POLARIZATION OF DEUTERONS ELASTICALLY SCATTERED ON ZERO-SPIN NUCLEI

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The most general form of the transition matrix is presented. The dependence of the transition matrix parameters on experimentally observed quantities is established. The general and explicit expressions are derived for the double scattering cross section and vector and tensor polarization, in which account is made of the mixing of the different waves. It is shown that by a choice of a special form of the potential (such as that employed in the optical model) phase shifts, and hence a description of the scattering, can be obtained. Calculations performed in the Born approximation are compared with the experimental results.

A large number of experimental and theoretical researches have been devoted to the study of polarized nucleons. In these researches attempts have been made to draw up a complete phase analysis and to establish the amplitudes of nucleon-nucleon scattering with the aim of investigating spin-dependent interactions.[1-4] Detailed information on the character of the nuclear forces can also be obtained from experiments on scattering of particles with spin 1 on zero spin targets. In this case the results can be obtained at energies of several Mev and the quantity of experiments necessary for completing the phase analysis is not large.

Consideration of the polarization of deuterons has attracted considerably less attention in the literature.[1-4] Expressions were derived in the research of Cheĭshvili[2] for the cross section and the polarization, which were obtained by use of projection operators; however, in this case the possibility of transitions with a change in the orbital momentum was not taken into account.

Extension of the method developed by Vol'fenshteĭn et al. for describing particles with spin $\frac{1}{2}$ (which is based on the use of the transition matrix M and the density ρ) to the case under consideration makes it possible to obtain results both in general form and in a form suitable for application. Calculation of transitions with change of orbital momentum does not raise any difficulties.

The most general form of the transition matrix is defined by the requirement that it be invariant under spatial rotations and reflections and under time reversal:

$$M = A(\vartheta, \varphi) + B_i(\vartheta, \varphi) S_i + C_{ij} S_{ij}. \quad (1)$$

Denoting the unit vectors in the directions of the incident and scattered beams by $\mathbf{k}_i$ and $\mathbf{k}_f$, and introducing the mutually-orthogonal vectors

$$\mathbf{P} = (\mathbf{k}_i + \mathbf{k}_f)/|\mathbf{k}_i + \mathbf{k}_f|,$$

$$\mathbf{K} = (\mathbf{k}_i - \mathbf{k}_f)|\mathbf{k}_i - \mathbf{k}_f|, \quad \mathbf{N} = [\mathbf{k}_i, \mathbf{k}_f]/|[\mathbf{k}_i, \mathbf{k}_f]|, \quad (1')$$

we can write M in the form

$$M = A + BS_N + C(S_P^2 + S_K^2 - \tfrac{4}{3}\delta_{ij}) + D(S_P^2 - S_K^2). \quad (1a)$$

Making use of these expressions for the transition matrix, we derive the vector and tensor polarizations for the cross section in the general form of the following formula:

$$I_0 = \tfrac{1}{3}(3AA^* + 2B_i B_i^* + C_{ij} C_{ij}^*),$$

$$I_0 P_n = \tfrac{1}{3}[4\,\mathrm{Re}(AB_n^*) + 2\,\mathrm{Re}(C_{ni} B_i^*) - \mathrm{Im}(B_i B_j^* \varepsilon_{ijn}) - \mathrm{Im}(C_{ik} C_{jk} \varepsilon_{ijn})],$$

$$I_0 T_{mn} = \tfrac{1}{3}[2\,\mathrm{Re}(AC_{mn}^*) + \mathrm{Re}(B_m B_n^*) - \tfrac{1}{3} B_i^* B_i \delta_{mn} - \mathrm{Re}(C_{mi} C_{in}^*) + \tfrac{1}{3} C_{ij} C_{ji} \delta_{mn} - \mathrm{Im}(C_{mi} B_j \varepsilon_{ijn}) - \mathrm{Im}(B_i C_{jn} \varepsilon_{mij})] \quad (2)$$

and accordingly in the chosen set of coordinates

$$I_0 = \tfrac{1}{3}\{3|A|^2 + 2|B|^2 + \tfrac{2}{3}|C|^2 + 2|D|^2\},$$

$$I_0 P_N = \tfrac{4}{3}\mathrm{Re}[(A - \tfrac{1}{3}C)B^*],$$

$$I_0(T_P T_K + T_K T_P) = -\tfrac{4}{3} i\,\mathrm{Re}\,BD^*, \quad (2a)$$

$$I_0 T_N = \tfrac{1}{3}\{2|A|^2 + 2|B|^2 + \tfrac{2}{9}|C|^2 + 2|D|^2 - \tfrac{4}{3}\mathrm{Re}\,AC^*\}.$$

In this case, if the y axis is perpendicular to the plane of the first scattering, or if the direction of motion of the beam of deuterons between the two collisions coincides with the z axis, the angular distribution of deuterons undergoing double scattering can be written in the form

$$I(\vartheta'\varphi') = I_0(\vartheta')[1 + \tfrac{1}{2}\alpha\alpha' + \tfrac{3}{2}(\beta\beta' + \gamma\gamma')\cos\varphi' + \tfrac{3}{2}\delta\delta'\cos 2\varphi'], \quad (3)$$

where ϑ and ϑ' are the angles of first and second scattering, respectively, φ' is the angle between the planes of the scattering.

The parameters α, β, γ, and δ are determined by the relations

$$I_0\alpha = -\tfrac{1}{3}|B|^2 + \tfrac{1}{9}|C|^2 - \tfrac{1}{3}|D|^2 + \tfrac{2}{3}\operatorname{Re} AC^* + 2\cos\vartheta \operatorname{Re}[D^*(A - \tfrac{1}{3}C - iB\tan\vartheta)],$$

$$I_0\beta = \tfrac{4}{3}\operatorname{Re}[B^*(A - \tfrac{1}{3}C)],$$

$$I_0\gamma = \tfrac{4}{3}\cos\vartheta \operatorname{Re}\{D^*[iB - (A + \tfrac{1}{3}C)\tan\vartheta]\},$$

$$I_0\delta = \tfrac{1}{3}(-|B|^2 + \tfrac{1}{3}|C|^2 - |D|^2 + 2\operatorname{Re} AC^*) + \tfrac{2}{3}\cos\vartheta \operatorname{Re}[D^*(-A + \tfrac{1}{3}C + iB\tan\vartheta)], \qquad (3a)$$

while the parameters α', β', γ', and δ' differ from these only by the fact that all the quantities are replaced by the corresponding values for the second scattering.

With the help of the well-known formula of Blatt and Biedenharn for the amplitude of the scattered waves[5]

$$\psi = i\lambda \frac{\exp(ikr)}{r} \sum_{m_{s'}} \chi_{m_{s'}} \sum_{m_s} M_{m_s m_{s'}} a_{m_s} = i\lambda \frac{\exp(ikr)}{r} \sum_{m_{s'}} \chi_{m_{s'}} \times \sum_{J,l,l'} i^{l-l'} \pi^{1/2} (2l+1)^{1/2} (l, s, 0, m_s | l, s, J, m_s) \times (l', s', \mu, m_{s'} | l', s', J, m_s)(\delta_{l',l} - S^J_{l',l}) Y_{l',\mu}$$

The explicit form of the dependence of the matrix M on the angle of scattering and phase shifts can be established:

$$M = \begin{pmatrix} a & b\exp(-i\varphi) & c\exp(-2i\varphi) \\ d\exp(i\varphi) & f & -d\exp(-i\varphi) \\ c\exp(2i\varphi) & -b\exp(i\varphi) & a \end{pmatrix}. \qquad (4)$$

where $S^J_{l',l}$ are the elements of the scattering matrix which has the form

$$S = \begin{pmatrix} \cos^2\varepsilon\exp(2i\Phi_{J-1}) + \sin^2\varepsilon\exp(2i\Phi_{J+1}) & \tfrac{1}{2}\sin 2\varepsilon[\exp(2i\Phi_{J-1}) - \exp(2i\Phi_{J+1})] \\ \tfrac{1}{2}\sin 2\varepsilon[\exp(2i\Phi_{J-1}) - \exp(2i\Phi_{J+1})] & \sin^2\varepsilon\exp(2i\Phi_{J-1}) + \cos^2\varepsilon\exp(2i\Phi_{J+1}) \end{pmatrix}$$

in the case under consideration.[5] Here ϵ is the so-called mixing parameter, $\Phi_{J\pm1}$ is the phase shift of waves with orbital momentum $l = J \pm 1$.

As a consequence of the invariance of the matrix M relative to time reversal, the additional condition

$$(a - c - f)/\cos\vartheta = \sqrt{2}(b + d)/\sin\vartheta \qquad (5)$$

is placed on its elements.

By making use of the explicit form of the spin operators, it is possible to establish the dependence of the cross section, and also the vector and tensor polarization, on the scattering angle and, on the phase shifts. For example,

$$I_0 = \tfrac{1}{3}\operatorname{Sp} MM^* = \tfrac{2}{3}(|a|^2 + |b|^2 + |c|^2 + |d|^2 + \tfrac{1}{2}|f|^2),$$

$$I_0 = P_N = \tfrac{1}{3}\operatorname{Sp} MM^*S_N = \tfrac{2}{3}\sqrt{2}\, i \operatorname{Im}[(a - c)d^* + bf^*],$$

$$I_0(T_P^2 + T_K^2 - \tfrac{4}{3}\delta_{ik}) = \tfrac{1}{3}\{\tfrac{1}{3}|a|^2 - \tfrac{2}{3}|b|^2 + \tfrac{1}{3}|c|^2 - \tfrac{2}{3}|d|^2 - \tfrac{1}{3}|f|^2 + 2\operatorname{Re} ac^*\}. \qquad (6)$$

One can also develop an explicit expression for the parameters entering into the expression for the cross section of the doubly scattered beam of deuterons:

Here

$$a = \frac{i\lambda}{2\sqrt{2}} \sum_J \Big\{ \sqrt{\frac{J}{2J+3}} [\sqrt{J}(1 - S^J_{J+1,J+1}) + \sqrt{J+1}\, S^J_{J+1,J-1}] P_{J+1,0} + \sqrt{\frac{J+1}{2J-1}} [\sqrt{J}\, S^J_{J-1,J+1} + \sqrt{J+1}(1 - S^J_{J-1,J-1})] \times P_{J-1,0} + \sqrt{2J+1}[1 - \exp(2i\delta_J)] P_{J,0} \Big\},$$

$$b = \frac{i\lambda}{2} \sum_J \Big\{ \sqrt{\frac{J+2}{2J+3}} [-\sqrt{J+1}(1 - S^J_{J+1,J+1}) + \sqrt{J}\, S^J_{J+1,J-1}] P_{J+1,-1} + \sqrt{\frac{J-1}{2J-1}} [-\sqrt{J+1}\, S^J_{J-1,J+1} + \sqrt{J}(1 - S^J_{J-1,J-1})] P_{J-1,-1} - \sqrt{\frac{2J+1}{J(J+1)}} [1 - \exp(2i\delta_J)] P_{J,-1} \Big\},$$

$$c = \frac{i\lambda}{2\sqrt{2}} \sum_J \Big\{ \sqrt{\frac{(J+2)(J+3)}{2J+3}} \Big[\sqrt{\frac{J}{J+1}}(1 - S^J_{J+1,J+1}) + S^J_{J+1,J-1} \Big] P_{J+1,2} + \sqrt{\frac{(J-2)(J-1)}{2J-1}} \Big[S^J_{J-1,J+1} + \sqrt{\frac{J+1}{J}}(1 - S^J_{J-1,J-1}) \Big] P_{J-1,2} - \sqrt{\frac{(2J+1)(J-1)(J+2)}{J(J+1)}} [1 - \exp(2i\delta_J)] P_{J,2} \Big\},$$

$$d = \frac{i\lambda}{2} \sum_J \Big\{ \sqrt{\frac{J(J+2)}{2J+3}} \Big[-\sqrt{\frac{J}{J+1}}(1 - S^J_{J+1,J+1}) - S^J_{J+1,J-1} \Big] P_{J+1,1} + \sqrt{\frac{(J-1)(J+1)}{2J-1}} \Big[S^J_{J-1,J+1} + \sqrt{\frac{J+1}{J}}(1 - S^J_{J-1,J-1}) \Big] P_{J-1,1} \Big\}.$$

$$f = \frac{i\lambda}{\sqrt{2}} \sum_J \Big\{ \sqrt{\frac{J+1}{2J+3}} \Big[\sqrt{J+1}(1 - S^J_{J+1,J+1}) - \sqrt{J}\, S^J_{J+1,J-1} \Big] P_{J+1,0} - \sqrt{\frac{J}{2J-1}} \Big[\sqrt{J+1}\, S^J_{J-1,J+1} - \sqrt{J}(1 - S^J_{J-1,J-1}) \Big] P_{J-1,0} \Big\},$$

$$I_0\alpha = -\tfrac{1}{3}[|a|^2+|b|^2+|c|^2+4|d|^2+2|f|^2]+\cos^2(\vartheta/2)[|a|^2+|b|^2+|c|^2]+\sin^2(\vartheta/2)[-|b|^2+2|d|^2+|f|^2 + 2\,\mathrm{Re}\,ac^*]+\sqrt{2}\sin\vartheta\,\mathrm{Re}(ad^*+bf^*-cd^*), \qquad I_0\beta=\tfrac{2}{3}\sqrt{2}\,i\,\mathrm{Im}[(a-c)d^*+bf^*],$$

$$I_0\gamma=\tfrac{1}{3}\sin\vartheta[|a|^2+2|b|^2+|c|^2-2|d|^2-|f|^2-2\,\mathrm{Re}\,ac^*]-\tfrac{2}{3}\sqrt{2}\cos\vartheta\,\mathrm{Re}(ad^*+bf^*-cd^*),$$

$$I_0\delta=\tfrac{1}{3}\{-|b|^2-2|d|^2-|f|^2-2\,\mathrm{Re}\,ac^*+\sin^2(\vartheta/2)(|a|^2+|b|^2+|c|^2)+\cos^2(\vartheta/2)(2|d|^2+|f|^2-|b|^2+2\mathrm{Re}\,ac^*) - \sqrt{2}\sin\vartheta\,\mathrm{Re}(ad^*+bf^*-cd^*)\}. \tag{7}$$

We note that these formulas can be obtained from Eqs. (2a) and (3a) by determining the coefficients in explicit form in dependence on M and the mean value of the spin operators

$$A=\tfrac{1}{3}\mathrm{Sp}\,M=\tfrac{1}{3}(2a+f),\quad B=\tfrac{1}{2}\mathrm{Sp}\,MS_N=i(b-d)/\sqrt{2},$$

$$C=\tfrac{3}{2}\mathrm{Sp}\,M(S_P^2+S_K^2-\tfrac{4}{3}\delta_{ik})=\tfrac{1}{2}(a-f)+\tfrac{3}{2}c,$$

$$D=\tfrac{1}{2}\mathrm{Sp}\,M(S_P^2-S_K^2)=(a-c-f)/2\cos\vartheta = \sqrt{1/2}\,(b+d)/\sin\vartheta.$$

Thus, to carry out phase analysis and to establish the scattering amplitude of deuterons on a zero-spin nucleus, it is necessary to determine four parameters from experiment. Measurement of the differential cross section of the doubly-scattered beam of deuterons makes it possible to determine three quantities: the coefficients for $\cos 2\varphi$, $\cos\varphi$ and the free term. Only one additional experiment is necessary. Consequently, the study of double scattering makes it possible to solve the problem that has been presented.

Another, and in a certain sense opposite, approach to this problem is also possible. Assuming a certain definite form for the interaction potential as, for example, is the case for the optical model, and setting

$$V=V_0\frac{1+i\zeta}{1+\exp[(r-r_0)/t]} - \frac{2\xi}{tr_0}V_0(1+i\zeta)\frac{\exp[(r-r_0)/t]}{\{1+\exp[(r-r_0)/t]\}^2}(\mathbf{SL}) + \begin{cases}(3r_0^2-r^2)(Ze^2/2r_0^3) & \text{for } r<r_0\\ Ze^2/r & \text{for } r>r_0\end{cases} \tag{8}$$

(ξ is the constant of spin-orbit coupling, and $r_0 = 1.28\,A^{1/3}\times 10^{-13}$ cm), one can find the phase shift of the different waves and thus describe the elastic scattering. In this case, the spin-orbit Coulomb interaction is not taken into account, since it is small in comparison with the spin-orbit nuclear interaction. For an exact solution, numerical calculations are necessary which can be completed only on electronic computing machines. An explicit expression for the amplitude of the scattered wave can be written down in the WKB approximation:

$$f=-\frac{\lambda\eta}{2\mu^2}\exp[-i\eta\ln\mu^2]+\sum_l k^{-1}(2l+1)\exp[i(2\eta_l+\Delta_l)] \times\sin\Delta_l P_l(\cos\vartheta)+\sum_{m_{s'}}\chi_{m_{s'}}\sum_{m_s}M_{m_s m_{s'}}(\vartheta,\varphi)\,a_{m_s}. \tag{9}$$

Here the phase shifts are determined by the potential (under the assumption $V \ll E$) in the following fashion:

$$\Phi_l=\frac{m}{(kh)^2}\int_{\varkappa}^{\infty}\frac{V_l(\tau)\,\tau d\tau}{[\tau^2-\varkappa^2]^{1/2}}\qquad\left(\tau=kr_0,\ \varkappa=l+\tfrac{1}{2}\right),$$

while the phase shift satisfying the Coulomb scattering,

$$\Delta_l=-\eta\ln\left[\tau+\sqrt{\tau^2-\varkappa^2}+\eta\sqrt{\tau^2-\varkappa^2}\left\{\frac{3}{2\tau}-\frac{\tau+2\varkappa^2}{\sigma\tau^3}\right\}\right],$$

where

$$\eta_l=\arg\Gamma(l+1+i\eta),\qquad \eta=\frac{mZe^2}{h^2k},\qquad \mu=\sin\frac{\vartheta}{2}.$$

However, for the form of potential chosen, only numerical methods are applicable, and at high energies, one must consider a large number of waves ($l_{max}\approx kr_0$) and the computations become cumbersome. Only in the Born approximation are the calculations carried out in elementary fashion.

According to the results of Fernbach, Heckrotte, and Lepore,[6] an interaction of type (8) is obtained if the nuclear potential is regarded as the result of an average pair interaction of the nucleons. Then V_d can be regarded as the sum of V_n and V_p. If the y axis is directed along the vector **N**, the parameters in the formula for the cross section of the doubly scattered beam take the form

$$\alpha=\delta=-\frac{1}{3}\frac{|H|^2}{|G|^2+{}^2/_3|H|^2},\qquad i\beta=-\frac{4i}{3}\frac{\mathrm{Im}(GH^*)}{|G|^2+{}^2/_3|H|^2}, \tag{10}$$

while the differential cross section for unpolarized deuterons is of the form

$$I_0=\left(\frac{2M}{h}\right)^2\left(\frac{4\alpha_1}{q}\right)^2\left(\tan^{-1}\frac{q}{4\alpha_1}\right)\left\{|G|^2+\frac{2}{3}|H|^2\right\},$$

$$G(\vartheta)=\frac{2V_0(1+i\zeta)}{q}\int_0^\infty J_1(qr)\frac{\partial\rho(r)}{\partial r}r^2dr+\frac{3}{2}\frac{Ze^2}{r_0}\left\{\frac{1}{q^3}\sin qr_0 - \frac{r_0}{q^2}\cos qr_0+\frac{r_0^2}{q^2}\left(1+\frac{2}{(qr_0)^2}\right)\right\}J_1(qr_0),$$

$$H(\vartheta)=\frac{2\xi V_0(1+i\zeta)}{q}k^2\sin\vartheta\int J_1(qr)\frac{\partial\rho(r)}{\partial r}r^2dr,\quad q=2k\sin\frac{\vartheta}{2}. \tag{11}$$

In the work of Chamberlain et al.,[4] 156-Mev deuterons undergo double scattering on carbon. At scattering angle $\vartheta = \vartheta' = 20°$, the following dependence of the differential cross section on the azimuthal angle φ was obtained experimentally:

$$I(20, \varphi) = u + v\cos\varphi + w\cos 2\varphi,$$

and the values of the parameters (in mb/sr) amounted to

$$u = 50.3 \pm 2.2, \quad v = 15.3 \pm 1.9, \quad w = -1.8 \pm 3.6.$$

The statistical errors for the ratios u:v:w are given.

Calculations carried out by these authors in the momentum approximation, and calculations with a square well, lead to results that differ several fold from the experimental. In the calculation with the potential (8), under the assumption that

$$\xi = 3.3 \cdot 10^{-27}\,\text{cm}^2, \quad t = 0.7 \cdot 10^{-13}\,\text{cm},$$

$$r_0 = 1.28\,A^{1/3} \cdot 10^{-13}\,\text{cm}, \quad V_0 = 42\,\text{Mev}, \quad \zeta = 0.3,$$

the following values were obtained for the parameters:

$$u = 42.6, \quad v = 10.9, \quad w = 6.6\,\text{mb/sr}$$

and if we put $V_0 = 30$ Mev and $\zeta = 0.4$, then they are shown to be equal to

$$u = 20.6, \quad v = 8.4, \quad w = 3.25\,\text{mb/sr}$$

Both the absolute values and the ratios of the values of the parameters that were obtained agree with the experimental data better than in the researches previously mentioned. Further divergence from the experimental values can evidently be eliminated by application of a more accurate method of calculation and consideration of the breakdown of the deuteron.

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