INSTABILITY OF LONGITUDINAL OSCILLATIONS OF AN ELECTRON-ION PLASMA

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The problem of the instability of longitudinal oscillations of low temperature electron-ion plasma is discussed. In an isotropic plasma the oscillations are always damped, while in an anisotropic one the ion motion may lead to the appearance of solutions that increase with time, i.e., to instability.

RECENTLY problems connected with the instability of a current flowing through a plasma, and particularly those connected with the instability of longitudinal plasma waves, have come in for ever increasing attention. Thus, Bohm and Gross[1] and Akhiezer and Faĭnberg[2] have investigated the oscillations of a plasma containing an electron beam and have demonstrated that the oscillations increase with time. In these studies ion motion was altogether neglected. In some cases, however, inclusion of ion motion may prove essential and produce new forms of plasma instability due to electron-ion interaction (see below).

Let $f_{0e}(\mathbf{v})$ and $f_{0i}(\mathbf{v})$ represent equilibrium electron and ion distribution functions (normalized to unity). For a given equilibrium distribution the asymptotic behavior of the solution is determined, as is known,[3] by the roots of the dispersion equation

$$1 = \frac{i\omega_{0e}^2}{k^2}\left\{\int d\mathbf{v}\,\frac{\mathbf{k}\partial f_{0e}/\partial\mathbf{v}}{s+i\mathbf{kv}} + \delta\int d\mathbf{v}\,\frac{\mathbf{k}\partial f_{0i}/\partial\mathbf{v}}{s+i\mathbf{kv}}\right\}, \qquad (1)$$

where $\omega_{0e}^2 = 2\pi e^2 n_e/m_e$, $\delta = zm_e/m_i$ (z is the ion charge), $\mathbf{k}$ is the wave vector, and $s = i\omega + \gamma$. Since we shall be concerned only with solutions that increase with time ($\gamma > 0$), the integrals in (1) can be taken along the real axis. Before proceeding to the discussion of a concrete case, we shall make a few general remarks.

If we transform Eq. (1) by integration by parts and separate the result into a real and imaginary part we obtain a system of two equations for ω and γ. One of these equations has the following form:

$$\gamma\int d\mathbf{v}\,[f_{0e}(\mathbf{v}) + \delta f_{0i}(\mathbf{v})]\,\frac{\omega+\mathbf{kv}}{[(\omega+\mathbf{kv})^2+\gamma^2]^2} = 0. \qquad (2)$$

Two conclusions follow directly from this equation.

First, oscillations with frequency $\omega > k\bar{v}$ are always stable, while oscillations with frequency $\omega < k\bar{v}$ may be unstable. Thus, the following inequality is a condition for instability:

$$\bar{v} > \omega/k = v_{\text{ph}}, \qquad (3)$$

where $\bar{v}$ represents some average value of the velocity. Condition (3) means that only waves whose phase speed is less than the average speed of the particles can be unstable. (This condition was first noted by Bohm and Gross.[1]) We shall show that condition (3) is necessary but insufficient; this will become apparent from an example discussed below.

Secondly, in the case of an isotropic plasma the oscillations are always stable for arbitrary distributions of the form $f_{0e}(v^2)$ and $f_{0i}(v^2)$. Actually, by integrating Eq. (2) over all directions in the velocity space, we find

$$\gamma\omega\int_0^\infty v^2 dv\,\frac{f_{0e}(v^2)+\delta f_{0i}(v^2)}{[\gamma^2-\omega^2+k^2v^2]^2+4\omega^2\gamma^2} = 0, \qquad (4)$$

whence it follows that $\gamma = 0$. The contrary assertion in the article by Bohm and Gross[1] is the result of an error in the computation of the integral appearing in the dispersion relation. Their result would have been incorrect none the less if only because of the fact that when the integration is along the real axis, as Landau has demonstrated,[3] it is in principle impossible to obtain damped solutions.

Let us examine a case where the electron gas is described by a drifting Maxwellian distribution, while the ion distribution is isotropic and likewise Maxwellian, i.e.,

$$f_{0e}(\mathbf{v}) = (m_e/2\pi\varkappa T_e)^{3/2}\exp\{-m_e(\mathbf{v}-\mathbf{u})^2/2\varkappa T_e\},$$
$$f_{0i}(\mathbf{v}) = (m_i/2\pi\varkappa T_i)^{3/2}\exp[-(m_i\mathbf{v}^2/2\varkappa T_i)]. \qquad (5)$$

Such a case might occur in a plasma located in a strong electric field.[4]

Substituting (5) in Eq. (1), we obtain the following dispersion equation:

$$F([s+i\mathbf{ku}]^2, \alpha_e) + \delta F(s^2, \alpha_i) + \omega_{0e}^{-2} = 0. \quad (6)$$

The following designations apply here

$$\alpha_e = m_e/2\varkappa T_e, \qquad \alpha_i = m_i/2\varkappa T_i;$$

$$F(s^2, \alpha) = \frac{2\alpha}{k^2}\left\{1 - \sqrt{\pi\alpha}\,\frac{s}{k}\,w\left(i\alpha^{1/2}\frac{s}{k}\right)\right\},$$

$$w(z) = e^{-z^2}\left(1 + \frac{2i}{\sqrt{\pi}}\int_0^z e^{t^2}dt\right). \quad (7)$$

Equation (6) with $u = 0$ was studied by Silin[5] and Kovrizhnykh[6] who showed that when k is small, two modes of plasma oscillations occur, which are entirely different in their properties. The first mode, called optical, is characterized by comparatively large oscillations of the charge density while the ion motion is insignificant. For this mode of plasma oscillations, $\omega(k) \neq 0$ with $k = 0$. The optical mode is obtained by solving Eq. (6) with $u = 0$ and with the second term neglected. The second, or acoustic mode, on the other hand, is characterized by small oscillations of the charge density and a linear dependence of s on k.

An analogous division into optical and acoustic modes may also be adopted for the case where $u \neq 0$. The optical mode will correspond, as for the case $u = 0$, to the solution of Eq. (6) with the second term omitted. Naturally the solution of this equation will differ from the corresponding solution where $u = 0$ due to the appearance of the imaginary term $i\mathbf{k}\cdot\mathbf{u}$, which corresponds to a Doppler effect due to the electron motion. The character of the oscillations specified by the real part of s does not change, i.e., the optical oscillations are attenuated regardless of the magnitude of u. This circumstance indicates the insufficiency of condition (3). It is a different matter with the second, or acoustic, mode of plasma oscillations. When $u \neq 0$, the acoustical solution of Eq. (6) may be unstable. The frequency region where $s^2\alpha_i/k^2 \gg 1$ corresponds to the unstable solutions of Eq. (6). Therefore, if we make use of the asymptotic representation of $\omega(z)$ in the region where $\mathrm{Re}(s) > 0$, we obtain from Eq. (6) the following dispersion equation, correct to terms $\sim k^2/s^2\alpha_i$:

$$[(s+i\mathbf{ku})^2 + k^2v_e^2]^{-1} + \delta/(s^2 + k^2v_i^2) + \omega_{0e}^{-2} = 0, \quad (8)$$

where $v_e^2 = 3kT_e/m_e$, $v_i^2 = 3kT_i/\overline{m}_i$, represent the thermal speed of the electrons and ions respectively. As was noted above, the linear dependence of s on k is characteristic of the acoustic mode. Therefore, it is always possible to isolate a region of k-space in which $s^2 \ll \omega_{0e}^2$. In this case the last term in Eq. (8) can be neglected. This also corresponds to neglecting oscillations of the charge density. Finally, we obtain the following solution for Eq. (8):

$$s = -i\mathbf{ku}\delta + [\delta(\mathbf{ku})^2 - k^2(\delta v_e^2 + v_i^2)]^{1/2}. \quad (9)$$

Because of the asymptotic representation used for $w(z)$, Eq. (9) is correct when $|\mathbf{k}\cdot\mathbf{u}|^2/k^2 \gg v_e^2$, v_i^2/δ and $(\mathbf{k}\cdot\mathbf{u})^2 \ll \omega_{0e}^2$. In this region Eq. (9) predicts unstable plasma oscillations.*

In addition to solution (9), an acoustic solution of Eq. (8) can be obtained when the term ω_{0e}^2 is included but with $|s| \ll |\mathbf{k}\cdot\mathbf{u}|$. In this case, neglecting terms of the order of $\sqrt{\delta}$ in comparison with unity, we find

$$s = \gamma = \omega_{0e}\delta^{1/2}(\mathbf{ku})\,[\omega_{0e}^2 - (\mathbf{ku})^2]^{-1/2}, \quad (10)$$

which is applicable when $\omega_{0e}^2 - (\mathbf{k}\cdot\mathbf{u})^2 > \omega_{0e}^2\delta^{1/3}$.

Thus in a plasma whose electrons have an average speed greater than their thermal speed, the acoustic oscillations may become unstable. In this case, the energy of the electron motion is converted into energy of acoustic oscillations and is the source of the instability. We note that a similar instability can occur in a plasma in which part of the electrons have a fast drift, i.e., in a plasma containing so-called "runaway" electrons. In this case, besides the excitation of unstable acoustic oscillations, there will be excitation of unstable optical oscillations with s given by an expression of the form of Eq. (10) but with δ equal to the ratio of the number of runaway electrons to the total number of electrons.

In conclusion we note that an instability of this type appears to have been observed experimentally.[8]

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[1] D. Bohm and E. P. Gross, Phys. Rev. **75**, 1864 (1949).

[2] A. I. Akhiezer and Ya. B. Faĭnberg, JETP **21**, 1262 (1951).

[3] L. D. Landau, JETP **16**, 574 (1946), see J. Phys. (U.S.S.R.) **10**, 25 (1946).

[4] L. M. Kovrizhnykh, JETP **37**, 1394 (1959), Soviet Phys. JETP **10**, 989 (1960).

[5] V. P. Silin, JETP **23**, 649 (1952).

[6] L. M. Kovrizhnykh, JETP **37**, 1692 (1959), Soviet Phys. JETP **10**, 1198 (1960).

[7] G. V. Gordeev, JETP **27**, 19 (1954).

[8] Bernstein, Chen, Heald, and Kranz, Phys. Fluids **1**, 430 (1958).

Translated by A. Skumanich
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*The case where $u^2 \ll v_e^2$ and $n_eT_e \gg n_iT_i$ has been investigated by Gordeev.[7]

ERRATA TO VOLUME 10

page	reads	should read
Article by A. S. Khaĭkin		
1044, title	. . . resonance in lead	. . . resonance in tin
6th line of article	~ 1000 oe	~ 1 oe
Article by V. L. Lyuboshitz		
1223, Eq. (13), second line	$\ldots - \mathrm{Sp}_{1,2}\mathscr{E}(e_1)$	$\ldots - \mathrm{Sp}_{1,2}\mathscr{E}(e_2)\ldots$
1226, Eq. (26), 12th line	$\cdots\{(p+q,\ p$	$\cdots\{(p+q,p)-(p+q,n)\cdot\cdot(pn)\}$
1227, Eqs. (38), (41), (41a) numerators and denominators	(p^2-q)	$(p^2-q^2)^2$
1228, top line	$m_2 = \frac{q_1-p_1}{q_1-p_1}$	$m_2 = [m_3\ m_1]$

ERRATA TO VOLUME 12

page	reads	should read
Article by Dzhelepov et al.		
205, figure caption	54	5.4
Article by M. Gavrila		
225, Eq. (2), last line	$-2\gamma\Theta^{-4}\,{}^1\!/_8$	$-2\gamma\Theta^{-4}-{}^1\!/_8$
Article by Dolgov-Savel'ev et al.		
291, caption of Fig. 5, 4th line	$p_0 = 50\times10^{-4}$ mm Hg	$p_0 = 5\times10^{-4}$ mm Hg.
Article by Belov et al.		
396, Eq. (24) second line	$\ldots-(4-2\eta)\sigma_1+\ldots$	$\ldots+(4-2\eta)\sigma_1+\ldots$
396, 17th line (r) from top	. . less than 0.7	. . less than 0.07
Article by Kovrizhnykh and Rukhadze		
615, 1st line after Eq. (1)	$\omega_{0e}^2 = 2\pi e^2 n_e/m_e,$	$\omega_{0e}^2 = 4\pi e^2 n_e/m_e,$
Article by Belyaev et al.		
686, Eq. (1), 4th line	$\cdots b_{\rho_2 m_2}(s_2')+\cdots$	$\cdots b_{\rho_1 m_1}(s_1')+\cdots$
Article by Zinov and Korenchenko		
798, Table X, heading of last column	$\sigma_{\pi^-\to\pi^+} =$	$\sigma_{\pi^-\to\pi^-} =$
Article by V. M. Shekhter		
967, 3d line after Eq. (3)	$\varepsilon \equiv 2m_pE + m_p^2$	$\varepsilon \equiv (2m_pE + m_p^2)^{1/2}$
967, Eq. (5), line 2	$+(B_V^2+B_A^2)\ldots$	$+(B_V^2+B_A^2)\,Q\ldots$
968, Eq. (7)	$\ldots(C_V^2+C_A^2).$	$\ldots C_V^2+C_A^2-Q^2(B_V^2+B_A^2).$
968, line after Eq. (7)	for $C_V^2+C_A^2\equiv\ldots$	for $C_V^2+C_A^2-Q^2(B_V^2+B_A^2)\equiv\ldots$
Article by Dovzhenko et al.		
983, 11th line (r)	$\gamma = 1.8\pm$	$\gamma = 1.8\pm0.2$
Article by Zinov et al.		
1021, Table XI, col. 4	−1,22	1,22
Article by V. I. Ritus		
1079, line 27 (l)	$-\Lambda_\pm(t),$	$\Lambda_\pm(t),$
1079, first line after Eq. (33)	$\frac{1}{2}(1+\beta).$	$\frac{1}{2}(1\pm\beta).$
1079, 3d line (l) from bottom	$\ldots\mathfrak{R}(q'p;\ pq')\ldots$	$\ldots\mathfrak{R}(p'q;\ pq')\ldots$
Article by R. V. Polovin		
1119, Eq. (8.2), fourth line	$U_{0x}u_xg(\gamma)-[\gamma\cdots$	$-U_{0x}u_xg(\gamma)[\gamma\cdots$
1119, Eq. (8.3)	$\cdots$sign $u.$	$\cdots$sign $u_g.$
Article by V. P. Silin		
1138, Eq. (18)	$\cdots+\frac{4}{5}c^2k^2$	$\cdots+\frac{6}{5}c^2k^2.$