Phi''_3(p)}{1 + ik_p a_{12}} p^2 dp. \tag{b}$$

For $p/\kappa \to \infty$,

$$\frac{\Phi''_{13}(p)}{1 + ik_p a_{12}} = \frac{a_{12}a_{13}}{1 + ik_{12}a_{12}} \left\{ \frac{1}{p(1 + p\gamma_{12}a_{12})} \cdot \right.$$

$$\times \sqrt{\frac{\mu_{12}}{\mu_{13}}} \int_{-1}^{1} \frac{dx}{4\pi^2} \frac{1}{\sqrt{1 - \mu_{12}\mu_{13}x^2/m_1^2}}$$

$$\left. - \frac{ik_{12}a_{12}}{2\pi^2 p^2 (1 + p\gamma_{12}a_{12})} + O\left(\frac{1}{p^4}\right) \right\}. \tag{c}$$

Therefore, if we write down

$$\frac{\Phi''_{13}(p)}{1 + ik_p a_{12}} = \frac{a_{12}a_{13}}{1 + ik_{12}a_{12}} \left\{ G_{13}(p) + \frac{C}{2\pi^2 p (1 + p\gamma_{12}a_{12})} \right.$$

$$\left. - \frac{ik_{12}a_{12}}{2\pi^2 p^2 (1 + p\gamma_{12}a_{12})} \right\}, \quad C = \frac{m_1}{\mu_{13}} \tan^{-1} \sqrt{\frac{\mu_{12}\mu_3}{m_1^2}}, \tag{d}$$

then $G_{13}(p) \sim 1/p^4$ for $p/\kappa \to \infty$ and, consequently, one can set $\rho_3 = 0$ in the term containing G_{13}, in the integral (b) for $\rho_3 \ll 1/\kappa$. Substituting (a) in (b) and taking into account this observation, we obtain the result given in the text [Eq. (49)].

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