# *DISPERSION IN HIGH-DENSITY, HIGH-TEMPERATURE MEDIA*

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The dispersion properties of high-density, high-temperature media are investigated; media of this kind may exist in the inner regions of irregular stars (white super dwarfs). Single photon annihilation and electron pair creation occur at electron densities $N_e \gtrsim 10^{32}$ cm$^{-3}$. At frequencies which satisfy the inequality in Eq. (31) electromagnetic waves are scattered on nucleons rather than electrons. The index of refraction at these frequencies and densities is given approximately by $n \approx 1 + 1.05 \times 10^{-41} N$ (N is the neutron density). Hard Cerenkov radiation can be excited in a medium of this kind. The energy of the Cerenkov photons satisfies the inequality in Eq. (35).

## SCATTERING OF ELECTROMAGNETIC WAVES BY ELECTRONS

We have shown earlier[1] that under certain physical conditions single photon annihilation and electron pair production $\gamma \rightleftarrows e^+ + e^-$ can occur in a scattering medium with a refractive index smaller than unity. In vacuum these processes can take place only in the presence of an additional particle. However, single photon annihilation and pair production do not occur in all media with refractive indices smaller than unity. In addition to this condition there are certain requirements on the density of particles and the temperature of the medium. The density requirement follows directly from a consideration of the meaning of the refractive index in a medium. Obviously, a refractive index is meaningful only if the following condition is satisfied:

$$\lambdabar \gg l, \tag{1}$$

where $l$ is the mean distance between electrons and $\lambdabar$ is the length of the electromagnetic wave divided by $2\pi$.

From (1) and the fact that the production and annihilation of electron pairs require a photon energy greater than $2mc^2$, where m is the mass of the electron, it follows that the effect being discussed here can occur only when the particle density in the medium is sufficiently high:

$$N \gg 8\lambdabar_e^{-3} \approx 1.4 \cdot 10^{32}\ \text{cm}^{-3}, \tag{2}$$

where $\lambdabar_e$ is the electron Compton wavelength divided by $2\pi$. It would appear that densities of this kind are in fact found in the inner regions of white super dwarfs.

It is apparent that atoms are completely ionized under these physical conditions so that the ensemble of electrons constitutes a relativistic ideal gas. At densities given by (2) the mean kinetic energy of electrons is of order $mc^2$ and higher while the ratio of potential energy $Ze^2/r$ to kinetic energy cp is $Ze^2/crp \approx Ze^2/\hbar c \ll 1$.

For the densities of interest here the degeneration temperature of the electron gas is

$$T_0 \gg 4 \cdot 10^{10}. \tag{3}$$

It is difficult to say whether the electron gas in a star is degenerate or nondegenerate. It is possible that there are white dwarfs of both types in nature. These two cases lead to completely different physical results. In the first case, at sufficiently high densities we are dealing with a medium whose dispersion properties at very high frequencies are determined by neutrons rather than by electrons. In a medium of this kind the refractive index is greater than unity; hence the process $\gamma \rightleftarrows e^+ + e^-$ is forbidden. On the other hand, another phenomenon can occur: charged particles moving with velocities exceeding the phase velocity of light can emit hard Cerenkov radiation, with photon energies up to 150 Mev. In the absence of degeneracy, however, the dispersion properties of a medium are always determined by the electrons. In a medium of this kind at densities given by (2) the process $\gamma \rightleftarrows e^+ + e^-$ is allowed while Cerenkov radiation is not. Both of these limiting cases will be discussed in detail below.

When the electron gas is not degenerate it is apparent that the dispersion is always determined by the electrons. As is well known, at frequencies greater than characteristic atomic frequencies the refractive index of a medium is given by the relation

$$n^2 = 1 - 4\pi Ne^2 / m\omega^2. \tag{4}$$

However, this relation cannot be used in the present case because its derivation assumes that the electrons are nonrelativistic whereas the electrons are relativistic in the case at hand.

At densities given by (2) electrons are quasi-classical particles. Provisionally, we may also assume that the quasi-classical approach is valid at relatively low electron densities; in particular we require that the mean distance between particles be somewhat smaller than the Fermi radius of the atom $l \ll Z^{1/3}\hbar^2/me^2$.

In order to calculate the refractive index we must find the additional velocity $\mathbf{v}' = \mathbf{v} - \mathbf{v}_0$ acquired by a free electron under the influence of the electromagnetic field. Solving the Hamiltonian-Jacoby equation for an electron in a field of a plane electromagnetic wave $\mathbf{A}(\mathbf{r}, t) = \mathbf{A}_0 \exp\{i(\mathbf{k}\cdot\mathbf{r} - \omega t)$

$$[\nabla S - (e/c)\,\mathbf{A}]^2 - c^{-2}(\partial S/\partial t)^2 + m^2c^2 = 0 \tag{5}$$

and then computing the velocity $\mathbf{v}$ from the formula

$$(1 - \beta^2)^{-1/2} m\mathbf{v} = \nabla S - (e/c)\,\mathbf{A}, \tag{6}$$

we obtain the additional velocity $\mathbf{v}'$:

$$\mathbf{v}' = -\frac{ec}{E_0}\mathbf{A} + \frac{ec^2(\mathbf{p}_0\mathbf{A})\,\mathbf{k}}{E_0[c(\mathbf{p}_0\mathbf{k}) + kE_0]} - \frac{ec^4(\mathbf{p}_0\mathbf{k})(\mathbf{p}_0\mathbf{A})\,\mathbf{p}_0}{E_0^3[c(\mathbf{p}_0\mathbf{k}) + kE_0]} + \frac{ec^3}{E_0^3}(\mathbf{p}_0\mathbf{A})\,\mathbf{p}_0, \tag{7}$$

where $p_0$ and $E_0 = c(p_0^2 + m^2c^2)^{1/2}$ are the initial momentum and energy.

We now determine the current induced by the electromagnetic field:

$$\mathbf{j}' = e\int \mathbf{v}' f(p_0)\,dp_0 \sin\vartheta\, d\vartheta\, d\varphi, \tag{8}$$

where $f(p)$ is the electron distribution in momentum space.* Substituting Eq. (7) in Eq. (8) and carrying out the integration over angles (i.e., over the initial direction of motion of the electron), we obtain

$$\mathbf{j}' = i\frac{2\pi e^2c^2}{\omega}\mathbf{E}\int\left(1 + \frac{m^2c^3}{Ep}\ln\frac{E + cp}{mc^2}\right)\frac{f(p)}{E}\,dp, \tag{9}$$

where $\mathbf{E}$ is the electric field intensity. Now, comparing the microscopic and macroscopic field equations and taking account of Eq. (9) we find the following formula for the index of refraction:

$$n^2 = 1 - \frac{8\pi^2e^2c^2}{\omega^2}\int\left(1 + \frac{m^2c^2}{Ep}\ln\frac{E + cp}{mc^2}\right)\frac{f(p)}{E}\,dp. \tag{10}$$

First we consider the case in which the gas is degenerate. In this case

$$f(p) = p^2/4\pi^3\hbar^3 \text{ for } p < p_f,$$
$$f(p) = 0 \text{ for } p > p_f; \tag{11}$$

where $p_f$ is the limiting Fermi momentum, which is given by

$$p_f = (3\pi^2)^{1/3}\hbar N^{1/3}. \tag{12}$$

When $p_f \ll mc$ we obtain the familiar formula (4); when $p_f \geq mc$ we have

$$n^2 = 1 - (\omega_0/\omega)^2, \tag{13}$$

$$\omega_0^2 \approx 3\pi ce^2N/p_f \approx 3c^2N^{2/3}/137. \tag{14}$$

Thus, for a relativistic degenerate gas the constant $\omega_0^2$ is proportional to $N^{2/3}$ rather than N.

We now consider the case in which the electron gas is not degenerate. Under these conditions we have[2]

$$f(p) = N\varphi(T)\,[p^2/4\pi(mc)^3]\exp(-E/\chi T), \tag{15}$$

$$\varphi(T) = [2u^{-2}K_1(u) + u^{-1}K_0(v)]^{-1}. \tag{16}$$

The quantities $K_1$ and $K_0$ are Bessel functions of the second kind of imaginary argument while $u = mc^2/\chi T$ ($\chi$ is the Boltzmann constant). When $u \gg 1$ we have

$$\varphi \approx 4\pi(mc)^3(2\pi m\chi T)^{-3/2}e^u,$$

and when $u \ll 1$ we have $\varphi \approx u^3/2$.

As in the preceding case, upon substitution of Eq. (15) in Eq. (10) we obtain an integral which cannot be integrated exactly. When $u \lesssim 1$, as an approximation we have

$$n^2(\omega) \approx 1 - \frac{4.8\,\pi e^2}{m\omega^2}\frac{N\varphi}{u}e^{-u}. \tag{17}$$

When $u \gg 1$, Eq. (17) no longer applies and does not give the proper transition to the familiar formula. As we see, in the absence of degeneracy the refractive index depends on temperature.

It is apparent that in the absence of degeneracy, in the region in which one can apply the notion of the dielectric constant of a medium, the dispersion is always determined by the electrons. Consequently, at electron densities $N_e \gtrsim 1.4 \times 10^{32}\ \text{cm}^{-3}$ photon annihilation and electron pair production can occur in a medium.

We now consider the second limiting case; here the electron gas is highly degenerate (temperatures $T \ll 4 \times 10^{10}$). (In this case there are still two possibilities: either the neutrons are not degenerate or the neutron gas is also degenerate.) Under these physical conditions it turns out to be more favorable thermodynamically to have reactions in which the nucleus captures an electron and simultaneously

*We omit from now on the subscript "zero" from the symbols for the momentum and energy.

emits a neutrino.[2] Because of this reaction the number of protons in the nucleus is reduced sharply and eventually the nucleus becomes unstable and decays. As a result we must in fact consider an ideal gas of electrons, protons, and neutrons.

The number of particles of each kind is determined from the equilibrium condition, that is to say, from the condition that the thermodynamic potential be a minimum for a given pressure and temperature. For the reactions

$$p + e^- \rightarrow n + \nu \text{ and } n \rightarrow p + e^- + \bar{\nu},$$

where $e^-$, p, n, $\nu$ and $\bar{\nu}$ denote the electron, the proton, the neutron, the neutrino and the antineutrino respectively, the equilibrium condition is

$$\mu_p + \mu_e = \mu_n, \tag{18}$$

where $\mu_e$, $\mu_p$ and $\mu_n$ are the chemical potentials for the corresponding particles. The electron chemical potential is

$$\mu_e = E_f \approx \pi c\hbar N_e^{1/3}, \tag{19}$$

where $E_f$ is the limiting Fermi energy.

If the nucleon gas is not degenerate, we have[2]

$$\mu_p = \chi T \ln\left[\frac{N_p}{2}\left(\frac{2\pi\hbar^2}{M\chi T}\right)^{3/2}\right]; \ \mu_n = \chi T \ln\left[\frac{N_n}{2}\left(\frac{2\pi\hbar^2}{M_n\chi T}\right)^{3/2}\right]. \tag{20}$$

Substituting the values of the chemical potentials in Eq. (18) and using the obvious relation $N_p = N_e$, we find*

$$N_n = N_e \exp(\pi c\hbar N_e^{1/3}/\chi T). \tag{21}$$

Since the electron gas is assumed to be highly degenerate $\chi T \ll \pi c\hbar N_e^{1/3}$ and, consequently, $N_n \gg N_e$.

Now we assume that at the densities being considered the temperatures are so low that the neutron and proton gases are also highly degenerate. In this case

$$\mu_p = Mc^2 + (\hbar^2/2M)(3\pi^2 N_e)^{2/3},$$
$$\mu_n = M_n c^2 + (\hbar^2/2M_n)(3\pi^2 N_n)^{2/3}, \tag{22}$$

where $M_n = M + \alpha m$ is the mass of the neutron and $\alpha \approx 2.54$ is the difference in the masses of the neutron and proton expressed in units of electron mass. Here we assume that the proton and neutron are nonrelativistic particles, as is the case for matter densities up to the order of nuclear densities. Substituting Eqs. (19) and (22) in the equilibrium relation (18) we find

$$N_e = \varkappa^{-3} N_0 \{[1 + \alpha\varkappa/\pi + \varkappa^2 (N_n/N_0)^{2/3}]^{1/2} - 1\}^3, \tag{23}$$

where we have introduced the notation $N_0 = 8\lambda_e^{-3} = 1.4 \times 10^{32}\ \text{cm}^{-3}$ and $\kappa = 2\pi m/M$.

If we limit ourselves to densities of the order of nuclear densities or smaller, $N_n \lesssim 10^{38}\ \text{cm}^{-3}$, Eq. (23) can be simplified:

$$N_e \approx (N_0/8)[\alpha/\pi + \varkappa (N_n/N_0)^{2/3}]^3. \tag{23'}$$

In order to ascertain whether the process $\gamma \rightleftarrows e^+ + e^-$ with the participation of the medium is possible or not, we must first understand the role played by the nucleons in the scattering of electromagnetic waves.

## 2. SCATTERING OF ELECTROMAGNETIC WAVES ON NUCLEONS

Under certain physical conditions, i.e., high matter densities, low temperatures (so that the electron gas may be considered highly degenerate), and high frequencies, electromagnetic waves are scattered on nucleons rather than on electrons, Under these conditions hard Cerenkov radiation is produced of necessity.

To understand the nature of this effect, we investigate the dispersion properties of a medium consisting of neutrons. We compute the refractive index of such a medium for electromagnetic waves and determine the frequencies and neutron densities for which the dispersion properties of the medium are determined by the neutrons rather than by electrons (we should speak generally of nucleons; however, under the physical conditions of interest the number of protons is small compared with the number of neutrons, so that we shall consider only the latter although the results also apply to protons).

Although it is electrically neutral, the neutron is capable of scattering $\gamma$ quanta. This effect is related to the well-known phenomenon of meson photoproduction. Both virtual and real mesons can be created in the interaction of photons with neutrons. The creation of real mesons corresponds to absorption of a wave; hence the refractive index of a medium consisting of neutrons is a complex quantity:

$$\sqrt{\varepsilon} = n + in_1. \tag{24}$$

The following relation obtains between the real and imaginary parts of the index of refraction:[3]

$$n(\omega) = 1 + \frac{2}{\pi}\int_0^\infty \frac{\varkappa n_1(\varkappa)}{\varkappa^2 - \omega^2} d\varkappa, \tag{25}$$

where the bar means that the integral is to be un-

*It is appropriate to note that there are no positrons in the medium at the temperatures being considered here. These particles appear only when $\chi T > mc^2$ (cf. reference 2, page 334). Because the electron gas is degenerate pair production is forbidden by the Pauli principle.

derstood in the sense of the principal value. Expressing $n_1(\kappa)$ in terms of the photoproduction cross section:

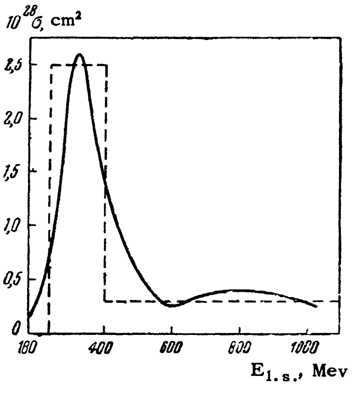
$$n_1(\varkappa) = (c/2\varkappa)\, N_n \sigma(\varkappa), \tag{26}$$

we find

$$n(\omega) = 1 + \frac{cN_n}{\pi}\int_0^\infty \frac{\sigma(\varkappa)\,d\varkappa}{\varkappa^2 - \omega^2}. \tag{27}$$

To compute n we start with the experimental curves for the cross sections for photoproduction of neutral pions on protons.[4-6] This curve exhibits two maxima: the first is very narrow and well defined ($\hbar\omega = 320$ Mev) while the second is very weak and broad ($E \sim 700$ Mev).

The neutron density increases towards the center of the star and, in accordance with (1), the region for which we can use the notion of a refractive index shifts towards the higher-energy quanta. In the limiting case we deal with nuclear densities. Under these conditions the mean distance between particles is $\hbar/\mu c$ ($\mu$ is the mass of the pion) and, correspondingly, the limiting photon energy should be of the order of 150 Mev. Hence, we can speak of a refractive index for neutron matter only at photon energies $\hbar\omega \lesssim 150$ Mev. In this energy region $\sigma(\omega) = 0$ and the principal maximum is found at an energy which is twice 150 Mev. It is thus clear that, since we are interested in frequencies $\hbar\omega \lesssim 150$ Mev, the detailed dependence of $\sigma(\omega)$ on $\omega$ is unimportant for the determination of $n(\omega)$. As a good approximation to the true curve shown in the figure we can use a discontinuous curve (shown by dashes); the dashed curve is chosen to make the area between it and the abscissa axis approximately the same as the area formed by the experimental curve. Further, we assume that the cross sections for photoproduction of neutral and positive pions on protons are the same (as is approximately the case[5,6]) and that the cross sections for photoproduction on neutrons and protons are also equal. Thus, for $\sigma(\omega)$ we have:




$$\sigma(\omega) = \begin{cases} 0 & \text{for } \omega < \omega_1, \\ \sigma_1 & \text{for } \omega_1 < \omega < \omega_2, \\ \sigma_2 & \text{for } \omega > \omega_2; \end{cases} \tag{28}$$

where $\sigma_1 = 5 \times 10^{-28}$ cm², $\sigma_2 = 0.62 \times 10^{-28}$ cm², $\omega_1 = 3.7 \times 10^{23}$ is the frequency corresponding to a photon energy of 240 Mev, and $\omega_2 = 6.1 \times 10^{23}$ is the same quantity for an energy of 400 Mev.

Since we are interested only in the frequency region $\hbar\omega \ll 150$ Mev $\sim \hbar\omega_1$ we can neglect $\omega$ in the integrand in Eq. (27). Then, using Eq. (28) we have

$$n^2 - 1 \approx \frac{2cN}{\pi}\left(\frac{\sigma_1}{\omega_2} - \frac{\sigma_1 - \sigma_2}{\omega_2}\right) = 2{,}1\cdot 10^{-41} N_n. \tag{29}$$

Thus, the dielectric constant of neutron matter is essentially independent of frequency and always very close to unity.

In the final analysis, to determine whether the dispersion properties of the medium are determined by the electrons or by the neutrons, we must compare Eq. (29) with Eq. (13). The dispersion is determined by the neutrons when

$$(n^2 - 1)/(1 - n_e^2) = 2{,}1\cdot 10^{-41} N_n (\omega/\omega_0)^2 > 1, \tag{30}$$

where the subscript e refers to the electron. From Eq. (14), substituting the value of $\omega_0$ we find

$$\omega^2 N_n > 10^{60} N_e^{4/3}. \tag{31}$$

First we consider the case in which the nucleon gas is not degenerate. According to Eqs. (21) and (31) the condition that dispersion be determined by electrons is given by the relation

$$N_e^{1/3}\exp(aN_e^{1/3}/T) < 3{,}6\cdot 10^{17}(\omega_0/\omega)^2, \tag{32}$$

where we have introduced the following notation: $\hbar\omega_0 = 2mc^2$, $a = \pi c\hbar/\chi \approx 0.72$. Taking account of the inequality in (1) $\omega \lesssim cN_e^{1/3}$, we find that when $aN_e^{1/3}/T \lesssim 15$ the process $e^+ + e^- \to \gamma$ is possible in the medium. At lower temperatures, however, this process is forbidden. The inverse process, $\gamma \to e^+ + e^-$, is forbidden by the Pauli principle; it becomes possible, however, if vacancies appear in the electron distribution because of various processes. The dispersion is determined by the neutrons when

$$3{,}6\cdot 10^{17}(N_0/N_e)^{4/3}\exp(-2aN_e^{1/3}/3T) \leqslant 3.6\cdot 10^{17}(\omega_0/\omega)^2 < N_e^{1/3}\exp(aN_e^{1/3}/T); \tag{33}$$

Here, the condition at the left means $\omega \lesssim cN_n^{1/3}$. Whence it is apparent that when $aN_e^{1/3}/T = 15$ and $\omega \gtrsim 7\omega_0$ the dispersion is determined by the neutrons; at frequencies below $7\omega_0$ the dispersion is determined by the electrons.

When the nucleon gas is also degenerate the electron density becomes a relatively weak function of neutron density. When $N_n < 10^{36}$ we have $N_e \sim N_0/8$ and $cN_e^{1/3} \approx 0.5\,\omega_0$ so that the processes $\gamma \rightleftarrows e^+ + e^-$ either do not occur at all in the medium or appear at the very threshold for the effect ($\hbar\omega \approx 2mc^2$). At densities of $N \gtrsim 8 \times 10^{35}$, however, the electron density becomes sufficient for creation and annihilation of pairs. As before, in order to determine whether this effect actually occurs we must find the frequencies at which dispersion is determined by the electrons and those at which it is determined by the neutrons. To find these frequencies we again use the condition in (31). According to Eqs. (23′) and (31) dispersion is determined by the neutrons when

$$N_n > 2.4 \cdot 10^{38} \nu^{-2} [\alpha/\pi + \varkappa (N_n/N_0)^{2/3}]^2, \qquad (34)$$

where $\nu = \omega/\omega_0$. As always, Eq. (34) must be supplemented by the criterion which must be satisfied if we are to use the idea of a medium, i.e., $\omega \lesssim cN_n^{1/3}$. When $N_n \lesssim 10^{36}$ the process $\gamma \rightleftarrows e^+ + e^-$ is possible and will occur only at threshold energy; the condition $N_n \gtrsim 10^{36}\ \mathrm{cm}^{-1/3}$ is necessary and sufficient (obviously we still must satisfy the requirement for this process $\omega \lesssim cN_e^{1/3}$).

On the basis of the material presented in this section we can conclude that for a given matter density there is a certain critical frequency above which dispersion is determined by neutrons; below this frequency the dispersion is due to electrons. According to Eq. (29), above the critical frequency the refractive index is greater than unity and remains essentially constant, being independent of frequency. It is clear that charged particles moving with velocities exceeding the phase velocity of light in such a medium will emit Cerenkov radiation. In accordance with the conditions in (1) and (31), only photons which satisfy the following inequality are radiated:

$$10^3 N_e^{1/3} N_n^{-1/2} < \hbar\omega \leqslant \hbar c N_n^{1/3}. \qquad (35)$$

In this energy range the number of photons radiated per energy interval is uniform. According to Eq. (29) the Cerenkov effect appears at particle energies $E \geq 2.2 \times 10^{20} N_n^{-1/2} Mc^2$ where $Mc^2$ is the self energy of the particle.

When the nucleon gas is not degenerate the relation between the densities $N_e$ and $N_n$ is given by Eq. (21). For purposes of illustration we take $aN_e^{1/3}/T = 10$; then, from Eq. (35) with $N_e = 10^{32}$ (i.e., $N_n = 2.2 \times 10^{36}$) we find $20 < \hbar\omega \lesssim 28$ Mev while with $N_e = 4.5 \times 10^{33}$ (i.e., $N_n = 10^{38}$) we have $10 < \hbar\omega \lesssim 100$ Mev.

When the nucleon gas is also degenerate the relation between $N_e$ and $N_n$ is given by Eq. (23′). In this case (35) assumes the following form:

$$2.6 \cdot 10^{13} N_n^{-1/2} [\alpha/\pi + \varkappa (N_n/N_0)^{2/3}] < \hbar\omega \leqslant hcN_n^{1/3}. \qquad (35')$$

Whence we find that the neutron density must be $N_n \gtrsim 8 \times 10^{35}$. With $N_n = 10^{36}\ \mathrm{cm}^{-3}$, from Eq. (35′) we find $13 < \hbar\omega \lesssim 20$ Mev while with $N_n = 10^{38}$ we find $1.6 < \hbar\omega \lesssim 100$ Mev.

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