THE EINSTEIN STATICS IN CONFORMAL SPACE

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A new definition is proposed for a three-dimensional spatial metric that corresponds to a given four-dimensional space-time metric. The space with a metric so defined is called conformal space. It is shown that the equations of the Einstein statics in conformal space are simple in form and lead to definite expressions for the energy density and for the gravitational tensions. Cases of spherical and axial symmetry are considered. It is pointed out that the concept of conformal space can be useful in the general non-static case as well.

INTRODUCTION

IN the Einstein theory of gravitation, the static is characterized, as is well known, by the fact that the fundamental tensor does not depend upon time and its mixed coefficient components are equal to zero. In this case, the time coordinate is determined uniquely (it is harmonic). For the spatial coordinates, there remains the group of their transformations among themselves. Therefore, it is natural here to make use of the apparatus of three-dimensional tensor analysis and to write the equations of gravity in corresponding fashion. This has also been done by Levi-Civita.[1] However, no attention was paid in these researches to the fact that complete separation of space and time, which is possible in the static case, can be achieved by different means. Usually a metric corresponding to the spatial part of ds^2 is attributed to the space. In other words, it is tacitly assumed that the square of the spatial interval must enter into the expression for the square of the four-dimensional interval with the coefficient (-1). But one can introduce another assumption. In the present paper, we shall show that the most natural one is this capability of separation of space and time, in which the metric of the space is conformally connected with the metric usually employed, in such a way that the coefficient mentioned will not be constant but will be a quantity inversely proportional to the coefficient in front of the square of the differential of the time.

A space with a metric defined in such a fashion — we shall call it conformal space — posseses many significant properties. First of all, it is approximately Euclidean; one can assume it to be Euclidean with much greater accuracy than in the usual definition of the spatial metric. In this case, the coordinates, harmonic in a four-dimensional sense, will be harmonic also in a three-dimensional sense (which, as is well known, does not hold in the usual definition of the metric of space). The well-known previous solutions of the equations of Einstein with spherical and axial symmetry (the solutions of Schwartzschild, Weyl, and Levi-Civita) are obtained in especially simple fashion if one starts out from the equations of gravity in conformal space. A further advantage of conformal space is the fact that the equations of Einstein written for it, without any artificial hypothesis, lead to definite expressions for the energy density and for gravitational tensions.

1. THE TRANSFORMATION OF LEVI-CIVITA

Let us consider the formulation of the Einstein statics proposed by Levi-Civita (see references 1 and 2).

The static case is characterized by the well-known equations*

$$\frac{\partial g_{\mu\nu}}{\partial t} = 0, \qquad g_{0i} = 0, \qquad (x_0 = t), \tag{1.1}$$

such that if we set†

$$g_{00} = c^2V^2, \qquad g_{ik} = -a_{ik}, \tag{1.2}$$

then one can write the quantity ds^2 in the form

$$ds^2 = c^2V^2\,dt^2 - a_{ik}\,dx_i\,dx_k. \tag{1.3}$$

*The Greek letters μ, ν, etc., take on the values 0, 1, 2, 3; the Latin letters i, k, etc., take on the values 1, 2, 3.

†In distinction from the notation used in references 1 and 2, we now write the coefficient for dt^2 in the form c^2V^2, and not in the form V^2.

Levi-Civita proposed to express all the four-dimensional quantities entering into the Einstein equations in terms of three-dimensional quantities referred to a space with the square of the linear element

$$dl^2 = a_{ik}\,dx_i\,dx_k. \tag{1.4}$$

By this means, in particular, a relation is obtained between the four-dimensional curvature tensor $(R_{\mu\nu})_g$, constructed from the fundamental tensor $g_{\mu\nu}$ and the three-dimensional curvature tensor $(R_{ik})_a$ constructed from the a_{ik}. For the spatial components, we have

$$(R_{ik})_g = (R_{ik})_a + \frac{(V_{ik})_a}{V}, \tag{1.5}$$

where $(V_{ik})_a$ is the second covariant derivative of V in the sense of the metric (1.4). The mixed components are equal to zero, while the time component is equal to

$$R_{00} = -c^2V(\Delta V)_a, \tag{1.6}$$

where

$$(\Delta V)_a = a^{ik}(V_{ik})_a \tag{1.7}$$

is the Laplace operator in the sense of (1.4). The gravitational equations of Einstein can be transformed by means of these expressions. The corresponding formulas are given in the book of Levi-Civita,[1] and also in our book.[2]

2. TRANSITION TO CONFORMAL SPACE

In contrast with Levi-Civita, we represent the square of the four-dimensional interval in the form

$$ds^2 = c^2V^2\,dt^2 - \frac{1}{V^2}h_{ik}\,dx_i\,dx_k \tag{2.1}$$

and write the spatial metric

$$d\sigma^2 = h_{ik}\,dx_i\,dx_k, \tag{2.2}$$

so that

$$ds^2 = c^2V^2dt^2 - \frac{1}{V^2}d\sigma^2. \tag{2.3}$$

Thus, we have

$$g_{00} = c^2V^2, \qquad g_{0i} = 0, \qquad g_{ik} = -\frac{h_{ik}}{V^2}, \tag{2.4}$$

and also

$$g^{00} = \frac{1}{c^2V^2}, \qquad g^{0i} = 0, \qquad g^{ik} = -V^2h^{ik} \tag{2.5}$$

and, finally,

$$\sqrt{-g} = \frac{c}{V^2}\sqrt{h}, \tag{2.6}$$

where h is the determinant composed of the h_{ik}. All the three-dimensional quantities will be related to the metric (2.2). From a comparison of (1.3) with (2.1), we obtain

$$dl^2 = \frac{1}{V^2}d\sigma^2 \tag{2.7}$$

and, consequently,

$$a_{ik} = \frac{1}{V^2}h_{ik}, \qquad a^{ik} = V^2h^{ik}. \tag{2.8}$$

Thus, both the spaces (1.4) and (2.2) are conformal to one another. According to the general formula of Finzi (see reference 1), for two such spaces, the covariant components of the curvature tensor of fourth rank are related by the equation

$$V^2(R_{kj,li})_a = R_{kj,li} + h_{kl}\frac{V_{ij}}{V} - h_{ki}\frac{V_{lj}}{V} + h_{ij}\frac{V_{kl}}{V} - h_{li}\frac{V_{ik}}{V} + (h_{ki}h_{jl} - h_{kl}h_{ji})\frac{V_mV^m}{V^2}. \tag{2.9}$$

On the right hand side of this formula, as in the following formulas, all the three-dimensional tensor symbols refer to the metric (2.2). It follows from Eq. (2.9) that

$$(R_{ik})_a = R_{ik} - \frac{V_{ik}}{V} + h_{ik}\left(-\frac{\Delta V}{V} + 2\frac{V_jV^j}{V^2}\right). \tag{2.10}$$

On the other hand, for the second covariant derivatives of some quantity W, we have

$$(W_{ik})_a = W_{ik} + \frac{V_i}{V}W_k + \frac{V_k}{V}W_i - h_{ik}\frac{V^jW_j}{V} \tag{2.11}$$

and, if $W = V$,

$$(V_{ik})_a = V_{ik} + \frac{2V_iV_k}{V} - h_{ik}\frac{V^jV_j}{V}. \tag{2.12}$$

Then,

$$(\Delta V)_a = V^2\left(\Delta V - \frac{V_jV^j}{V}\right). \tag{2.13}$$

With the help of (2.10) and (2.12), Eq. (1.5) gives the following expression for the spatial components of the four-dimensional curvature tensor of second rank

$$(R_{ik})_g = R_{ik} + \frac{2V_iV_k}{V^2} + h_{ik}\left(-\frac{\Delta V}{V} + \frac{V_jV^j}{V^2}\right). \tag{2.14}$$

In accordance with (1.6), the time component is equal to

$$(R_{00})_g = c^2V^2(-V\Delta V + V_jV^j). \tag{2.15}$$

The mixed components are equal to zero, as has already been pointed out. The four-dimensional invariant is equal to

$$(R) = 2V\Delta V - 4V_jV^j - V^2R. \tag{2.16}$$

Hence we obtain for the components of the Einstein tensor

$$G_{\mu\nu} = (R_{\mu\nu})_g - \frac{1}{2}g_{\mu\nu}(R)_g \tag{2.17}$$

the following expressions

$$G_{ik} = H_{ik} + 2\frac{V_i V_k}{V^2} - h_{ik}\frac{V_j V^j}{V^2}, \qquad (2.18)$$

$$G_{00} = c^2 V^2 \{-V^2 H - 2V\Delta V + 3V_j V^j\}. \qquad (2.19)$$

Here H_{ik} is the three-dimensional conservative tensor of conformal space, and H is its invariant

$$H_{ik} = R_{ik} - \frac{1}{2} h_{ik} R, \qquad H = -\frac{1}{2} R. \qquad (2.20)$$

In this case

$$R_{ik} = H_{ik} - h_{ik} H. \qquad (2.21)$$

We also write down the expression for the four-dimensional D'Alembert operator

$$\Box\psi = \frac{1}{c^2 V^2}\frac{\partial^2 \psi}{\partial t^2} - V^2 \Delta\psi. \qquad (2.22)$$

Here $\Delta\psi$ denotes the Laplace operator in conformal space, as before,

$$\Delta\psi = \frac{1}{\sqrt{h}}\frac{\partial}{\partial x_i}\left(\sqrt{h}\, h^{ik}\frac{\partial\psi}{\partial x_k}\right). \qquad (2.23)$$

It is then seen that the coordinates which are harmonic in a four-dimensional sense will be harmonic also in the sense of three-dimensional conformal space.

An important difference between our formulas (2.18) for the spatial components of the conservative tensor and the formulas of Levi-Civita following from (1.5) and (1.6) is the fact that our formulas, which are written for conformal space, do not contain second derivatives of the quantity V.

3. THE EINSTEIN EQUATIONS IN CONFORMAL SPACE

Introducing the expressions (2.15) and (2.16) for the Einstein tensor in the equation of gravity

$$G_{\mu\nu} = -\varkappa T_{\mu\nu}, \qquad \varkappa = \frac{8\pi\gamma}{c^2}, \qquad (3.1)$$

we obtain

$$H_{ik} + 2\frac{V_i V_k}{V^2} - h_{ik}\frac{V_j V^j}{V^2} = -\varkappa T_{ik}, \qquad (3.2)$$

$$H + 2\frac{\Delta V}{V} - 3\frac{V_j V^j}{V^2} = \frac{1}{c^2 V^4}\varkappa T_{00} = \varkappa c^2 T^{00} \qquad (3.3)$$

Eliminating the quantity H from (3.3) by means of (3.2), we can write

$$2\left(\frac{\Delta V}{V} - \frac{V_j V^j}{V^2}\right) = \varkappa\mu, \qquad (3.4)$$

where

$$\mu = c^2 T^{00} + h^{ik} T_{ik} \qquad (3.5)$$

or

$$\mu = \frac{1}{V^2}(T_0^0 - T_1^1 - T_2^2 - T_3^3). \qquad (3.6)$$

In the last equation, the quantities T^{α}_{β} are also understood in the four-dimensional space. The physical meaning of the quantity μ is the mass density in conformal space.*

In what follows it is advantageous to replace the quantity V by Φ through the formula

$$V = e^{-\Phi}, \qquad (3.7)$$

so that the connection of the metric of space-time with the metric of conformal space will have the form

$$ds^2 = c^2 e^{-2\Phi} dt^2 - e^{2\Phi} d\sigma^2. \qquad (3.8)$$

As a consequence of

$$\Phi_i = -\frac{V_i}{V}, \qquad \Delta\Phi = -\frac{\Delta V}{V} + \frac{V_j V^j}{V^2}, \qquad (3.9)$$

the equations of gravity are written

$$H_{ik} = -2\Phi_i\Phi_k + h_{ik}\Phi_j\Phi^j - \varkappa T_{ik}, \qquad (3.10)$$

$$\Delta\Phi = -\frac{1}{2}\varkappa\mu. \qquad (3.11)$$

Since μ is the mass density, and the Einstein constant κ is connected with the Newtonian constant γ by the relation (3.1), the the latter equation is essentially the equation for the Newtonian potential U. We can write approximately

$$\Phi = \frac{U}{c^2} \qquad (3.12)$$

and, consequently,

$$\Phi_i = \frac{g_i}{c^2}, \qquad (3.13)$$

where g_i is the component of the acceleration due to gravity.

We shall now make clear the physical meaning of Eq. (3.10). We set

$$Q_{ik} = \frac{1}{\varkappa}(2\Phi_i\Phi_k - h_{ik}\Phi_j\Phi^j) \qquad (3.14)$$

and write the equation (3.10) in the form

$$H_{ik} = -\varkappa(Q_{ik} + T_{ik}). \qquad (3.15)$$

The divergence of the tensor Q_{ik} is equal to

$$\nabla^k Q_{ik} = \frac{2}{\varkappa}\Phi_i\Delta\Phi = -\mu\Phi_i \qquad (3.16)$$

by virtue of (3.11). On the other hand, since H_{ik} is the conservative tensor of conformal space, then its divergence is equal to zero. Therefore, it follows from Eq. (3.15) that

$$\nabla^k T_{ik} = \mu\Phi_i. \qquad (3.17)$$

It is not difficult to see that this equation represents a generalization of the ordinary equations of the statics of an elastic body in a gravitational

*The value obtained for μ agrees with the well-known expression of Tolman (see the book of Moeller,[3] page 341).

field. It is natural to explain the expression Q_{ik} as the spatial part of the energy-momentum tensor of the gravitational field or, inasmuch as we are dealing with statics here, as a quantity proportional to the tensor of gravitational tension. From this viewpoint the equations (3.15) written down for conformal space form an analogue with the Einstein equations (3.1) written down for space-time. In the present and other equations, the left hand side is a conservative tensor while the right hand is the energy-momentum tensor or the stress tensor. Here the gravitational stresses appear in explicit form only after separation of space from time and after a transition to conformal space.

The gravitational equations in the form (3.10) show that conformal space will be almost Euclidean. In fact, as is seen from (3.13), the right hand side of (3.10) will be of the order of g_i^2/c^4; this leads to the result that the deviation of h_{ik} from the Euclidean values will be of the order of U^2/c^4, and not of order U/c^2, as for a space with the metric (1.4).

All the formulas of this section [except the estimates (3.12) and (3.13)] contain no approximations if we do not consider the approximation of the static character of the field. All our equations are covariant relative to choice of coordinates in conformal space.

For empty space $T_{\alpha\beta} = 0$ we shall have $\mu = 0$ and Eq. (3.11) will be the consequence of (3.10).

4. SCHWARTZSCHILD'S SOLUTION IN CONFORMAL SPACE

In the case of spherical symmetry we can set

$$d\sigma^2 = dr^2 + \rho^2 (d\vartheta^2 + \sin^2\vartheta d\varphi^2). \tag{4.1}$$

We shall assume the coefficient of dr^2 equal to unity, which presents no limitations. The quantity ρ is a function of the single variable r. The Laplace operator in a space with the metric (4.1) is written

$$\Delta\psi = \frac{1}{\rho^2}\left(\frac{\partial}{\partial r}\left(\rho^2 \frac{\partial\psi}{\partial r}\right) + \Delta^*\psi\right), \tag{4.2}$$

where $\Delta^*\psi$ is the Laplace operator on a sphere. Combining the quantities H_{ik}, we obtain

$$H_{rr} = \frac{1-\rho'^2}{\rho^2}, \qquad H_{\vartheta\vartheta} = -\rho\rho'', \qquad H_{\varphi\varphi} = -\rho\rho''\sin^2\vartheta, \tag{4.3}$$

while the nondiagonal elements H_{ik} are equal to zero. The equation for empty space has the form

$$H_{rr} = -\Phi_r^2, \qquad H_{\vartheta\vartheta} = \rho^2\Phi_r^2. \tag{4.4}$$

Since we have $\Delta\Phi = 0$, then

$$\rho^2\Phi_r = -\alpha, \tag{4.5}$$

where α is a constant (the gravitational mass radius). As a result of (4.3) and (4.5), the first equation of (4.2) gives

$$\rho'^2 = 1 + \frac{\alpha^2}{\rho^2}, \qquad \rho\rho' = \sqrt{\rho^2+\alpha^2}. \tag{4.6}$$

Since

$$\rho\rho'' + \rho'^2 = 1, \tag{4.7}$$

the second equation here is also satisfied. Hence

$$r = \sqrt{\rho^2+\alpha^2}, \qquad \rho = \sqrt{r^2-\alpha^2}. \tag{4.8}$$

We shall set the additive constant equal to zero. Then,

$$d\sigma^2 = dr^2 + (r^2-\alpha^2)(d\vartheta^2 + \sin^2\vartheta d\varphi^2). \tag{4.9}$$

With the help of (4.2), it is easy to prove that the coordinates

$$x_1 = r\sin\vartheta\cos\varphi, \qquad x_2 = r\sin\vartheta\sin\varphi,$$
$$x_3 = r\cos\vartheta \tag{4.10}$$

will be harmonic. In order to derive the solution of the equations, it suffices to find C. We have from (4.5)

$$\frac{d\Phi}{dr} = -\frac{\alpha}{r^2-\alpha^2}, \qquad \Phi = \frac{1}{2}\log\frac{r+\alpha}{r-\alpha}, \tag{4.11}$$

since for $r = \infty$, Φ should be equal to zero. Thus, finally,

$$ds^2 = c^2\frac{r-\alpha}{r+\alpha}dt^2 - \frac{r+\alpha}{r-\alpha}d\sigma^2, \tag{4.12}$$

where $d\sigma^2$ has the form (4.9).

The computations that have been written out have the purpose of showing how the Schwartzschild problem is easily solved in conformal space.

5. WEYL'S SOLUTION IN CONFORMAL SPACE

For axial symmetry the value of $d\sigma^2$ can be written in the form

$$d\sigma^2 = f^2(dz^2 + dr^2) + \rho^2 d\varphi^2, \tag{5.1}$$

where f and ρ are functions of z and r, but not of φ. Computing the quantities H_{ik}, we have

$$H_{zz} = -\frac{1}{\rho}\frac{\partial^2\rho}{\partial z^2} - \frac{1}{f}\frac{\partial f}{\partial r}\cdot\frac{1}{\rho}\frac{\partial\rho}{\partial r} + \frac{1}{f}\frac{\partial f}{\partial z}\cdot\frac{1}{\rho}\frac{\partial\rho}{\partial z},$$

$$H_{rr} = -\frac{1}{\rho}\frac{\partial^2\rho}{\partial r^2} + \frac{1}{f}\frac{\partial f}{\partial r}\cdot\frac{1}{\rho}\frac{\partial\rho}{\partial r} - \frac{1}{f}\frac{\partial f}{\partial z}\cdot\frac{1}{\rho}\frac{\partial\rho}{\partial z},$$

$$H_{rz} = \frac{1}{\rho}\frac{\partial^2\rho}{\partial r\partial z} - \frac{1}{f}\frac{\partial f}{\partial r}\cdot\frac{1}{\rho}\frac{\partial\rho}{\partial z} - \frac{1}{f}\frac{\partial f}{\partial z}\cdot\frac{1}{\rho}\frac{\partial\rho}{\partial r}, \tag{5.2}$$

$$H_{\varphi\varphi} = -\frac{\rho^2}{f^2}\left\{\frac{\partial}{\partial r}\left(\frac{1}{f}\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{f}\frac{\partial f}{\partial z}\right)\right\}. \tag{5.3}$$

It then follows that

$$R_{\varphi\varphi} = \frac{\rho}{f^2}\left(\frac{\partial^2\rho}{\partial r^2} + \frac{\partial^2\rho}{\partial z^2}\right). \tag{5.4}$$

If $R_{\varphi\varphi} = 0$, then we can replace z by such a function $\zeta(z, r)$ that $\zeta + i\rho$ would be a function of the complex variable $z + ir$. Then the quantity $d\sigma^2$ takes the form

$$d\sigma^2 = F^2(d\zeta^2 + d\rho^2) + \rho^2 d\varphi^2. \tag{5.5}$$

The variables ζ and ρ represent the cyclindrical coordinates of Weyl (see Sec. 35 of Weyl's book[4]). In these coordinates, the expressions for H_{ik} are simplified and can be written

$$H_{\zeta\zeta} = -H_{\rho\rho} = \frac{1}{\rho F}\frac{\partial F}{\partial \rho}, \qquad H_{\rho\zeta} = -\frac{1}{\rho F}\frac{\partial F}{\partial \zeta}, \tag{5.6}$$

$$H_{\varphi\varphi} = -\frac{\rho^2}{F^2}\left\{\frac{\partial}{\partial\rho}\left(\frac{1}{F}\frac{\partial F}{\partial\rho}\right) + \frac{\partial}{\partial\zeta}\left(\frac{1}{F}\frac{\partial F}{\partial\zeta}\right)\right\}. \tag{5.7}$$

The corresponding metric (5.5) of the Laplace operator is equal to

$$\Delta\psi = \frac{1}{F^2}\left\{\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{\partial^2\psi}{\partial\zeta^2}\right\} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2} \tag{5.8}$$

and for functions which do not depend upon φ, the Laplace equation will have the same form as in Euclidean space.

Let us write down the conditions for the stress tensor, which follow from our assumption that $R_{\varphi\varphi} = 0$ and that the function Φ entering into (3.10) does not depend on φ. We must then have

$$T_\rho^\rho + T_\zeta^\zeta = T_r^r + T_z^z = 0, \qquad T_\varphi^\varphi = 0. \tag{5.9}$$

Moreover, the stress tensor must apparently satisfy the equations of statics (3.17).

For empty space, the equations of gravity (3.10) take the form

$$\frac{1}{F}\frac{\partial F}{\partial\rho} = \rho(\Phi_\rho^2 - \Phi_\zeta^2), \qquad \frac{1}{F}\frac{\partial F}{\partial\zeta} = 2\rho\Phi_\rho\Phi_\zeta, \tag{5.10}$$

where Φ is the solution of the equation

$$\frac{\partial^2\Phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{\partial^2\Phi}{\partial\zeta^2} = 0. \tag{5.11}$$

When Φ is given, the function F is obtained from (5.10) by integration of the total differential (under the condition that $F = 1$ when $\rho = 0$). The Weyl solution consists of this also. We shall see that, in conformal space, this solution is obtained in an extremely simple way.*

*Here it is appropriate to turn our attention to the fact that in the well-known book of Weyl[4] there are typographical errors in the equations for the axially symmetric solution which can lead to misunderstandings, inasmuch as the derivation of the formula is not given in the book.

6. CONNECTION BETWEEN THE WEYL AND THE SCHWARTZSCHILD SOLUTIONS

Let us express the solution with spherical symmetry considered earlier in the Weyl coordinates. If we mean by ρ, ϑ, φ the spherical coordinates of Sec. 4, and by ζ, ρ, φ the Weyl coordinates, then obviously φ will be the same in both cases. As a consequence of (5.8), $\zeta = r\cos\vartheta$ since the coordinate ζ is harmonic. The value of ρ can be obtained by equating the coefficients for $d\varphi^2$ in (5.5) and (4.8). Then

$$\zeta = r\cos\vartheta, \qquad \rho = \sqrt{r^2 - \alpha^2}\sin\vartheta. \tag{6.1}$$

Conversely,

$$r = \frac{1}{2}\left(\sqrt{\rho^2 + (\zeta+\alpha)^2} + \sqrt{\rho^2 + (\zeta-\alpha)^2}\right),$$

$$\alpha\cos\vartheta = \frac{1}{2}\left(\sqrt{\rho^2 + (\zeta+\alpha)^2} - \sqrt{\rho^2 + (\zeta-\alpha)^2}\right). \tag{6.2}$$

Spheres in harmonic coordinates will correspond to ellipsoids of revolution in Weyl coordinates with foci at the points $\rho = 0$, $\zeta = \alpha$, and $\zeta = -\alpha$. If we set

$$r = \alpha\cosh u, \tag{6.3}$$

then u, ϑ will be the ordinary ellipsoidal coordinates. In these coordinates, Eq. (5.11) is written

$$\frac{\partial^2\Phi}{\partial u^2} + \frac{\partial^2\Phi}{\partial\vartheta^2} + \coth u\frac{\partial\Phi}{\partial u} + \cot\vartheta\frac{\partial\Phi}{\partial\vartheta} = 0. \tag{6.4}$$

The desired solution will be a function depending only on u, namely

$$\Phi = \frac{1}{2}\log\frac{\cosh u + 1}{\cosh u - 1}. \tag{6.5}$$

This solution coincides with (4.11). Integration of the total differential gives

$$F^2 = \frac{\sinh^2 u}{\sinh^2 u + \sin^2\vartheta}, \tag{6.6}$$

and, since

$$d\rho^2 + d\zeta^2 = \alpha^2(\sinh^2 u + \sin^2\vartheta)(du^2 + d\vartheta^2), \tag{6.7}$$

then

$$d\sigma^2 = \alpha^2\sinh^2 u\,(du^2 + d\vartheta^2 + \sin^2\vartheta\,d\varphi^2). \tag{6.8}$$

We see that in the Weyl coordinates the spherical symmetry is lost and is replaced by axial symmetry. If we take the solution in which Φ is spherically symmetric in the Weyl coordinates,

$$\Phi = \frac{\alpha}{\sqrt{\rho^2 + \zeta^2}}, \qquad \log F = -\frac{\alpha^2\rho^2}{2(\rho^2 + \zeta^2)^2},$$

then it is shown that it has a singularity in the plane $\zeta = 0$ for $\rho \to 0$, which it is difficult to explain physically. Therefore, in the given case, the Weyl coordinates correspond less to the character of the problem than the harmonic coordinates.

7. CONCLUDING REMARKS

The possibility of a simple physical interpretation of the gravitational equations, which are written down for conformal space, the simplicity of these equations and their solution speak in convincing fashion on behalf of the expediency of the introduction of the concept of conformal space. To this one can add the fact that this concept makes it possible to interpret very simply the known fact that the deviation of a light ray passing by a heavy mass is twice as great as one would expect on the basis of the so-called principle of equivalence. This fact is explained simply in this way, that, according to (2.22), the velocity of light in conformal space is equal to cV^2, and not to cV as in the ordinary definition of the spatial metric.

In the present work we have considered only the static case. For the non-static case the transformation of the Einstein equations to the three-dimensional form (without transition to conformal space) was carried out by Zel'manov.[5] The transition to conformal space in this case is not entirely unique, since it depends on the choice of the time coordinate. For corresponding additional conditions laid on the time coordinate, such a transition can nevertheless be shown to be expedient, especially in approximate calculations, when the fact that the conformal space is almost Euclidean can introduce material simplifications.

An attempt to construct a theory of the gravitational field with a single function similar to our function Φ has been made by Papapetrou.[6] The theory of Papapetrou corresponds to the assumption that the conformal space is Euclidean. There is no doubt that in the static case, the theory of Papapetrou represents a reasonable approximation to the Einstein theory. However, in the general, non-static case, a one field function is clearly insufficient. As a subsequent calculation, one could introduce four functions — the scalar and vector potentials of gravity. The approximate solutions of the gravitational equations given by us in reference 2 demonstrates this. There it was shown that, with the relative error of the order of U^2/c^4 in g_{ik}, and of the order of U^3/c^6 in g_{00}, the gravitational field can be expressed in terms of four functions U^* and U_i, which satisfy the equations

$$\Delta U^* - \frac{1}{c^2}\frac{\partial^2 U^*}{\partial t^2} = -4\pi\gamma\left(c^2 T^{00} + T^{kk}\right), \tag{7.1}$$

$$\Delta U_i = -4\pi\gamma\rho v_i, \tag{7.2}$$

where ρv_i is the density of mass flow [Eqs. (68.30) and (68.19) in reference 3]. Equation (7.1) corresponds in the static case to Eq. (3.11), while the right-hand side of (7.1) is proportional to the quantity μ from (3.05). In the given approximation, the conformal space is Euclidean and we have

$$g_{00} \approx c^2\frac{c^2 - U^*}{c^2 + U^*}, \qquad g_{0i} \approx \frac{4}{c^2}U_i. \tag{7.3}$$

It is not excluded that the concept of conformal space can find application in an exact non-static theory.

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